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Identifying inverse source for Diffusion equation with Conformable time derivative by Fractional Tikhonov method

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Abstract

In this paper, we study inverse source for diffusion equation with conformable derivative:

$$CoD_t^{(\gamma)}u - \Delta u = \Phi(t)\mathcal{F}(x) \text{ where } 0 < \gamma < 1,\ (x,t) \in \Omega \times (0,T).$$

We survey the following issues: The error estimate between the sought solution and the regularized solution under a priori parameter choice rule; The error estimate between the sought solution and the regularized solution under a posteriori parameter choice rule; Regularization and $\mathscr{L}_p$ estimate by Truncation method.

Keywords: Fractional diffusion equation; Inverse source problem; Conformable derivative; Regularization methods; Fractional Tikhonov method.
2010 MSC: 26A33, 35B65,35B05, 35R11.

1. Introduction

In this article, for the equation

$$CoD_t^{(\gamma)}u(x,t) - \Delta u(x,t) = \Phi(t)\mathcal{F}(x), \quad x \in \Omega, t \in (0,T) \tag{1}$$

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accompanied with boundary conditions

$$u(x,t)|_{x\in\partial\Omega} = 0,\ t \in (0,T), \tag{2}$$

and the initial condition

$$u(x,0) = u_0(x),\ x \in \Omega, \tag{3}$$

and later final condition

$$u(x,T) = \ell(x),\ x \in \Omega, \tag{4}$$

The function $u = u(x,t)$ represents a concentration of contaminant at a position x and time t. $CoD_t^{(\gamma)}$ is the symbol of representation the conformable time derivative with order $\gamma \in (0,1)$ (see Khalil et al. [1]): for a given function $G : [0,\infty) \to \mathbb{R}$, the Conformable fractional of order $\gamma \in (0,1]$ is defined by

$$CoD_t^{(\gamma)} G(t) = \lim_{\rho\to 0} \frac{G(t+\rho t^{1-\gamma}) - G(t)}{\rho}, \tag{5}$$

for all $t > 0$. For some $(0,t_0), t_0 > 0$ and the $\lim_{t\to t_0^+} CoD_t^{(\gamma)} G(t)$ exists, then

$$CoD_t^{(\gamma)} G(t_0) = \lim_{t\to t_0^+} CoD_t^{(\gamma)} G(t).$$

Equations with fractional derivatives and inverse problems to them appear in different branches of science and engineering. Fractional calculus has many applications in the real world interested [2–6]. There are many types of fractional derivatives: Riemann-Liouville, Caputo, Conformable, Grunwald-Letnikov fractional operators, ... (see [7–15] and references therein). Each defines fractional derivatives with properties that are advantageous in certain applications. Many properties of Conformable fractional can be found more details in [16],[17] and references therein. Consider the inverse source problem (1). By the definition of Hadamard [18] a problem is well-posed if it satisfies: the existence, the uniqueness, and the stability of the solution. This implies that if one of the three properties is not satisfied, the problem is ill-posed. According to our research experience, the stability property of the sought solution is most often violated. Therefore, to overcome this difficulty, a regularization method is required. We do not know observe the data Φ, ℓ, and using approximate data Φ^ϵ, ℓ^ϵ satisfies

$$\|\ell - \ell^\epsilon\|_{\mathscr{L}_2(\Omega)} + \|\Phi - \Phi^\epsilon\|_{\mathscr{L}_\infty(0,T)} \le \epsilon. \tag{6}$$

where $\epsilon > 0$ is the noise level. There are a lot of research results for an inverse source problem of a time-fractional diffusion equation. To do that, during the past decades, a lot of technical developments by mathematicians around the world: Quasi-Reversibility method, see [19], Quasi-Boundary Value method, which readers can see in [20, 21], the Landweber iterative method (see [22, 23]), the Fractional Landweber method (see [24]), a Tikhonov regularization method (see [25]), a Fourier truncation method (see [26]). However, the object of this topic is to restore the source function $\mathcal{F}(x)$ of the problem (1) by the Fractional Tikhonov method. Daniel Gerth introduced this method, see [27]. The fractional Tikhonov method is like being in the middle between the Quasi-Boundary Value method and the classical Tikhonov method (see [28]).

The next sections of the paper are divided into 3 sections. Section 2 provides the preliminary results to be used in this article, In Subsection 2.1, it gives the formula of source function $\mathcal{F}(x)$, in Subsection 2.2, we have the ill-posedness of problem (1)-(4) and the conditional stability is shown in Subsection 2.3. In Section 3, we consider the Fractional Tikhonov method by choosing a priori parameter choice (Subsection 3.1), an a posteriori parameter choice rule (Subsection 3.2). In Subsection 3.3, we receive the regularization and error in $\mathscr{L}_p$.

2. Preliminaries

Definition 2.1. Let $\langle\cdot\rangle$ be an inner product in $\mathscr{L}_2(\Omega)$. The notation $\|\cdot\|_X$ stands for in the norm in the Banach space X. We denote by $\mathscr{L}_p(0,T;X)$, $1 \le p \le \infty$, the Banach space of real-valued functions $u:(0,T)\to X$ measurable, providing that

$$\|u\|_{\mathscr{L}_p(0,T;X)} = \left(\int_0^T \big\|u(t)\big\|_p dt\right)^{\frac{1}{p}} < \infty, \text{ for } 1 \le p < \infty,$$

and

$$\|u\|_{\mathscr{L}_\infty(0,T;X)} = \text{ess}\sup_{t\in(0,T)} \|u(t)\|_X, \text{ for } p = \infty.$$

We begin this subsection by introducing a few properties of the eigenvalues of the operator Δ, see [30]. We have the following equality

$$\Delta \mathrm{e}_j(x) = -\lambda_j \mathrm{e}_j(x),\ x\in\Omega;\ \mathrm{e}_j = 0,\ x\in\partial\Omega,\ j\in\mathbb{N},$$

where $\{\lambda_j\}_{j=1}^{\infty}$denotes the set of eigenvalues of Δ satisfying

$$0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_j \le \ldots,$$

and $\lim_{j\to\infty}\lambda_j = \infty$. For any $m \ge 0$, we also define the space

$$\mathscr{H}^m(\Omega) = \Big\{u\in\mathscr{L}_2(\Omega) : \sum_{j=1}^{\infty}\lambda_j^{2m}\big|\langle u,\mathrm{e}_j\rangle\big|^2 < +\infty\Big\},$$

then $\mathscr{H}^m(\Omega)$ is a Hilbert space endowed with the norm

$$\|u\|_{\mathscr{H}^m(\Omega)} = \Big(\sum_{j=1}^{\infty}\lambda_j^{2m}|\langle u,e_j\rangle|^2\Big)^{\frac{1}{2}}.$$

Lemma 2.2. *Let $\underline{\Phi},\overline{\Phi}$ are positive constants such that $\underline{\Phi}\le\Phi\le\overline{\Phi}$. Let choose $\epsilon\in\Big(0,\dfrac{\Phi}{4}\Big)$, we obtain*

$$\frac{\Phi}{4} \le \big|\Phi^\epsilon(t)\big| \le \mathcal{B}\big(|\underline{\Phi}|,|\overline{\Phi}|\big). \tag{7}$$

Proof. The proof is completed in [28]. □

Lemma 2.3 (See [29])**.** *The following inclusions hold true:*

$$\begin{aligned}&\mathscr{L}_p(\Omega)\hookrightarrow\mathcal{D}(\mathcal{A}^s),\ \text{if } -\frac{d}{4}<s\le 0,\ p\ge\frac{2d}{d-4s},\\ &\mathcal{D}(\mathcal{A}^s)\hookrightarrow\mathscr{L}_p(\Omega),\ \text{if } -0<s\le\frac{d}{4},\ p\le\frac{2d}{d-4s}.\end{aligned} \tag{8}$$

2.1. The formula of source term $\mathcal{F}$

In this subsection, we introduce the mild solution of the following initial value problem

$$\begin{cases} CoD_t^\gamma u(x,t) - \Delta u(x,t) = \Phi(t)\mathcal{F}(x), & x \in \Omega, x \in (0,T), \\ u(x,t) = 0, & x \in \partial\Omega, t \in (0,T], \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{9}$$

Using the separation of variables to yield the solution of (9). Suppose that the exact u is defined by Fourier series

$$u(x,t) = \sum_{j=1}^{\infty} u_j(t)\mathrm{e}_j(x), \text{ with } u_j(t) = \big\langle u(\cdot,t), \mathrm{e}_j(\cdot)\big\rangle. \tag{10}$$

From (10), we get

$$u_j(t) = \sum_{j=1}^{\infty}\Big[\exp\big(-\lambda_j t^\gamma \gamma^{-1}\big)u_{0,j} + \big\langle \mathcal{F}(\cdot), \mathrm{e}_j(\cdot)\big\rangle \int_0^t \varsigma^{\gamma-1}\exp\big(-\lambda_j\big(t^\gamma - \varsigma^\gamma\big)\gamma^{-1}\big)\Phi(\varsigma)d\varsigma\Big]\mathrm{e}_j(x).$$

Letting $t = T$ and $u_{0,j} = 0$, we get

$$\ell_j(x) = u_j(T) = \sum_{j=1}^{\infty}\Big[\big\langle \mathcal{F}(\cdot), \mathrm{e}_j(\cdot)\big\rangle \int_0^T \varsigma^{\gamma-1}\exp\big(-\lambda_j(T^\gamma - \varsigma^\gamma)\gamma^{-1}\big)\Phi(\varsigma)d\varsigma\Big]\mathrm{e}_j(x). \tag{11}$$

From (11), it gives

$$\mathcal{F}(x) = \sum_{j=1}^{\infty}\frac{\big\langle \ell(\cdot), \mathrm{e_j}(\cdot)\big\rangle}{\int_0^T \varsigma^{\gamma-1}\exp\big(-\lambda_j(T^\gamma - \varsigma^\gamma)\gamma^{-1}\big)\Phi(\varsigma)d\varsigma}.$$

this implies that

$$\mathcal{F}(x) = \sum_{j=1}^{\infty}\frac{\big\langle \ell(\cdot), \mathrm{e_j}(\cdot)\big\rangle \mathrm{e_j(x)}}{\int_0^T \varsigma^{\gamma-1}\exp\big(-\lambda_j(T^\gamma - \varsigma^\gamma)\gamma^{-1}\big)\Phi(\varsigma)d\varsigma}. \tag{12}$$

2.2. The ill-posedness of inverse source problem

Theorem 2.4. *The inverse source problem is ill-posed.*

Proof. Defining a linear operator $\mathcal{L} : \mathscr{L}_2(\Omega) \to \mathscr{L}_2(\Omega)$ as follows:

$$\mathcal{L}\mathcal{F}(x) = \sum_{j=1}^{\infty}\Big[\int_0^T \varsigma^{\gamma-1}\exp\big(-\lambda_j(T^\gamma - \varsigma^\gamma)\gamma^{-1}\big)\Phi(\varsigma)d\varsigma\Big]\big\langle \mathcal{F}(\cdot), \mathrm{e}_j(\cdot)\big\rangle \mathrm{e}_j(x) = \int_\Omega k(x,\xi)\mathcal{F}(\xi)d\xi, \tag{13}$$

whereby

$$k(x,\xi) = \sum_{j=1}^{\infty}\Big(\int_0^T \varsigma^{\gamma-1}\exp\big(-\lambda_j(T^\gamma - \varsigma^\gamma)\gamma^{-1}\big)\Phi(\varsigma)d\varsigma\Big)\mathrm{e}_j(x)\mathrm{e}_j(\xi).$$

Due to $k(x,\xi) = k(\xi, x)$, we know $\mathcal{L}$ is self-adjoint operator. Next, we are going to prove its compactness. Let us define $\mathcal{L}_N$ as follows

$$\mathcal{L}_N\mathcal{F}(x) = \sum_{j=1}^{N}\Big(\int_0^T \varsigma^{\gamma-1}\exp\big(-\lambda_j(T^\gamma-\varsigma^\gamma)\gamma^{-1}\big)\Phi(\varsigma)d\varsigma\Big)\big\langle \mathcal{F}(\cdot), \mathrm{e}_j(\cdot)\big\rangle \mathrm{e}_j(x). \tag{14}$$

It is easy to check that $\mathcal{L}_N$ is a finite rank operator. Then, from (13) and (14), we have:

$$\begin{aligned}\big\|\mathcal{L}_N\mathcal{F} - \mathcal{L}\mathcal{F}\big\|^2_{\mathscr{L}_2(\Omega)} &= \sum_{j=N+1}^{\infty}\Big(\int_0^T \varsigma^{\gamma-1}\exp\big(-\lambda_j(T^\gamma-\varsigma^\gamma)\gamma^{-1}\big)\Phi(\varsigma)d\varsigma\Big)^2\big|\big\langle\mathcal{F}(\cdot), \mathrm{e}_j(\cdot)\big\rangle\big|^2 \\ &\le \overline{\Phi}^2\sum_{j=N+1}^{\infty}\underbrace{\Big(\int_0^T \varsigma^{\gamma-1}\exp\big(-\lambda_j(T^\gamma-\varsigma^\gamma)\gamma^{-1}\big)d\varsigma\Big)^2}_{\mathcal{V}_\gamma^2}\big|\big\langle\mathcal{F}(\cdot), e_j(\cdot)\big\rangle\big|^2. \end{aligned}\tag{15}$$

We consider the integral $\mathcal{V}_\gamma = \int_0^{T^\gamma} \varsigma^{\gamma-1}\exp\big(-\lambda_j(T^\gamma-\varsigma^\gamma)\gamma^{-1}\big)d\varsigma$ as follows, by denoting $\varsigma^\gamma = \omega$, using the variable transformation method, we get

$$\mathcal{V}_\gamma = \frac{1}{\gamma}\int_0^{T^\gamma}\exp\big(-\lambda_j(T^\gamma-\varsigma^\gamma)\gamma^{-1}\big)d\varsigma \le \frac{1}{\lambda_j}\Big(1-\exp\big(-\lambda_j T^\gamma\gamma^{-1}\big)\Big) \le \frac{1}{\lambda_j}\cdot \tag{16}$$

Combining (15) and (16), one has

$$\big\|\mathcal{L}_N\mathcal{F} - \mathcal{L}\mathcal{F}\big\|^2_{\mathscr{L}_2(\Omega)} \le \overline{\Phi}^2\sum_{j=N+1}^{\infty}\frac{1}{\lambda_j^2}\big|\big\langle\mathcal{F}(\cdot), e_j(\cdot)\big\rangle\big|^2. \tag{17}$$

This implies that $\big\|\mathcal{L}_N\mathcal{F} - \mathcal{L}\mathcal{F}\big\|_{\mathscr{L}_2(\Omega)} \le \dfrac{\overline{\Phi}}{\lambda_N}\|\mathcal{F}\|_{\mathscr{L}_2(\Omega)}$. Therefore, $\|\mathcal{L}_N - \mathcal{L}\|_{\mathscr{L}_2(\Omega)} \to 0$ in the sense of operator norm in $\mathscr{L}(\mathscr{L}_2(\Omega);\mathscr{L}_2(\Omega))$ as $N\to\infty$. $\mathcal{L}$ is a compact operator. Next, the singular values for the linear self-adjoint compact operator $\mathcal{L}$ are

$$\Lambda_j = \int_0^T \varsigma^{\gamma-1}\exp\big(-\lambda_j(T^\gamma-\varsigma^\gamma)\gamma^{-1}\big)\Phi(\varsigma)d\varsigma, \tag{18}$$

and in the $\mathscr{L}_2(\Omega)$ space, eigenvectors e_j are an orthonormal its basis. From (13), the inverse source problem (1) can be rewritten as an operator equation

$$\mathcal{L}\mathcal{F}(x) = \ell(x), \tag{19}$$

and by Kirsch ([18]), we conclude that problem is ill-posed. We will make the following assumptions $\ell^k(\cdot) = \dfrac{\mathrm{e}_k(\cdot)}{\sqrt{\lambda_k}}\cdot$ The source term corresponding to ℓ^k is

$$\mathcal{F}^k(x) = \sum_{j=1}^{\infty}\frac{\big\langle \ell^k(\cdot), \mathrm{e}_j(\cdot)\big\rangle\mathrm{e}_j(x)}{\int_0^T \varsigma^{\gamma-1}\exp\big(-\lambda_j(T^\gamma-\varsigma^\gamma)\gamma^{-1}\big)\Phi(\varsigma)d\varsigma} = \frac{\mathrm{e}_k(x)}{\sqrt{\lambda_k}\int_0^T \varsigma^{\gamma-1}\exp\big(-\lambda_k(T^\gamma-\varsigma^\gamma)\gamma^{-1}\big)\Phi(\varsigma)d\varsigma}\cdot$$

If the final data $\ell = 0$ then $\mathcal{F} = 0$, ℓ and ℓ^k have estimated:

$$\|\ell^k - \ell\|_{\mathscr{L}_2(\Omega)} = \Big\|\frac{\mathrm{e}_k(\cdot)}{\sqrt{\lambda_k}}\Big\|_{\mathscr{L}_2(\Omega)} = \frac{1}{\sqrt{\lambda_k}}, \text{ which leads to } \lim_{k\to+\infty} \|\ell^k - \ell\|_{\mathscr{L}_2(\Omega)} = \lim_{k\to+\infty} \frac{1}{\sqrt{\lambda_k}} = 0. \tag{20}$$

Estimates errors between $\mathcal{F}^k$ and $\mathcal{F}$ is given as follow

$$\begin{aligned}\big\|\mathcal{F}^k - \mathcal{F}\big\|_{\mathscr{L}_2(\Omega)} &= \frac{1}{\sqrt{\lambda_k}}\Big(\int_0^T \varsigma^{\gamma-1}\exp\big(-\lambda_k(T^\gamma - \varsigma^\gamma)\gamma^{-1}\big)\Phi(\varsigma)d\varsigma\Big)^{-1} \\ &\geq \frac{1}{\overline{\Phi}\sqrt{\lambda_k}}\Big(\int_0^T \varsigma^{\gamma-1}\exp\big(-\lambda_k(T^\gamma - \varsigma^\gamma)\gamma^{-1}\big)d\varsigma\Big)^{-1} = \frac{\sqrt{\lambda_k}}{\overline{\Phi}}\cdot\end{aligned} \tag{21}$$

From estimation above, we receive

$$\big\|\mathcal{F}^k - \mathcal{F}\big\|_{\mathscr{L}_2(\Omega)} \geq \frac{\sqrt{\lambda_k}}{\overline{\Phi}}, \text{ this leads to } \lim_{k\to+\infty}\big\|\mathcal{F}^k - \mathcal{F}\big\|_{\mathscr{L}_2(\Omega)} > \lim_{k\to+\infty}\frac{\sqrt{\lambda_k}}{\overline{\Phi}} = +\infty. \tag{22}$$

Combining (20) and (22), we conclude that the inverse source problem is ill-posed. □

2.3. Conditional stability of source term f

Theorem 2.5. *Let $\mathcal{M} > 0$, $s > 0$ and we have been working under the assumption that $\|\mathcal{F}\|_{\mathscr{H}^m(\Omega)} \leq \mathcal{M}$, one has*

$$\|\mathcal{F}\|_{\mathscr{L}_2(\Omega)} \leq C(m,\mathcal{M})\|\ell\|_{\mathscr{L}_2(\Omega)}^{\frac{m}{m+1}}, \tag{23}$$

whereby

$$C(m,\mathcal{M}) = \Big(|\underline{\Phi}|\big|1 - \exp(-\lambda_1 T^\gamma\gamma^{-1})\big|\Big)^{-\frac{m}{m+1}}\mathcal{M}^{\frac{1}{m+1}}. \tag{24}$$

Proof. Using the Hölder inequality, form now on, for a shorter,

$$\mathcal{S}(\lambda_j,\gamma,\Phi) = \int_0^T \varsigma^{\gamma-1}\exp\big(-\lambda_j(T^\gamma - \varsigma^\gamma)\gamma^{-1}\big)\Phi(\varsigma)d\varsigma,$$

we have

$$\begin{aligned}\|\mathcal{F}\|^2_{\mathscr{L}_2(\Omega)} &= \sum_{j=1}^\infty \Big|\frac{\langle \ell(\cdot), \mathrm{e}_j(\cdot)\rangle}{\mathcal{S}(\lambda_j,\gamma,\Phi)}\Big|^2 = \sum_{j=1}^\infty \frac{\big|\langle \ell(\cdot), \mathrm{e}_j(\cdot)\rangle\big|^{\frac{2}{m+1}}\big|\langle \ell(\cdot), \mathrm{e}_j(\cdot)\rangle\big|^{\frac{2m}{m+1}}}{\big|\mathcal{S}(\lambda_j,\gamma,\Phi)\big|^2} \\ &\leq \Big[\sum_{j=1}^\infty \frac{\big|\langle \ell(\cdot), \mathrm{e}_j(\cdot)\rangle\big|^2}{\big|\mathcal{S}(\lambda_j,\gamma,\Phi)\big|^{2m+2}}\Big]^{\frac{1}{m+1}}\Big[\sum_{j=1}^\infty \big|\langle \ell(\cdot), \mathrm{e}_j(\cdot)\rangle\big|^2\Big]^{\frac{m}{m+1}} \\ &\leq \Big[\sum_{j=1}^\infty \frac{\big|\langle \mathcal{F}(\cdot), \mathrm{e}_j(\cdot)\rangle\big|^2}{\big|\mathcal{S}(\lambda_j,\gamma,\Phi)\big|^{2m}}\Big]^{\frac{1}{m+1}}\|\ell\|_{\mathscr{L}_2(\Omega)}^{\frac{2m}{m+1}}.\end{aligned} \tag{25}$$

From (25), we have

$$\big|\mathcal{S}(\lambda_j,\gamma,\Phi)\big|^{2m} \geq |\underline{\Phi}|^{2m}\big|\mathcal{S}(\lambda_j,\gamma)\big|^{2m} \geq |\underline{\Phi}|^{2m}|\lambda_j|^{-2m}\big|1 - \exp\big(-\lambda_j T^\gamma\gamma^{-1}\big)\big|^{2m}, \tag{26}$$

and this inequality leads to

$$\sum_{j=1}^{\infty} \frac{\left|\langle \mathcal{F}(\cdot), \mathrm{e}_j(\cdot)\rangle\right|^2}{\left|\mathcal{S}(\lambda_j, \gamma, \Phi)\right|^{2m}} \le \sum_{j=1}^{\infty} \frac{\lambda_j^{2m}\left|\langle \mathcal{F}(\cdot), \mathrm{e}_j(\cdot)\rangle\right|^2}{|\underline{\Phi}|^{2m}\left|1-\exp(-\lambda_1 T^{\gamma}\gamma^{-1})\right|^{2m}}. \tag{27}$$

Combining (26) and (27), we get

$$\|\mathcal{F}\|^2_{\mathscr{L}_2(\Omega)} \le \Big(|\underline{\Phi}|\left|1-\exp(-\lambda_1 T^{\gamma}\gamma^{-1})\right|\Big)^{-\frac{2m}{m+1}} \|\mathcal{F}\|^{\frac{2}{m+1}}_{\mathscr{H}^m(\Omega)} \|\ell\|^{\frac{2m}{m+1}}_{\mathscr{L}_2(\Omega)} \le [C(m,\mathcal{M})]^2 \|\ell\|^{\frac{2m}{m+1}}_{\mathscr{L}_2(\Omega)}. \tag{28}$$

□

3. The Fractional Tikhonov method

Due to singular value decomposition for compact self-adjoint operator $\mathcal{K}$, as in (13). If the measured data ℓ^{ϵ} and ℓ with a noise level of ϵ satisfy$\|\ell-\ell^{\epsilon}\|_{\mathscr{L}_2(\Omega)} \le \epsilon$ then we can present a regularized solution as follows:

$$\mathcal{F}^{\epsilon}_{\beta(\epsilon)}(x) = \sum_{j=1}^{\infty} \frac{\left|\mathcal{S}(\lambda_j, \gamma, \Phi^{\epsilon})\right|^{2\xi-1}}{[\beta(\epsilon)]^2 + \left|\mathcal{S}(\lambda_j, \gamma, \Phi^{\epsilon})\right|^{2\xi}} \langle \ell^{\epsilon}(x), \mathrm{e}_j(x)\rangle \mathrm{e}_j(x), \ \frac{1}{2} \le \xi \le 1, \tag{29}$$

$$\mathcal{F}_{\beta(\epsilon)}(x) = \sum_{j=1}^{\infty} \frac{\left|\mathcal{S}(\lambda_j, \gamma, \Phi)\right|^{2\xi-1}}{\left[\beta(\epsilon)\right]^2 + \left|\mathcal{S}(\lambda_j, \gamma, \Phi)\right|^{2\xi}} \langle \ell(\cdot), \mathrm{e}_j(\cdot)\rangle \mathrm{e}_j(x), \ \frac{1}{2} \le \xi \le 1, \tag{30}$$

and $\beta(\epsilon)$ is a parameter regularization.

Case 1: If $\xi = \frac{1}{2}$ then the Fractional Tikhonov is called the Quasi-Boundary Value Method.

Case 2: When $\xi = 1$, it is the classic Tikhonov method.

3.1. An a priori parameter choice

Afterwards, the estimation for $\left\|\mathcal{F}(\cdot) - \mathcal{F}^{\epsilon}_{\beta(\epsilon)}(\cdot)\right\|_{\mathscr{L}_2(\Omega)}$ is established by our next Theorem and show convergence rate under a suitable choice for the regularization parameter. To do this, we need the following lemma

Lemma 3.1. *For constants $z \ge \lambda_1$ and $\frac{1}{2} \le \xi \le 1$, we have*

$$G_1(z) = \frac{z}{A^{2\xi} + \beta z^{2\xi}} \le \overline{B}(\xi, A)\beta^{-\frac{1}{2\xi}}. \tag{31}$$

where $\overline{B}(\xi, A)$ are independent on β, z.

Proof. For $\dfrac{1}{2} < \xi < 1$, from (31), solve the equation $G_1'(z) = 0$, we can know that

$$z_0 = A(2\xi-1)^{-\frac{1}{2\xi}}\beta^{-\frac{1}{2\xi}}.$$

Replacing the z_0 into equation (31), we see that

$$G_1(z) \le G_1(z_0) \le \overline{B}(\xi, A)\beta^{-\frac{1}{2\xi}} \text{ in which } \overline{B}(\xi, A) = \frac{A^{1-2\xi}(2\xi-1)^{-\frac{1}{2\xi}}}{2\xi}. \tag{32}$$

□

Lemma 3.2. *Let the constant $z \geq \lambda_1$ and $\frac{1}{2} \leq \xi \leq 1$, one has*

$$G_2(z) = \frac{\beta^2 z^{2\xi-m}}{A^{2\xi} + \beta^2 z^{2\xi}} \leq \begin{cases} (2\xi)^{-1}\big((2\xi - m)^{\frac{2\xi-m}{2\xi}} A^{-m} m^{\frac{m}{2\xi}}\big)\beta^{\frac{m}{\xi}}, & 0 < m < 2\xi, \\ \big(A^{2\xi}\lambda_1^{m-2\xi}\big)^{-1}\beta^2, & m \geq 2\xi. \end{cases} \tag{33}$$

Proof. The proof is completed in [31]. □

Theorem 3.3. *Let $\mathcal{F}$ be as* (25) *and the noise assumption* (6) *holds. Then, we have the following estimate:*

- *If $0 < m < 2\xi$, by choosing $\beta(\epsilon) = \Big(\dfrac{\epsilon}{\mathcal{M}}\Big)^{\frac{\xi}{m+2}}$ then*

$$\big\|\mathcal{F}(\cdot) - \mathcal{F}^{\epsilon}_{\beta(\epsilon)}(\cdot)\big\|_{\mathscr{L}_2(\Omega)} \text{ is of order } \epsilon^{\frac{m}{m+2}}. \tag{34}$$

- *If $m \geq 2\xi$, by choosing $\beta(\epsilon) = \Big(\dfrac{\epsilon}{\mathcal{M}}\Big)^{\frac{\xi}{\xi+1}}$ then*

$$\big\|\mathcal{F}(\cdot) - \mathcal{F}^{\epsilon}_{\beta(\epsilon)}(\cdot)\big\|_{\mathscr{L}_2(\Omega)} \text{ is of order } \epsilon^{\frac{\xi}{\xi+2}}. \tag{35}$$

Proof. By the triangle inequality, we know

$$\big\|\mathcal{F}(\cdot) - \mathcal{F}^{\epsilon}_{\beta(\epsilon)}(\cdot)\big\|_{\mathscr{L}_2(\Omega)} \leq \underbrace{\big\|\mathcal{F}_{\beta(\epsilon)}(\cdot) - \mathcal{F}^{\epsilon}_{\beta(\epsilon)}(\cdot)\big\|_{\mathscr{L}_2(\Omega)}}_{\mathcal{A}_1 := \mathcal{Q}_1 + \mathcal{Q}_2} + \underbrace{\big\|\mathcal{F}(\cdot) - \mathcal{F}_{\beta(\epsilon)}(\cdot)\big\|_{\mathscr{L}_2(\Omega)}}_{\mathcal{A}_2}. \tag{36}$$

inwhich

$$\begin{aligned} \mathcal{Q}_1 &= \sum_{j=1}^{\infty} \frac{\big|\mathcal{S}(\lambda_j, \gamma, \Phi^{\epsilon})\big|^{2\xi-1}}{[\beta(\epsilon)]^2 + \big|\mathcal{S}(\lambda_j, \gamma, \Phi^{\epsilon})\big|^{2\xi}} \big\langle \ell^{\epsilon}(x) - \ell(x), \mathrm{e}_j(x)\big\rangle \mathrm{e}_j(x), \\ \mathcal{Q}_2 &= \sum_{j=1}^{\infty} \frac{\big|\mathcal{S}(\lambda_j, \gamma, \Phi^{\epsilon})\big|^{2\xi-1}}{[\beta(\epsilon)]^2 + \big|\mathcal{S}(\lambda_j, \gamma, \Phi^{\epsilon})\big|^{2\xi}} - \frac{\big|\mathcal{S}(\lambda_j, \gamma, \Phi)\big|^{2\xi-1}}{\big[\beta(\epsilon)\big]^2 + \big|\mathcal{S}(\lambda_j, \gamma, \Phi)\big|^{2\xi}} \big\langle \ell(\cdot), \mathrm{e}_j(\cdot)\big\rangle \mathrm{e}_j(x), \\ \mathcal{A}_2 &= \sum_{j=1}^{\infty} \left(\frac{\big|\mathcal{S}(\lambda_j, \gamma, \Phi)\big|^{2\xi-1}}{[\beta(\epsilon)]^2 + \big|\mathcal{S}(\lambda_j, \gamma, \Phi)\big|^{2\xi}} - \frac{1}{\big|\mathcal{S}(\lambda_j, \gamma, \Phi)\big|} \right) \big\langle \ell(\cdot), \mathrm{e}_j(\cdot)\big\rangle \mathrm{e}_j(x). \end{aligned} \tag{37}$$

The proof falls naturally into some steps.

Step 1: Estimation for $\|\mathcal{Q}_1\|_{\mathcal{L}^2(\Omega)}$, we receive Because of estimation (16), and

$$\Big| \int_0^T \mathcal{D}(\lambda_j, T, \gamma)\Phi^{\epsilon}(\varsigma) d\varsigma \Big| \geq \frac{\underline{\Phi}}{4} \frac{1 - \exp(-\lambda_1 T^{\gamma}\gamma^{-1})}{\lambda_j}.$$

From now on, for a shorter, by denoting

$$\frac{\underline{\Phi}}{4}\big(1 - \exp\big(-\lambda_1 T^{\gamma}\gamma^{-1}\big)\big) = \overline{A}(\underline{\Phi}, \lambda_1, T, \gamma).$$

Next, from (3.1), we know that

$$\begin{aligned} \|\mathcal{Q}_1\|_{\mathscr{L}_2(\Omega)} &\leq \sum_{j=1}^{\infty} \frac{\Big|\frac{\mathcal{B}(|\underline{\Phi}|, |\overline{\Phi}|)}{\lambda_j}\Big|^{2\xi-1}}{[\beta(\epsilon)]^2 + \Big|\dfrac{\overline{A}(\underline{\Phi}, \lambda_1, T, \gamma)}{\lambda_j}\Big|^{2\xi}} \big\langle \ell^{\epsilon}(\cdot) - \ell(\cdot), \mathrm{e}_j(\cdot)\big\rangle \mathrm{e}_j(x) \\ &\leq \sum_{j=1}^{\infty} \frac{\big|\mathcal{B}(|\underline{\Phi}|, |\overline{\Phi}|)\big|^{2\xi-1}\lambda_j}{[\beta(\epsilon)]^2\lambda_j^{2\xi} + \big|\overline{A}(\underline{\Phi}, \lambda_1, T, \gamma)\big|^{2\xi}} \big\langle \ell^{\epsilon}(\cdot) - \ell(\cdot), \mathrm{e}_j(\cdot)\big\rangle \mathrm{e}_j(x) \\ &\leq \epsilon \,\Big|\mathcal{B}(|\underline{\Phi}|, |\overline{\Phi}|)\Big|^{2\xi-1} \sup_{j\in\mathbb{N}} \lambda_j \Big[[\beta(\epsilon)]^2\lambda_j^{2\xi} + \big|\overline{A}(\underline{\Phi}, \lambda_1, T, \gamma)\big|^{2\xi}\Big]^{-1}. \end{aligned} \tag{38}$$

Applying the Lemma 3.1, it gives

$$\|\mathcal{Q}_1\|_{\mathscr{L}_2(\Omega)} \leq \epsilon\ [\beta(\epsilon)]^{-\frac{1}{\xi}}\ \left(\big|\mathcal{B}(|\underline{\Phi}|,|\overline{\Phi}|)\big|^{2\xi-1}\ \overline{B}\big(\xi,\overline{A}(\underline{\Phi},\lambda_1,T,\gamma)\big) \right). \tag{39}$$

Step 2: Next, $\mathcal{Q}_2$ have seen estimate

$$\mathcal{Q}_2 = \overbrace{\sum_{j=1}^{\infty} \frac{[\beta(\epsilon)]^2 \big|\mathcal{S}(\lambda_j,\gamma,\Phi^\epsilon-\Phi)\big|^{2\xi-1}}{\Big([\beta(\epsilon)]^2+\big|\mathcal{S}(\lambda_j,\gamma,\Phi^\epsilon)\big|^{2\xi}\Big)\ \Big([\beta(\epsilon)]^2+\big|\mathcal{S}(\lambda_j,\gamma,\Phi)\big|^{2\xi}\Big)} \big\langle \ell(\cdot),\mathrm{e}_j(\cdot)\big\rangle \mathrm{e}_j(x)}^{\mathcal{L}_1}$$
$$+\overbrace{\sum_{j=1}^{\infty} \frac{\big|\mathcal{S}(\lambda_j,\gamma,\Phi^\epsilon)\big|^{2\xi}\ \big|\mathcal{S}(\lambda_j,\gamma,\Phi)\big|^{2\xi}\ \Big(\big|\mathcal{S}(\lambda_j,\gamma,\Phi^\epsilon)\big|^{-1}-\big|\mathcal{S}(\lambda_j,\gamma,\Phi)\big|^{-1}\Big)}{\Big([\beta(\epsilon)]^2+\big|\mathcal{S}(\lambda_j,\gamma,\Phi^\epsilon)\big|^{2\xi}\Big)\ \Big([\beta(\epsilon)]^2+\big|\mathcal{S}(\lambda_j,\gamma,\Phi)\big|^{2\xi}\Big)} \big\langle \ell(\cdot),\mathrm{e}_j(\cdot)\big\rangle \mathrm{e}_j(x)}^{\mathcal{L}_2}. \tag{40}$$

From (40), we have estimate for $\big\|\mathcal{L}_1\big\|_{\mathscr{L}_2(\Omega)}$ and $\big\|\mathcal{L}_2\big\|_{\mathscr{L}_2(\Omega)}$

$$\begin{aligned}\|\mathcal{L}_1\|_{\mathscr{L}_2(\Omega)} &\leq \sum_{j=1}^{\infty} \frac{[\beta(\epsilon)]^2 \big|\mathcal{S}(\lambda_j,\gamma,\Phi^\epsilon-\Phi)\big|^{2\xi-1}}{\Big([\beta(\epsilon)]^2+\big|\mathcal{S}(\lambda_j,\gamma,\Phi^\epsilon)\big|^{2\xi}\Big)\ \big|\mathcal{S}(\lambda_j,\gamma,\Phi)\big|^{2\xi-1}} \frac{\big\langle \ell(\cdot),\mathrm{e}_j(\cdot)\big\rangle \mathrm{e}_j(x)}{\mathcal{S}(\lambda_j,\gamma,\Phi)} \\ &\leq \frac{\big|\Phi^\epsilon-\Phi\big|^{2\xi-1}_{\mathscr{L}_\infty(0,T)}}{\big|\underline{\Phi}\big|^{2\xi-1}} \sum_{j=1}^{\infty} \frac{\big\langle \ell(\cdot),\mathrm{e}_j(\cdot)\big\rangle \mathrm{e}_j(x)}{\mathcal{S}(\lambda_j,\gamma,\Phi)} \leq \frac{\big|\Phi^\epsilon-\Phi\big|^{2\xi-1}_{\mathscr{L}_\infty(0,T)}}{\big|\underline{\Phi}\big|^{2\xi-1}} \|\mathcal{F}\|_{\mathscr{L}_2(\Omega)}.\end{aligned} \tag{41}$$

$$\begin{aligned}\|\mathcal{L}_2\|_{\mathscr{L}_2(\Omega)} &\leq \sum_{j=1}^{\infty} \frac{\big|\mathcal{S}(\lambda_j,\gamma,\Phi^\epsilon)\big|^{2\xi-1}\ \mathcal{S}(\lambda_j,\gamma,\Phi^\epsilon-\Phi)}{\Big([\beta(\epsilon)]^2+\big|\mathcal{S}(\lambda_j,\gamma,\Phi^\epsilon)\big|^{2\xi}\Big)} \frac{\big\langle \ell(\cdot),\mathrm{e}_j(\cdot)\big\rangle \mathrm{e}_j(x)}{\big|\mathcal{S}(\lambda_j,\gamma,\Phi)\big|} \\ &\leq \sum_{j=1}^{\infty} \frac{\mathcal{S}(\lambda_j,\gamma,\Phi^\epsilon-\Phi)}{\big|\mathcal{S}(\lambda_j,\gamma,\Phi^\epsilon)\big|} \frac{\big\langle \ell(\cdot),\mathrm{e}_j(\cdot)\big\rangle \mathrm{e}_j(x)}{\big|\mathcal{S}(\lambda_j,\gamma,\Phi)\big|} \\ &\leq \frac{4\|\Phi^\epsilon-\Phi\|_{\mathscr{L}_\infty(0,T)}}{\underline{\Phi}} \sum_{j=1}^{\infty} \frac{\big\langle \ell(\cdot),\mathrm{e}_j(\cdot)\big\rangle \mathrm{e}_j(x)}{\big|\mathcal{S}(\lambda_j,\gamma,\Phi)\big|} \leq \frac{4\|\Phi^\epsilon-\Phi\|_{\mathscr{L}_\infty(0,T)}}{\underline{\Phi}} \|\mathcal{F}\|_{\mathscr{L}_2(\Omega)}.\end{aligned} \tag{42}$$

Combining (40) to (42), we see that

$$\begin{aligned}\|\mathcal{Q}_2\|_{\mathscr{L}_2(\Omega)} &\leq \frac{\big|\Phi^\epsilon-\Phi\big|^{2\xi-1}_{\mathscr{L}_\infty(0,T)}}{\big|\underline{\Phi}\big|^{2\xi-1}} \|\mathcal{F}\|_{\mathscr{L}_2(\Omega)} + \frac{4\|\Phi^\epsilon-\Phi\|_{\mathscr{L}_\infty(0,T)}}{\underline{\Phi}} \|\mathcal{F}\|_{\mathscr{L}_2(\Omega)} \\ &\leq 2\|\Phi^\epsilon-\Phi\|_{\mathscr{L}_\infty(0,T)} \max\Big\{\frac{1}{|\underline{\Phi}|^{2\xi-1}}, \frac{4}{\underline{\Phi}}\Big\} \|\mathcal{F}\|_{\mathscr{L}_2(\Omega)}.\end{aligned} \tag{43}$$

Step 3: Next, we have to estimate $\|\mathcal{A}_2\|_{\mathscr{L}_2(\Omega)}$, we get

$$
\begin{aligned}
\|\mathcal{A}_2\|^2_{\mathscr{L}_2(\Omega)} &\leq \sum_{j=1}^{\infty}\left(\frac{\big|\mathcal{S}(\lambda_j,\gamma,\Phi)\big|^{2\xi-1}}{[\beta(\epsilon)]^2+\big|\mathcal{S}(\lambda_j,\gamma,\Phi)\big|^{2\xi}}-\frac{1}{\big|\mathcal{S}(\lambda_j,\gamma,\Phi)\big|}\right)^2\big|\langle\ell(\cdot),\mathrm{e}_j(\cdot)\rangle\big|^2\\
&\leq \sum_{j=1}^{\infty}\left(\frac{[\beta(\epsilon)]^2}{\big|\mathcal{S}(\lambda_j,\gamma,\Phi)\big|\Big([\beta(\epsilon)]^2+\big|\mathcal{S}(\lambda_j,\gamma,\Phi)\big|^{2\xi}\Big)}\right)^2\big|\langle\ell(\cdot),\mathrm{e}_j(\cdot)\rangle\big|^2,\\
&\leq \sum_{j=1}^{\infty}\frac{[\beta(\epsilon)]^4}{\Big([\beta(\epsilon)]^2+\big|\mathcal{S}(\lambda_j,\gamma,\Phi)\big|^{2\xi}\Big)}\frac{\big|\langle\ell(\cdot),\mathrm{e}_j(\cdot)\rangle\big|^2}{\big|\mathcal{S}(\lambda_j,\gamma,\Phi)\big|^2}\\
&\leq \sum_{j=1}^{\infty}\frac{[\beta(\epsilon)]^4\lambda_j^{-2m}}{\Big([\beta(\epsilon)]^2+\big|\mathcal{S}(\lambda_j,\gamma,\Phi)\big|^{2\xi}\Big)}\frac{\lambda_j^{2m}\big|\langle\ell(\cdot),\mathrm{e}_j(\cdot)\rangle\big|^2}{\big|\mathcal{S}(\lambda_j,\gamma,\Phi)\big|^2}\\
&\leq \sup_{j\in\mathbb{N}}\big|G_2(\lambda_j)\big|^2\sum_{j=1}^{\infty}\frac{\lambda_j^{2m}\big|\langle\ell(\cdot),\mathrm{e}_j(\cdot)\rangle\big|^2}{\big|\mathcal{S}(\lambda_j,\gamma,\Phi)\big|^2}=\sup_{j\in\mathbb{N}}\big|G_2(\lambda_j)\big|^2\big\|\mathcal{F}\big\|^2_{\mathscr{H}^m(\Omega)}.
\end{aligned}
\tag{44}
$$

Hence, $G_2(\lambda_j)$ has been estimated

$$
G_2(\lambda_j)=\frac{[\beta(\epsilon)]^2\lambda_j^{-m}}{[\beta(\epsilon)]^2+\big|\mathcal{S}(\lambda_j,\gamma,\Phi)\big|^{2\xi}}\leq\frac{[\beta(\epsilon)]^2\lambda_j^{2\xi-m}}{[\beta(\epsilon)]^2\lambda_j^{2\xi}+\big|\overline{A}(\underline{\Phi},\lambda_1,T,\gamma)\big|^{2\xi}}. \tag{45}
$$

With the Lemma 3.2, replace A by $\overline{A}(\underline{\Phi},\lambda_1,T,\gamma)$, $G_2(\lambda_j)$ can be bounded as follow

$$
G_2(\lambda_j)\leq\begin{cases}(2\xi)^{-1}\big((2\xi-m)^{\frac{2\xi-m}{2\xi}}A^{-m}m^{\frac{m}{2\xi}}\big)[\beta(\epsilon)]^{\frac{m}{\xi}}, & 0<m<2\xi,\\ \big(A^{2\xi}\lambda_1^{m-2\xi}\big)^{-1}[\beta(\epsilon)]^2, & m\geq 2\xi.\end{cases} \tag{46}
$$

Combining (44) to (46), we conclude that

$$
\|\mathcal{A}_2\|_{\mathscr{L}_2(\Omega)}\leq\begin{cases}(2\xi)^{-1}\big((2\xi-m)^{\frac{2\xi-m}{2\xi}}A^{-m}m^{\frac{m}{2\xi}}\big)\mathcal{M}[\beta(\epsilon)]^{\frac{m}{\xi}}, & 0<m<2\xi,\\ \big(A^{2\xi}\lambda_1^{m-2\xi}\big)^{-1}\mathcal{M}[\beta(\epsilon)]^2, & m\geq 2\xi.\end{cases} \tag{47}
$$

Next, combining the above **three steps**, we obtain

$$
\begin{aligned}
\big\|\mathcal{F}(\cdot)-\mathcal{F}^{\epsilon}_{\beta(\epsilon)}(\cdot)\big\|_{\mathscr{L}_2(\Omega)} &\leq \epsilon\,[\beta(\epsilon)]^{-\frac{1}{\xi}}\left(\big|\mathcal{B}(|\underline{\Phi}|,|\overline{\Phi}|)\big|^{2\xi-1}\,\overline{B}\big(\xi,\overline{A}(\underline{\Phi},\lambda_1,T,\gamma)\big)\right)\\
&+2\|\Phi^{\epsilon}-\Phi\|_{\mathscr{L}_\infty(0,T)}\max\left\{\frac{1}{|\underline{\Phi}|^{2\xi-1}},\frac{4}{\underline{\Phi}}\right\}\|\mathcal{F}\|_{\mathscr{L}_2(\Omega)}\\
&+\begin{cases}(2\xi)^{-1}\big((2\xi-m)^{\frac{2\xi-m}{2\xi}}|\overline{A}(\underline{\Phi},\lambda_1,T,\gamma)|^{-m}m^{\frac{m}{2\xi}}\big)\mathcal{M}[\beta(\epsilon)]^{\frac{m}{\xi}}, & 0<m<2\xi,\\ \big(|\overline{A}(\underline{\Phi},\lambda_1,T,\gamma)|^{2\xi}\lambda_1^{m-2\xi}\big)^{-1}\mathcal{M}[\beta(\epsilon)]^2, & m\geq 2\xi.\end{cases}
\end{aligned}
\tag{48}
$$

Choose the regularization parameter $\beta(\epsilon)$ as follows:

$$
\beta(\epsilon)=\begin{cases}\left(\dfrac{\epsilon}{\mathcal{M}}\right)^{\frac{\xi}{m+2}}, & 0<m<2\xi,\\[2ex] \left(\dfrac{\epsilon}{\mathcal{M}}\right)^{\frac{\xi}{\xi+1}}, & m\geq 2\xi.\end{cases} \tag{49}
$$

From the selection of β as in the formula (49), we receive

Case 1: If $0 < m \le 2\xi$ then

$$\big\|\mathcal{F}(\cdot) - \mathcal{F}^{\epsilon}_{\beta(\epsilon)}(\cdot)\big\|_{\mathscr{L}_2(\Omega)} \le \epsilon^{\frac{m}{m+2}}\Bigg[\epsilon^{\frac{1}{m+2}}\mathcal{M}^{\frac{1}{m+2}}\bigg(\big|\mathcal{B}(|\underline{\Phi}|,|\overline{\Phi}|)\big|^{2\xi-1}\,\overline{B}\big(\xi,\overline{A}(\underline{\Phi},\lambda_1,T,\gamma)\big)\bigg)$$
$$+ 2\epsilon^{\frac{2}{m+2}}\max\Big\{\frac{1}{|\underline{\Phi}|^{2\xi-1}},\frac{4}{\underline{\Phi}}\Big\}\|\mathcal{F}\|_{\mathscr{L}_2(\Omega)} + \mathcal{M}^{\frac{m+1}{m+2}}(2\xi)^{-1}\Big((2\xi-m)^{\frac{2\xi-m}{2\xi}}|\overline{A}(\underline{\Phi},\lambda_1,T,\gamma)|^{-m}m^{\frac{m}{2\xi}}\Big)\Bigg]. \tag{50}$$

Case 2: If $m > 2\xi$ then

$$\big\|\mathcal{F}(\cdot)-\mathcal{F}^{\epsilon}_{\beta(\epsilon)}(\cdot)\big\|_{\mathscr{L}_2(\Omega)} \le \epsilon^{\frac{\xi}{\xi+1}}\Bigg[\mathcal{M}^{\frac{1}{\xi+1}}\bigg(\big|\mathcal{B}(|\underline{\Phi}|,|\overline{\Phi}|)\big|^{2\xi-1}\,\overline{B}\big(\xi,\overline{A}(\underline{\Phi},\lambda_1,T,\gamma)\big)\bigg)$$
$$+ 2\epsilon^{\frac{1}{\xi+1}}\max\Big\{\frac{1}{|\underline{\Phi}|^{2\xi-1}},\frac{4}{\underline{\Phi}}\Big\}\|\mathcal{F}\|_{\mathscr{L}_2(\Omega)} + \big(|\overline{A}(\underline{\Phi},\lambda_1,T,\gamma)|^{2\xi}\big)^{-1}\epsilon^{\frac{\xi}{\xi+1}}\mathcal{M}^{\frac{1-\xi}{\xi+1}}\Bigg]. \tag{51}$$

□

3.2. An a posteriori parameter choice

In this subsection, we study an a posteriori regularization parameter choice in Morozov's discrepancy principle, readers can see [14].

$$\bigg\|\frac{\big|\mathcal{S}(\lambda_j,\gamma,\Phi)\big|^{2\xi}}{\beta^2+\big|\mathcal{S}(\lambda_j,\gamma,\Phi)\big|^{2\xi}}\ell^{\epsilon}(\cdot) - \ell^{\epsilon}(\cdot)\bigg\|_{\mathscr{L}_2(\Omega)} = \delta\epsilon, \tag{52}$$

whereby $\frac{1}{2} \le \xi \le 1$, $\delta > 1$.

Lemma 3.4. *Let $\lambda_j > \lambda_1 > 0$ and $\frac{1}{2} \le \xi < 1$, and $G_3(\lambda_j)$ is defined by*

$$\begin{aligned} G_3(\lambda_j) &= \frac{\overline{\Phi}\beta^2\lambda_j^{2\xi-(m+1)}}{\beta^2\lambda_j^{2\xi}+\big|4\overline{A}(\underline{\Phi},\lambda_1,T,\gamma)\big|^{m+1}} \\ &\le \begin{cases} \dfrac{\overline{\Phi}(2\xi-m-1)}{2\xi|4\overline{A}(\underline{\Phi},\lambda_1,T,\gamma)|^{m+1}}\Big(\dfrac{m+1}{2\xi-m-1}\Big)^{\frac{m+1}{2\xi}}\beta^{\frac{m+1}{\xi}}, & 0<m<2\xi-1, \\ \overline{\Phi}\big(|4\overline{A}(\underline{\Phi},\lambda_1,T,\gamma)|^{2\xi}\lambda_1^{(m+1)-2\xi}\big)^{-1}\beta^2, & m \ge 2\xi-1. \end{cases} \end{aligned} \tag{53}$$

Proof. The proof is completed in [31]. □

Lemma 3.5. *Assume that*

$$\mathcal{K}(\beta) = \bigg(\sum_{j=1}^{\infty}\Big(\frac{\beta^2}{\beta^2+\big|\mathcal{S}(\lambda_j,\gamma,\Phi)\big|^{2\xi}}\Big)^2\big|\langle\ell^{\epsilon}(\cdot),\mathrm{e}_j(\cdot)\rangle\big|^2\bigg)^{\frac{1}{2}}. \tag{54}$$

If $0 < \delta\epsilon < \|\ell^{\epsilon}\|_{\mathscr{L}_2(\Omega)}$, then the following results hold:

- *(a) $\mathcal{K}(\beta)$ is a continuous function;*
- *(b) $\mathcal{K}(\beta) \to 0$ as $\beta \to 0$;*
- *(c) $\mathcal{K}(\beta) \to \|\ell^{\epsilon}\|_{\mathscr{L}_2(\Omega)}$ as $\beta \to \infty$;*
- *(d) $\mathcal{K}(\beta)$ is a strictly increasing function.*

Lemma 3.6. *Let β be the solution of* (52), *one has*

$$\frac{1}{\beta^{\frac{1}{\xi}}} \le \begin{cases} \dfrac{\left(\frac{1}{2}\right)^{\frac{1}{2(m+1)}}\left(\dfrac{\overline{\Phi}(2\xi-m-1)}{\xi|4\overline{A}(\underline{\Phi},\lambda_1,T,\gamma)|^{m+1}}\right)^{\frac{1}{m+1}}\left(\dfrac{m+1}{2\xi-m-1}\right)^{\frac{1}{2\xi}}}{(\delta^2-2)^{\frac{1}{2(m+1)}}}\left(\dfrac{\mathcal{M}}{\epsilon}\right)^{\frac{1}{m+1}}, & 0<m<2\xi-1, \\ \dfrac{8^{\frac{1}{4\xi}}(\overline{\Phi})^{\frac{1}{2\xi}}\left(|4\overline{A}(\underline{\Phi},\lambda_1,T,\gamma)|^{2\xi}\lambda_1^{(m+1)-2\xi}\right)^{-\frac{1}{2\xi}}}{(\delta^2-2)^{\frac{1}{4\xi}}}\left(\dfrac{\mathcal{M}}{\epsilon}\right)^{\frac{1}{2\xi}}, & m\ge 2\xi-1. \end{cases} \tag{55}$$

which gives the required results.

Proof. **Step 1:** We receive

$$\begin{aligned}\delta^2\epsilon^2 &\le 2\sum_{j=1}^{\infty}\left(\frac{\beta^2}{\beta^2+\left|\mathcal{S}(\lambda_j,\gamma,\Phi)\right|^{2\xi}}\right)^2\left|\left\langle \ell^{\epsilon}(\cdot)-\ell(\cdot),\mathrm{e}_i(\cdot)\right\rangle\right|^2 \\ &+2\sum_{j=1}^{\infty}\left(\frac{\beta^2\ \lambda_j^{-m}\left|\mathcal{S}(\lambda_j,\gamma,\Phi)\right|}{\beta^2+\left|\mathcal{S}(\lambda_j,\gamma,\Phi)\right|^{2\xi}}\right)^2\frac{\lambda_j^{2m}\left|\left\langle \ell(\cdot),\mathrm{e}_i(\cdot)\right\rangle\right|}{\mathcal{S}(\lambda_j,\gamma,\Phi)} \le 2\epsilon^2+2\sum_{j=1}^{\infty}\left|\mathcal{H}_j\right|^2\frac{\lambda_j^{2m}\left|\left\langle \ell(\cdot),\mathrm{e}_j(\cdot)\right\rangle\right|^2}{\left|\mathcal{S}(\lambda_j,\gamma,\Phi)\right|^2}.\end{aligned} \tag{56}$$

From inequalities above, we can see that

$$\mathcal{H}_j = \frac{\beta^2\left|\mathcal{S}(\lambda_j,\gamma,\Phi)\right|}{\left|\beta^2+\left|\mathcal{S}(\lambda_j,\gamma,\Phi)\right|^{2\xi}\right|\lambda_j^m} \le \frac{\overline{\Phi}\beta^2\lambda_j^{2\xi-(m+1)}}{\beta^2\lambda_j^{2\xi}+\left|4\overline{A}(\underline{\Phi},\lambda_1,T,\gamma)\right|^{2\xi}}. \tag{57}$$

From (57), using Lemma 53, we have

$$\mathcal{H}_j \le \begin{cases} \dfrac{\overline{\Phi}(2\xi-m-1)}{2\xi|4\overline{A}(\underline{\Phi},\lambda_1,T,\gamma)|^{m+1}}\left(\dfrac{m+1}{2\xi-m-1}\right)^{\frac{m+1}{2\xi}}\beta^{\frac{m+1}{\xi}}, & 0<m<2\xi-1, \\ \overline{\Phi}\left(|4\overline{A}(\underline{\Phi},\lambda_1,T,\gamma)|^{2\xi}\lambda_1^{(m+1)-2\xi}\right)^{-1}\beta^2, & m\ge 2\xi-1. \end{cases} \tag{58}$$

Because of (58), we know that

$$\delta^2\epsilon^2 \le 2\epsilon^2+2\begin{cases} \left(\dfrac{\overline{\Phi}(2\xi-m-1)}{2\xi|4\overline{A}(\underline{\Phi},\lambda_1,T,\gamma)|^{m+1}}\right)^2\left(\dfrac{m+1}{2\xi-m-1}\right)^{\frac{m+1}{\xi}}\mathcal{M}^2\beta^{2\frac{m+1}{\xi}}, & 0<m<2\xi-1, \\ (2\overline{\Phi})^2\left(|4\overline{A}(\underline{\Phi},\lambda_1,T,\gamma)|^{2\xi}\lambda_1^{(m+1)-2\xi}\right)^{-2}\mathcal{M}^2\beta^4, & m\ge 2\xi-1. \end{cases} \tag{59}$$

From (59), it is very easy to see that

$$\left(\delta^2-2\right)\epsilon^2 \le 2\begin{cases} \left(\dfrac{\overline{\Phi}(2\xi-m-1)}{2\xi|4\overline{A}(\underline{\Phi},\lambda_1,T,\gamma)|^{m+1}}\right)^2\left(\dfrac{m+1}{2\xi-m-1}\right)^{\frac{m+1}{\xi}}\mathcal{M}^2\beta^{2\frac{m+1}{\xi}}, & 0<m<2\xi-1, \\ (2\overline{\Phi})^2\left(|4\overline{A}(\underline{\Phi},\lambda_1,T,\gamma)|^{2\xi}\lambda_1^{(m+1)-2\xi}\right)^{-2}\mathcal{M}^2\beta^4, & m\ge 2\xi-1. \end{cases} \tag{60}$$

So,

$$\frac{1}{\beta^{\frac{1}{\xi}}} \le \begin{cases} \dfrac{\left(\dfrac{1}{2}\right)^{\frac{1}{2(m+1)}}\left(\dfrac{\overline{\Phi}(2\xi-m-1)}{\xi|4\overline{A}(\underline{\Phi},\lambda_1,T,\gamma)|^{m+1}}\right)^{\frac{1}{m+1}}\left(\frac{m+1}{2\xi-m-1}\right)^{\frac{1}{2\xi}}}{(\delta^2-2)^{\frac{1}{2(m+1)}}}\left(\dfrac{\mathcal{M}}{\epsilon}\right)^{\frac{1}{m+1}}, & 0<m<2\xi-1, \\ \dfrac{8^{\frac{1}{4\xi}}(\overline{\Phi})^{\frac{1}{2\xi}}\left(|4\overline{A}(\underline{\Phi},\lambda_1,T,\gamma)|^{2\xi}\lambda_1^{(m+1)-2\xi}\right)^{-\frac{1}{2\xi}}}{(\delta^2-2)^{\frac{1}{4\xi}}}\left(\dfrac{\mathcal{M}}{\epsilon}\right)^{\frac{1}{2\xi}}, & m\ge 2\xi-1. \end{cases} \tag{61}$$

The estimation of $\left\|\mathcal{F}(\cdot)-\mathcal{F}^{\epsilon}_{\beta(\epsilon)}(\cdot)\right\|_{\mathscr{L}_2(\Omega)}$ is established by our next Theorem. □

Theorem 3.7. *Assume the a priori condition and the noise assumption (6) hold, and there exists $\delta > 1$ such that $0 < \delta\epsilon < \|\ell^\epsilon\|_{\mathscr{L}_2(\Omega)}$. This Theorem shows the convergent estimate between the exact solution and the regularized solution such that*

- *If $0 < m < 2\xi - 1$, it gives*

$$\left\|\mathcal{F}(\cdot) - \mathcal{F}^\epsilon_{\beta(\epsilon)}(\cdot)\right\|_{\mathscr{L}_2(\Omega)} \text{ is of order } \epsilon^{\frac{m}{m+1}}. \tag{62}$$

- *If $m \geq 2\xi - 1$, it gives*

$$\left\|\mathcal{F}(\cdot) - \mathcal{F}^\epsilon_{\beta(\epsilon)}(\cdot)\right\|_{\mathscr{L}_2(\Omega)} \text{ is of order } \epsilon^{1-\frac{1}{2\xi}}. \tag{63}$$

Proof. Applying the triangle inequality, we get

$$\left\|\mathcal{F}(\cdot) - \mathcal{F}^\epsilon_{\beta(\epsilon)}(\cdot)\right\|_{\mathscr{L}_2(\Omega)} \leq \underbrace{\left\|\mathcal{F}(\cdot) - \mathcal{F}_{\beta(\epsilon)}\right\|_{\mathscr{L}_2(\Omega)}}_{\mathcal{A}_2} + \left\|\mathcal{F}_{\beta(\epsilon)}(\cdot) - \mathcal{F}^\epsilon_{\beta(\epsilon)}(\cdot)\right\|_{\mathscr{L}_2(\Omega)}. \tag{64}$$

Case 1: If $0 < m \leq 2\xi - 1$, we have

$$\begin{aligned}\left\|\mathcal{F}_{\beta(\epsilon)}(\cdot) - \mathcal{F}^\epsilon_{\beta(\epsilon)}(\cdot)\right\|_{\mathscr{L}_2(\Omega)} &\leq \epsilon\, [\beta(\epsilon)]^{-\frac{1}{\xi}} \left(\left|\mathcal{B}(|\underline{\Phi}|, |\overline{\Phi}|)\right|^{2\xi-1} \overline{B}\big(\xi, \overline{A}(\underline{\Phi}, \lambda_1, T, \gamma)\big)\right) \\ &\quad + 2\|\Phi^\epsilon - \Phi\|_{\mathscr{L}_\infty(0,T)} \max\left\{\frac{1}{|\underline{\Phi}|^{2\xi-1}}, \frac{4}{\underline{\Phi}}\right\}\|\mathcal{F}\|_{\mathscr{L}_2(\Omega)}.\end{aligned} \tag{65}$$

We get

$$\begin{aligned}\|\mathcal{A}_2\|_{\mathscr{L}_2(\Omega)} &= \Big\|\sum_{j=1}^\infty \left(\frac{\left|\mathcal{S}(\lambda_j, \gamma, \Phi)\right|^{2\xi-1}}{[\beta(\epsilon)]^2 + \left|\mathcal{S}(\lambda_j, \gamma, \Phi)\right|^{2\xi}} - \frac{1}{\left|\mathcal{S}(\lambda_j, \gamma, \Phi)\right|}\right)\langle \ell(\cdot), \mathrm{e}_j(\cdot)\rangle \mathrm{e}_j(x)\Big\|_{\mathscr{L}_2(\Omega)} \\ &= \Big\|\sum_{j=1}^\infty \frac{[\beta(\epsilon)]^2 \mathcal{S}(\lambda_j, \gamma, \Phi)}{[\beta(\epsilon)]^2 + \left|\mathcal{S}(\lambda_j, \gamma, \Phi)\right|^{2\xi}}\langle \mathcal{F}(\cdot), \mathrm{e}_j(\cdot)\rangle \mathrm{e}_j(x)\Big\|_{\mathscr{L}_2(\Omega)}.\end{aligned} \tag{66}$$

The Hölder inequality gives us the result

$$\begin{aligned}\|\mathcal{A}_2\|_{\mathscr{L}_2(\Omega)} &\leq \Big\|\sum_{j=1}^\infty \frac{[\beta(\epsilon)]^2 \mathcal{S}(\lambda_j, \gamma, \Phi)}{[\beta(\epsilon)]^2 + \left|\mathcal{S}(\lambda_j, \gamma, \Phi)\right|^{2\xi}}\langle \mathcal{F}(\cdot), \mathrm{e}_j(\cdot)\rangle \mathrm{e}_j(\cdot)\Big\|^{\frac{m}{m+1}}_{\mathscr{L}_2(\Omega)} \\ &\quad \times \Big\|\sum_{j=1}^\infty \frac{[\beta(\epsilon)]^2 \mathcal{S}(\lambda_j, \gamma, \Phi)}{[\beta(\epsilon)]^2 + \left|\mathcal{S}(\lambda_j, \gamma, \Phi)\right|^{2\xi}} \frac{\langle \mathcal{F}(\cdot), \mathrm{e}_j(\cdot)\rangle}{\left|\mathcal{S}(\lambda_j, \gamma, \Phi)\right|^{m+1}} \mathrm{e}_j(x)\Big\|^{\frac{1}{m+1}}_{\mathscr{L}_2(\Omega)} \\ &\leq \underbrace{\Big\|\sum_{j=1}^\infty \frac{[\beta(\epsilon)]^2}{[\beta(\epsilon)]^2 + \left|\mathcal{S}(\lambda_j, \gamma, \Phi)\right|^{2\xi}}\langle \ell(\cdot), \mathrm{e}_j(\cdot)\rangle \mathrm{e}_j(x)\Big\|^{\frac{m}{m+1}}_{\mathscr{L}_2(\Omega)}}_{\mathcal{Z}_1} \\ &\quad \times \underbrace{\Big\|\sum_{j=1}^\infty \frac{[\beta(\epsilon)]^2}{[\beta(\epsilon)]^2 + \left|\mathcal{S}(\lambda_j, \gamma, \Phi)\right|^{2\xi}} \frac{\langle \mathcal{F}(\cdot), \mathrm{e}_j(\cdot)\rangle}{\left|\mathcal{S}(\lambda_j, \gamma, \Phi)\right|^{m}} \mathrm{e}_j(x)\Big\|^{\frac{1}{m+1}}_{\mathscr{L}_2(\Omega)}}_{\mathcal{Z}_2}.\end{aligned} \tag{67}$$

From (67), using Lemma 3.6, one has

$$\mathscr{Z}_1 \leq \bigg(\Big\| \sum_{j=1}^{\infty} \frac{[\beta(\epsilon)]^2}{[\beta(\epsilon)]^2 + \big|\mathcal{S}(\lambda_j,\gamma,\Phi)\big|^{2\xi}} \big\langle \ell(\cdot) - \ell^{\epsilon}(\cdot), \mathrm{e}_j(x) \big\rangle \mathrm{e}_j(x) \Big\|_{\mathscr{L}_2(\Omega)} + \Big\| \sum_{j=1}^{\infty} \frac{[\beta(\epsilon)]^2}{[\beta(\epsilon)]^2 + \big|\mathcal{S}(\lambda_j,\gamma,\Phi)\big|^{2\xi}} \big\langle \ell^{\epsilon}(\cdot), \mathrm{e}_j(\cdot) \big\rangle \mathrm{e}_j(x) \Big\|_{\mathscr{L}_2(\Omega)} \bigg)^{\frac{m}{m+1}} \leq \epsilon^{\frac{m}{m+1}} (1+\delta)^{\frac{m}{m+1}}.$$

Next, using the priori condition, we have

$$\begin{aligned} \mathscr{Z}_2 &= \Big\| \sum_{j=1}^{+\infty} \frac{[\beta(\epsilon)]^2}{[\beta(\epsilon)]^2 + \big|\mathcal{S}(\lambda_j,\gamma,\Phi)\big|^{2\xi}} \frac{\big\langle \mathcal{F}(\cdot), \mathrm{e}_j(\cdot) \big\rangle}{\big|\mathcal{S}(\lambda_j,\gamma,\Phi)\big|^{m}} \mathrm{e}_j(x) \Big\|_{\mathscr{L}_2(\Omega)}^{\frac{1}{m+1}} \\ &\leq \Big\| \sum_{j=1}^{+\infty} \frac{\big\langle \mathcal{F}(\cdot), \mathrm{e}_j(\cdot) \big\rangle}{|\mathcal{S}(\lambda_j,\gamma,\Phi)|^m} \mathrm{e}_j(x) \Big\|_{\mathscr{L}_2(\Omega)}^{\frac{1}{m+1}} \leq \Big\| \sum_{j=1}^{\infty} \frac{\lambda_j^m \big\langle \mathcal{F}_j(\cdot), \mathrm{e}_j(\cdot) \big\rangle}{\big|4\overline{A}(\underline{\Phi},\lambda_1,T,\gamma)\big|^m} \mathrm{e}_j(x) \Big\|_{\mathscr{L}_2(\Omega)}^{\frac{1}{m+1}} \\ &\leq \frac{\mathcal{M}^{\frac{1}{m+1}}}{\big|4\overline{A}(\underline{\Phi},\lambda_1,T,\gamma)\big|^{\frac{m}{m+1}}}. \end{aligned} \tag{68}$$

Combining (66) to (68), we conclude that

$$\big\|\mathcal{F}(\cdot) - \mathcal{F}_{\beta(\epsilon)}(\cdot)\big\|_{\mathscr{L}_2(\Omega)} \leq \epsilon^{\frac{m}{m+1}} (1+\delta)^{\frac{m}{m+1}} \frac{\mathcal{M}^{\frac{1}{m+1}}}{\big|4\overline{A}(\underline{\Phi},\lambda_1,T,\gamma)\big|^{\frac{m}{m+1}}}. \tag{69}$$

Combining (65) to (69), we know that

$$\big\|\mathcal{F}(\cdot) - \mathcal{F}^{\epsilon}_{\beta(\epsilon)}(\cdot)\big\|_{\mathscr{L}_2(\Omega)} \leq \epsilon^{\frac{m}{m+1}} \mathcal{M}^{\frac{1}{m+1}} \mathcal{X}_1(\underline{\Phi}, \overline{\Phi}, \delta, \overline{B}, \overline{A}), \tag{70}$$

whereby

$$\begin{aligned} \mathcal{X}_1(\underline{\Phi}, \overline{\Phi}, \delta, \overline{B}, \overline{A}) &= 2\epsilon^{\frac{1}{m+1}} \max\Big\{ \frac{1}{|\underline{\Phi}|^{2\xi-1}}, \frac{4}{\underline{\Phi}} \Big\} \bigg(\frac{\|\ell\|_{\mathscr{L}_2(\Omega)}}{\big|4\overline{A}(\underline{\Phi},\lambda_1,T,\gamma)\big|} \bigg)^{\frac{m}{m+1}} + \frac{(1+\delta)^{\frac{m}{m+1}}}{|4\overline{A}(\underline{\Phi},\lambda_1,T,\gamma)|^{\frac{m}{m+1}}} \\ &+ \frac{(\frac{1}{2})^{\frac{1}{2(m+1)}} \Big(\frac{\overline{\Phi}(2\xi - m - 1)}{\xi |4\overline{A}(\underline{\Phi},\lambda_1,T,\gamma)|^{m+1}} \Big)^{\frac{1}{m+1}} \big(\frac{m+1}{2\xi - m - 1} \big)^{\frac{1}{2\xi}}}{(\delta^2 - 2)^{\frac{1}{2(m+1)}}} \bigg(\big|\mathcal{B}(|\underline{\Phi}|, |\overline{\Phi}|)\big|^{2\xi-1} \; \overline{B}\big(\xi, \overline{A}(\underline{\Phi},\lambda_1,T,\gamma)\big) \bigg). \end{aligned}$$

Case 2: Our next goal is to determine the estimation of $\big\|\mathcal{F}_{\beta(\epsilon)}(\cdot) - \mathcal{F}^{\epsilon}_{\beta(\epsilon)}(\cdot)\big\|_{\mathscr{L}_2(\Omega)}$ in case $m \geq 2\xi - 1$, we get

$$\begin{aligned} &\big\|\mathcal{F}_{\beta(\epsilon)}(\cdot) - \mathcal{F}^{\epsilon}_{\beta(\epsilon)}(\cdot)\big\|_{\mathscr{L}_2(\Omega)} \\ &\leq \epsilon^{1-\frac{1}{2\xi}} \mathcal{M}^{\frac{1}{2\xi}} \frac{8^{\frac{1}{4\xi}} (\overline{\Phi})^{\frac{1}{2\xi}} \Big(|4\overline{A}(\underline{\Phi},\lambda_1,T,\gamma)|^{2\xi} \lambda_1^{(m+1)-2\xi} \Big)^{-\frac{1}{2\xi}}}{(\delta^2 - 2)^{\frac{1}{4\xi}}} \bigg(\big|\mathcal{B}(|\underline{\Phi}|, |\overline{\Phi}|)\big|^{2\xi-1} \; \overline{B}\big(\xi, \overline{A}(\underline{\Phi},\lambda_1,T,\gamma)\big) \bigg) \\ &+ \epsilon^{1-\frac{1}{2\xi}} \mathcal{M}^{\frac{1}{2\xi}} \bigg(2 \max\Big\{ \frac{1}{|\underline{\Phi}|^{2\xi-1}}, \frac{4}{\underline{\Phi}} \Big\} \frac{1}{\big|4\overline{A}(\underline{\Phi},\lambda_1,T,\gamma)\big|^{1-\frac{1}{2\xi}}} \bigg). \end{aligned} \tag{71}$$

Next, $\left\|\mathcal{F}(\cdot)-\mathcal{F}_{\beta(\epsilon)}(\cdot)\right\|_{\mathscr{L}_2(\Omega)}$ can be bounded as follows

$$\begin{aligned}\left\|\mathcal{F}(\cdot)-\mathcal{F}_{\beta(\epsilon)}(\cdot)\right\|_{\mathscr{L}_2(\Omega)} &= \Big\|\sum_{j=1}^{\infty}\frac{[\beta(\epsilon)]^2}{[\beta(\epsilon)]^2+\left|\mathcal{S}(\lambda_j,\gamma,\Phi)\right|^{2\xi}}\big\langle\mathcal{F}(\cdot),\mathrm{e}_j(\cdot)\big\rangle\mathrm{e}_j(x)\Big\|_{\mathscr{L}_2(\Omega)}\\ &= \Big\|\sum_{j=1}^{+\infty}\frac{[\beta(\epsilon)]^2\mathcal{S}(\lambda_j,\gamma,\Phi)}{\alpha^2+\left|\mathcal{S}(\lambda_j,\gamma,\Phi)\right|^{2\xi}}\frac{\big\langle\mathcal{F}(\cdot),\mathrm{e}_j(\cdot)\big\rangle}{\mathcal{S}(\lambda_j,\gamma,\Phi)}\mathrm{e}_j(x)\Big\|_{\mathscr{L}_2(\Omega)}\\ &\le \underbrace{\Big\|\sum_{j=1}^{\infty}\frac{[\beta(\epsilon)]^2\mathcal{S}(\lambda_j,\gamma,\Phi)}{[\beta(\epsilon)]^2+\left|\mathcal{S}(\lambda_j,\gamma,\Phi)\right|^{2\xi}}\big\langle\mathcal{F}(\cdot),\mathrm{e}_j(\cdot)\big\rangle\mathrm{e}_j(x)\Big\|^{1-\frac{1}{2\xi}}_{\mathscr{L}_2(\Omega)}}_{\mathcal{J}_1}\\ &\quad\times\underbrace{\Big\|\sum_{j=1}^{\infty}\frac{[\beta(\epsilon)]^2\mathcal{S}(\lambda_j,\gamma,\Phi)}{[\beta(\epsilon)]^2+\left|\mathcal{S}(\lambda_j,\gamma,\Phi)\right|^{2\xi}}\frac{\big\langle\mathcal{F}(\cdot),\mathrm{e}_j(\cdot)\big\rangle}{\left|\mathcal{S}(\lambda_j,\gamma,\Phi)\right|^{2\xi}}\mathrm{e}_j(x)\Big\|^{\frac{1}{2\xi}}_{\mathscr{L}_2(\Omega)}}_{\mathcal{J}_2}.\end{aligned}\tag{72}$$

From (72), repeated application of Lemma 3.6 Part (b) enables us to write $\mathcal{J}_1$, it is easy to check that

$$\mathcal{J}_1\le(\epsilon+\delta\epsilon)^{1-\frac{1}{2\xi}}=\epsilon^{1-\frac{1}{2\xi}}(1+\delta)^{1-\frac{1}{2\xi}}.$$

In the same way as in $\mathcal{Z}_2$, it follows easily that

$$\frac{[\beta(\epsilon)]^2}{[\beta(\epsilon)]^2+|\mathcal{S}(\lambda_j,\gamma,\Phi)|^{2\xi}}<1,$$

we now proceed by induction

$$\begin{aligned}\mathcal{J}_2&\le\Big\|\sum_{j=1}^{\infty}\frac{\big\langle\mathcal{F}(\cdot),\mathrm{e}_j(\cdot)\big\rangle}{|\mathcal{S}(\lambda_j,\gamma,\Phi)|^{2\xi-1}}\mathrm{e}_j(x)\Big\|^{\frac{1}{2\xi}}_{\mathscr{L}_2(\Omega)}\le\Big\|\sum_{j=1}^{\infty}\Big(\frac{\lambda_j}{|4\overline{A}(\underline{\Phi},\lambda_1,T,\gamma)|}\Big)^{2\xi-1}\lambda_j^{-m}\lambda_j^{m}\mathcal{F}_j(\cdot)\mathrm{e}_j(x)\Big\|^{\frac{1}{2\xi}}_{\mathscr{L}_2(\Omega)}\\&\le|4\overline{A}(\underline{\Phi},\lambda_1,T,\gamma)|^{\frac{1}{2\xi}-1}\lambda_1^{2\xi-m-1}\mathcal{M}^{\frac{1}{2\xi}}.\end{aligned}\tag{73}$$

Combining (72) to (73), it may be concluded that

$$\left\|\mathcal{F}(\cdot)-\mathcal{F}_{\beta(\epsilon)}(\cdot)\right\|_{\mathscr{L}_2(\Omega)}\le\epsilon^{1-\frac{1}{2\xi}}\mathcal{M}^{\frac{1}{2\xi}}\Big((1+\delta)^{1-\frac{1}{2\xi}}|4\overline{A}(\underline{\Phi},\lambda_1,T,\gamma)|^{\frac{1}{2\xi}-1}\lambda_1^{2\xi-m-1}\Big).$$

Finally, from (70) and (71), we can assert that

$$\left\|\mathcal{F}(\cdot)-\mathcal{F}^{\epsilon}_{\beta(\epsilon)}(\cdot)\right\|_{\mathscr{L}_2(\Omega)}\le\epsilon^{1-\frac{1}{2\xi}}\mathcal{M}^{\frac{1}{2\xi}}\mathcal{X}_2(\underline{\Phi},\overline{\Phi},\delta,\overline{B},\overline{A}).$$

whereby

$$\begin{aligned}\mathcal{X}_2(\underline{\Phi},\overline{\Phi},\delta,\overline{B},\overline{A})&=\Bigg(\frac{2\max\Big\{\frac{1}{|\underline{\Phi}|^{2\xi-1}},\frac{4}{\underline{\Phi}}\Big\}}{\left|4\overline{A}(\underline{\Phi},\lambda_1,T,\gamma)\right|^{1-\frac{1}{2\xi}}}\Bigg)+\Big((1+\delta)^{1-\frac{1}{2\xi}}|4\overline{A}(\underline{\Phi},\lambda_1,T,\gamma)|^{\frac{1}{2\xi}-1}\lambda_1^{2\xi-m-1}\Big)\\&+\frac{8^{\frac{1}{4\xi}}(\overline{\Phi})^{\frac{1}{2\xi}}\Big(|4\overline{A}(\underline{\Phi},\lambda_1,T,\gamma)|^{2\xi}\lambda_1^{(m+1)-2\xi}\Big)^{-\frac{1}{2\xi}}}{(\delta^2-2)^{\frac{1}{4\xi}}}\Big(\left|\mathcal{B}(|\underline{\Phi}|,|\overline{\Phi}|)\right|^{2\xi-1}\overline{B}\big(\xi,\overline{A}(\underline{\Phi},\lambda_1,T,\gamma)\big)\Big).\end{aligned}$$

The proof is completed by showing that (70) and (71). □

3.3. Regularization and $\mathscr{L}_p$ stimate by Truncation method

In this subsection, we assume that ℓ^ϵ is noisy data and satisfied that

$$\|\ell^\epsilon - \ell\|_{\mathscr{L}_p(\Omega)} \le \epsilon. \tag{74}$$

Theorem 3.8. *Let ℓ^ϵ be as in (74). Assume that $\mathcal{F}$ belongs to $\mathcal{D}(\mathcal{A}^\zeta)$ for any $\zeta > 0$. Let us gives a regularized solution as follows. Let us give a regularized solution as follows:*

$$\mathcal{F}^\epsilon_{\mathcal{N}_\epsilon}(x) = \sum_{j=1}^{\mathcal{N}_\epsilon} \frac{\langle \ell^\epsilon(\cdot), \mathrm{e}_j(\cdot)\rangle \mathrm{e}_j(x)}{|\mathcal{S}(\lambda_j, \gamma, \Phi^\epsilon)|}, \text{ and } \mathcal{F}_{\mathcal{N}_\epsilon}(x) = \sum_{j=1}^{\mathcal{N}_\epsilon} \frac{\langle \ell(\cdot), \mathrm{e}_j(\cdot)\rangle \mathrm{e}_j(x)}{|\mathcal{S}(\lambda_j, \gamma, \Phi)|}. \tag{75}$$

By choosing

$$\mathcal{N}_\epsilon = \epsilon^{(h-1)(\zeta - m+1)}, \; 0 < h < 1, \tag{76}$$

in which

$$-\frac{\omega}{4} < p \le \min\left\{0, \frac{(p-2)\omega}{4p}\right\}, \; 0 \le \zeta < \frac{\omega}{4}.$$

Then we have

$$\left\|\mathcal{F}^\epsilon_{\mathcal{N}_\epsilon}(\cdot) - \mathcal{F}(\cdot)\right\|_{\mathscr{L}^{\frac{2\omega}{\omega-4\zeta}}(\Omega)} \to 0 \text{ when } \epsilon \to 0.$$

Proof. Since the Sobolev embedding $\mathscr{L}_p(\Omega) \hookrightarrow \mathcal{D}(\mathcal{A}^m)$, we find that there exists a positive constant,

$$\|\ell^\epsilon - \ell\|_{\mathcal{D}(\mathcal{A}^m)} \le C_{m,p}\|\ell^\epsilon - \ell\|_{\mathscr{L}_p(\Omega)} \le C_{m,p}\epsilon. \tag{77}$$

For $\zeta > 0$, using the triangle inequality, we get

$$\left\|\mathcal{F}^\epsilon_{\mathcal{N}_\epsilon}(\cdot) - \mathcal{F}(\cdot)\right\|_{\mathcal{D}(\mathcal{A}^\zeta)} \le \left\|\mathcal{F}^\epsilon_{\mathcal{N}_\epsilon}(\cdot) - \mathcal{F}_{\mathcal{N}_\epsilon}(\cdot)\right\|_{\mathcal{D}(\mathcal{A}^\zeta)} + \left\|\mathcal{F}_{\mathcal{N}_\epsilon}(\cdot) - \mathcal{F}(\cdot)\right\|_{\mathcal{D}(\mathcal{A}^\zeta)}. \tag{78}$$

In the following, we first consider the term $\left\|\mathcal{F}^\epsilon_{\mathcal{N}_\epsilon}(\cdot) - \mathcal{F}_{\mathcal{N}_\epsilon}(\cdot)\right\|_{\mathcal{D}(\mathcal{A}^\zeta)}$ for any $0 < \zeta < \dfrac{\omega}{4}$. Indeed, we get

$$\begin{aligned}
&\mathcal{F}^\epsilon_{\mathcal{N}_\epsilon}(x) - \mathcal{F}_{\mathcal{N}_\epsilon}(x) \\
&= \sum_{j=1}^{\lambda_j \le \mathcal{N}_\epsilon} \frac{\langle \ell^\epsilon(\cdot) - \ell(\cdot), \mathrm{e}_j(\cdot)\rangle \mathrm{e}_j(x)}{\mathcal{S}(\lambda_j, \gamma, \Phi^\epsilon)} + \sum_{j=1}^{\lambda_j \le \mathcal{N}_\epsilon} \Big[|\mathcal{S}(\lambda_j, \gamma, \Phi^\epsilon)|^{-1} - |\mathcal{S}(\lambda_j, \gamma, \Phi)|^{-1}\Big] \langle \ell(\cdot), \mathrm{e}_j(\cdot)\rangle \mathrm{e}_j(x).
\end{aligned} \tag{79}$$

From (79), using the triangle inequality $(a+b)^2 \le 2a^2 + 2b^2, \forall a, b \ge 0$, we have

$$\begin{aligned}
&\left\|\mathcal{F}^\epsilon_{\mathcal{N}_\epsilon}(\cdot) - \mathcal{F}_{\mathcal{N}_\epsilon}(\cdot)\right\|^2_{\mathcal{D}(\mathcal{A}^\zeta)} \\
&\le 2\sum_{j=1}^{\lambda_j \le \mathcal{N}_\epsilon} \lambda_j^{2\zeta - 2m} \frac{\lambda_j^{2m}\left|\langle \ell^\epsilon(\cdot) - \ell(\cdot), \mathrm{e}_j(\cdot)\rangle\right|^2}{|\mathcal{S}(\lambda_j, \gamma, \Phi^\epsilon)|^2} + 2\sum_{j=1}^{\lambda_j \le \mathcal{N}_\epsilon} \frac{|\mathcal{S}(\lambda_j, \gamma, \Phi - \Phi^\epsilon)|^2}{|\mathcal{S}(\lambda_j, \gamma, \Phi^\epsilon)|^2} \frac{\lambda_j^{2\zeta}\left|\langle \ell(\cdot), \mathrm{e}_j(\cdot)\rangle\right|^2}{|\mathcal{S}(\lambda_j, \gamma, \Phi)|^2} \\
&\le 2\sum_{j=1}^{\lambda_j \le \mathcal{N}_\epsilon} \lambda_j^{2\zeta - 2m+2} \frac{\lambda_j^{2m}\left|\langle \ell^\epsilon(\cdot) - \ell(\cdot), \mathrm{e}_j(\cdot)\rangle\right|^2}{|\overline{A}(\underline{\Phi}, \lambda_1, T, \gamma)|^2} + \frac{32\|\Phi^\epsilon - \Phi\|^2_{\mathscr{L}_\infty(0,T)}}{|\underline{\Phi}|^2} \sum_{j=1}^{\lambda_j \le \mathcal{N}_\epsilon} \frac{\lambda_j^{2\zeta}\left|\langle \ell(\cdot), \mathrm{e}_j(\cdot)\rangle\right|^2}{|\mathcal{S}(\lambda_j, \gamma, \Phi)|^2} \\
&\le 2\sum_{j=1}^{\lambda_j \le \mathcal{N}_\epsilon} \lambda_j^{2\zeta - 2m+2} \frac{\|\ell^\epsilon - \ell\|^2_{\mathcal{D}(\mathcal{A}^m)}}{|\overline{A}(\underline{\Phi}, \lambda_1, T, \gamma)|^2} + \frac{32\|\Phi^\epsilon - \Phi\|^2_{\mathscr{L}_\infty(0,T)}}{|\underline{\Phi}|^2} \sum_{j=1}^{\infty} \lambda_j^{2\zeta}\left|\langle \mathcal{F}(\cdot), \mathrm{e}_j(\cdot)\rangle\right|^2.
\end{aligned} \tag{80}$$

Using the condition (77), we can know that

$$\left\|\mathcal{F}_{\mathcal{N}_\epsilon}^{\epsilon}(\cdot)-\mathcal{F}_{\mathcal{N}_\epsilon}(\cdot)\right\|_{\mathcal{D}(\mathcal{A}^\zeta)}^2 \le \frac{2C_{m,p}^2\epsilon^2}{|\overline{A}(\underline{\Phi},\lambda_1,T,\gamma)|^2}(\mathcal{N}_\epsilon)^{2\zeta-2m+2}+\frac{32\epsilon^2}{|\underline{\Phi}|^2}\|\mathcal{F}\|_{\mathcal{D}(\mathcal{A}^\zeta)}^2.$$

Next, we continue to get the following estimate

$$\begin{aligned}\left\|\mathcal{F}(\cdot)-\mathcal{F}_{\mathcal{N}_\epsilon}(\cdot)\right\|_{\mathcal{D}(\mathcal{A}^\zeta)}^2 &\le \sum_{\lambda_j\ge\mathcal{N}_\epsilon}^{\infty}\lambda_j^{-2\zeta}\lambda_j^{2\zeta}\frac{\left|\left\langle \ell(\cdot),\mathrm{e}_j(\cdot)\right\rangle\right|^2}{|\mathcal{S}(\lambda_j,\gamma,\Phi)|^2}\le \sum_{\lambda_j\ge\mathcal{N}_\epsilon}^{\infty}\lambda_j^{-2\zeta}\lambda_j^{2\zeta}\left|\left\langle \mathcal{F}(\cdot),\mathrm{e}_j(\cdot)\right\rangle\right|^2\\ &\le \left(\mathcal{N}_\epsilon\right)^{-2\zeta}\sum_{\lambda_j\ge\mathcal{N}_\epsilon}^{\infty}\lambda_j^{2\zeta}\left|\left\langle \mathcal{F}(\cdot),\mathrm{e}_j(\cdot)\right\rangle\right|^2\le\left(\mathcal{N}_\epsilon\right)^{-2\zeta}\left\|\mathcal{F}\right\|_{\mathcal{D}(\mathcal{A}^\zeta)}^2. \end{aligned} \tag{81}$$

Since the Sobolev embedding $\mathcal{D}(\mathcal{A}^\zeta)\hookrightarrow \mathscr{L}^{\frac{2\omega}{\omega-4\zeta}}(\Omega)$, combining (78) to (81), we conclude that

$$\begin{aligned}\left\|\mathcal{F}_{\mathcal{N}_\epsilon}^{\epsilon}(\cdot)-\mathcal{F}(\cdot)\right\|_{\mathscr{L}^{\frac{2\omega}{\omega-4\zeta}}(\Omega)} &\le C\left\|\mathcal{F}_{\mathcal{N}_\epsilon}^{\epsilon}(\cdot)-\mathcal{F}(\cdot)\right\|_{\mathcal{D}(\mathcal{A}^\zeta)}\le C\left\|\mathcal{F}_{\mathcal{N}_\epsilon}^{\epsilon}(\cdot)-\mathcal{F}_{\mathcal{N}_\epsilon}(\cdot)\right\|_{\mathcal{D}(\mathcal{A}^\zeta)}+C\left\|\mathcal{F}_{\mathcal{N}_\epsilon}(\cdot)-\mathcal{F}(\cdot)\right\|_{\mathcal{D}(\mathcal{A}^\zeta)}\\ &\le \frac{\sqrt{2}CC_{m,p}\epsilon}{|\overline{A}(\underline{\Phi},\lambda_1,T,\gamma)|}(\mathcal{N}_\epsilon)^{\zeta-m+1}+\left(\frac{4\sqrt{2}C\epsilon}{|\underline{\Phi}|}+C\left(\mathcal{N}_\epsilon\right)^{-\zeta}\right)\left\|\mathcal{F}\right\|_{\mathcal{D}(\mathcal{A}^\zeta)}\\ &\le \epsilon^h\frac{\sqrt{2}CC_{m,p}}{|\overline{A}(\underline{\Phi},\lambda_1,T,\gamma)|}+\left(\frac{4\sqrt{2}C\epsilon}{|\underline{\Phi}|}+C\epsilon^{\zeta(1-h)(\zeta-m+1)}\right)\left\|\mathcal{F}\right\|_{\mathcal{D}(\mathcal{A}^\zeta)}.\end{aligned} \tag{82}$$

□

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