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Research Article

Completeness of fuzzy quasi-pseudometric spaces

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Abstract

The purpose of this paper is to present the relations among the completeness of sequences, of filters and of nets in the framework of fuzzy quasi-pseudometric spaces. In particular, we show that right completeness of filters and of sequences are equivalent under special conditions of fuzzy quasi-pseudometrics. By introducing a kind of more general right K-Cauchy nets in fuzzy quasi-pseudometric spaces, the equivalence between the completeness of the nets and the sequential completeness is established.

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1. Introduction

The notion of quasi-metrics was introduced by Wilson [29] in 1931, which is relaxed by omitting the symmetry condition. The lose of symmetry usually causes a lot of troubles in which the appropriate generalizations of metric results are no longer valid, particularly these for completeness [1, 4, 17, 22, 23]. In metric spaces, the notions of completeness of sequences, of nets and of filters all coincide. But in the quasi-metric setting, there exist situations such that these notions are different apart from changing certain conditions. A kind of situation was emphasized by Stoltenberg [26], who proposed a notion of Cauchy nets for which completeness coincides with the sequential completeness. Later Gregori and Ferrer [9] further proposed a new version of right K-Cauchy nets for which the corresponding completeness and sequential completeness coincide. In a recent paper [4], Cobzaş revisited Gregori and Ferrer's notion of Cauchy nets and proposed a new notion of Cauchy nets for which the equivalence with sequential completeness holds.

As a significant generalization of ordinary metric spaces, the theory of probabilistic metric spaces can be traced back to the work of Menger [20] and Wald [28]. In the theory, the notion of distance has a probabilistic nature. Namely, the distance between two points is represented by a distribution function, assigning any positive number x the probability

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that the distance is less than x. Such a probabilistic generalization is well adapted for the investigation of physical quantities and practical problems. More information on this subject can be found in the textbook [24].

In 1975, Kramosil and Michálek [19] extended the concept of probabilistic metric spaces to the fuzzy context and gave a notion of fuzzy metric spaces. In particular, they observed that the class of fuzzy metric spaces is "equivalent" to the class of probabilistic metric spaces under continuous t-norms. In order to provide rich topological structures, George and Veeramani [7] introduced and studied a notion of fuzzy metrics which constitutes a modification of Kramosil and Michálek's fuzzy metrics. Subsequently, several properties of classical metrics were extended to the fuzzy setting in [8]. Gregori and Romaguera [13] proved the topology generated by any (complete) fuzzy metric space is (completely) metrizable. Nevertheless, the completion theory of George and Veeramani's fuzzy metrics is quite different from that of ordinary or probabilistic metric spaces. Indeed, there exist fuzzy metric spaces which are non-completable [10, 14]. Further, Gregori et al. [11] provided a characterization of those fuzzy metric spaces admitting a fuzzy metric completion.

The notion of fuzzy quasi-metric spaces was formally introduced by Gregori and Romaguera in 2004 [15]. They presented some preliminary concepts and facts for solving the problem of bicompletion of fuzzy quasi-metric spaces in the sense of George and Veeramani. Subsequently, Gregori et al. [16] obtained an internal characterization of those fuzzy quasi-metric spaces that admit a fuzzy quasi-metric bicompletion. Following the idea of Doitchinov [6], Gregori et al. [12] introduced a notion of Cauchy sequence in fuzzy quasi-metric spaces and studied a completion for a special class of such spaces. By extending the Sherwood's results on completion of probabilistic metric spaces [25], Castro-Company et al. [3] showed that every fuzzy quasi-metric space (in the sense of Kramosil and Michálek) has a unique bicompletion up to isometry.

To our knowledge, the relations among various notions of sequential completeness and the corresponding notions of completeness of nets or of filters have not been investigated efficiently in fuzzy quasi-pseudometric spaces. The aim of this paper is to present the several versions of the completeness of sequences, nets and filters in fuzzy quasi-pseudometric spaces in the sense of George and Veeramani, as well as their relations of these completeness.

The outline of this paper is organized as follows. In the next section, the preliminary notions on fuzzy metric spaces are recalled. In Section 3, following the idea of Reilly et al. [22], we define seven kinds of Cauchy sequences, yielding fourteen different notions of completeness in fuzzy quasi-metric spaces, all coinciding with the usual one in the fuzzy metric case. Moreover, we exploit the relations between these notions and provide some characterizations for sequential completeness in part. In Section 4, we present the relations between completeness of sequences, of nets and of filters in fuzzy quasi-pseudometric spaces. For some notions of completeness they coincide, but they can be different for others, particularly in the study of right completeness. More precisely, we first give some properties and show that the right completeness of filters coincide with the sequential right completeness under these properties. Further, we show that a kind of more general right K-Cauchy nets in fuzzy quasi-pseudometric spaces for which the corresponding completeness are all equivalent to the sequential completeness. A brief conclusion is given in Section 5.

2. Preliminaries

In this section, we recall some basic concepts and results which will be used throughout this paper. The letters $\mathbb{R}$ and $\mathbb{N}$ always denotes the set of real numbers and the set of positive integer numbers, respectively.

A quasi-pseudometric on a set X is a map $d : X \times X \longrightarrow [0, +\infty)$ such that for all $x, y, z \in X$:

(i) $d(x,x) = 0$;
(ii) $d(x,y) \leq d(x,z) + d(z,y)$.

A quasi-metric on X is a quasi-pseudometric d satisfying the condition:

(iii) $d(x,y) = 0 = d(y,x)$ implies that $x = y$.

The pair (X,d) is called a quasi-metric space.

Definition 2.1 ([18])**.** A t-norm $*$ on $[0,1]$ is a binary operation on $[0,1]$ which is commutative (i.e., $a*b = b*a$ whenever $a,b \in [0,1]$), associative (i.e., $a*(b*c) = (a*b)*c$ whenever $a,b,c \in [0,1]$), monotone (i.e., $a*c \leq b*d$ whenever $a \leq b$ and $c \leq d$ for $a,b,c,d \in [0,1]$) and has the top element 1 as the unit (i.e., $b*1 = b$ whenever $b \in [0,1]$). A t-norm is said to be continuous if $* : [0,1] \times [0,1] \longrightarrow [0,1]$ is a continuous function.

Example 2.2 ([18])**.** Three basic continuous t-norms:

(1) the minimum t-norm $*_m : a *_m b = a \wedge b$;
(2) the product t-norm $*_p : a *_p b = ab$;
(3) the Łukasiewicz t-norm $*_L : a *_L b = 0 \vee (a + b - 1)$.

Lemma 2.3 ([7])**.** *Suppose that $*$ is a continuous t-norm.*

(1) *For any $r_1, r_2 \in (0,1]$, if $r_1 > r_2$, then there exists $r_3 \in (0,1)$ such that $r_1 * r_3 > r_2$.*
(2) *If $r \in [0,1)$, then there exists $s \in (r,1)$ such that $s * s > r$.*

Proof. (1) Let $r_1, r_2 \in (0,1]$ such that $r_1 > r_2$. Define a map $f : [0,1] \longrightarrow [0,1]$ by $f(x) = r_1 * x$. Since

$$r_1 = f(1) = f\left(\lim_{n\to\infty} 1 - \frac{1}{n}\right) = \lim_{n\to\infty} f\left(1 - \frac{1}{n}\right) > r_2,$$

there exists $r_3 \in \left\{1 - \frac{1}{n} : n \geq 2\right\}$ such that $f(r_3) > r_2$, i.e., $r_1 * r_3 > r_2$.

(2) Let $r \in [0,1)$ and define a map $g : [0,1] \longrightarrow [0,1]$ by $g(x) = x * x$. Since

$$1 = g(1) = g\left(\lim_{n\to\infty} 1 - \frac{1}{n}\right) = \lim_{n\to\infty} g\left(1 - \frac{1}{n}\right) = \lim_{n\to\infty} \left(1 - \frac{1}{n}\right) * \left(1 - \frac{1}{n}\right) > r,$$

there exists $s \in \left\{1 - \frac{1}{n} : n \geq 2\right\}$ such that $s * s > r$. □

Definition 2.4 ([15])**.** Let X be a nonempty set and let $*$ be a continuous t-norm. A fuzzy quasi-pseudometric on X is a pair $(M,*)$ (or just M) such that the map $M : X \times X \times (0,\infty) \longrightarrow (0,1]$ satisfying the following conditions:

(FM1) $M(x,y,t) > 0$ for all $x,y \in X$ and $t > 0$;
(FM2) $M(x,x,t) = 1$ for all $t > 0$;
(FM3) $M(x,y,r) * M(y,z,s) \leq M(x,z,r+s)$ for all $x,y,z \in X$ and $r,s > 0$;
(FM4) $M(x,y,\cdot) : (0,\infty) \longrightarrow (0,1]$ is continuous.

A fuzzy quasi-metric on X is a fuzzy quasi-pseudometric $(M,*)$ additionally satisfying the condition:

(FM2′) $M(x,y,t) = M(y,x,t) = 1$ for all $x,y \in X$ and $t > 0$ implies $x = y$.

In this case, the triple $(X,M,*)$ is called a fuzzy quasi-metric space.

Remark 2.5. The value of $M(x,y,t)$ is usually understood as the degree of certainty that the distance between x and y is less than t. It is easy to check that $M(x,y,\cdot)$ is nondecreasing for all $x,y \in X$ [15].

Remark 2.6. If a fuzzy quasi-metric M on X satisfies:

(FM5) $M(x,y,t) = M(y,x,t)$ for all $x,y \in X$ and $t > 0$,

then M is a fuzzy metric in the sense of George and Veeramani [7].

The conjugate of a fuzzy quasi-metric M is the fuzzy quasi-metric M^{-1} defined by $M^{-1}(x,y,t) = M(y,x,t)$ $(\forall x,y \in X,\ \forall t > 0)$. The map $M^s(x,y,t) = M(x,y,t) \wedge M^{-1}(x,y,t)$ $(\forall x,y \in X,\ \forall t > 0)$ is a fuzzy metric on X.

Let $(X, M, *)$ be a fuzzy quasi-pseudometric space. Then for $x \in X, \lambda \in (0,1), t > 0$, define the open and closed balls in X by the formulae, respectively,

$$B_M(x, \lambda, t) = \{y \in X : M(x, y, t) > 1 - \lambda\} \text{ and } B_M[x, \lambda, t] = \{y \in X : M(x, y, t) \geq 1 - \lambda\}.$$

It is easy to check that the following inclusions hold:

$$B_{M^s}(x, \lambda, t) \subseteq B_M(x, \lambda, t) \text{ and } B_{M^s}(x, \lambda, t) \subseteq B_{M^{-1}}(x, \lambda, t).$$

The similar inclusions are valid for the closed balls as well.

The topology τ_M of a fuzzy quasi-pseudometric $(X, M, *)$ can be defined through the neighborhood system $\{\mathcal{N}_M(x) : x \in X\}$:

$$\begin{aligned} V \in \mathcal{N}_M(x) &\Longleftrightarrow \exists \lambda \in (0,1), \exists t > 0 \text{ such that } B_M(x, \lambda, t) \subseteq V \\ &\Longleftrightarrow \exists \lambda \in (0,1), \exists t > 0 \text{ such that } B_M[x, \lambda, t] \subseteq V. \end{aligned}$$

Obviously, the family $\left\{B_M\left(x, \frac{1}{n}, \frac{1}{n}\right) : n \in \mathbb{N}\right\}$ is a base of the neighborhood $\mathcal{N}_M(x)$ for all $x \in X$. Hence, the topology τ_M is first countable. The topological notions corresponding to M will be prefixed by M- (e.g., M-open, M-closure, etc.). The ball $B_M(x, \lambda, t)$ is M-open, and the ball $B_M[x, \lambda, t]$ is indeed M^{-1}-closed but not M-closed [7, 15].

If a sequence $\{x_n\}$ converges to x with respect to τ_M, then $\{x_n\}$ is called M-convergent and denoted by $x_n \xrightarrow{M} x$. It can be characterized in the following way:

$$\begin{aligned} x_n \xrightarrow{M} x &\Longleftrightarrow \lim_{n\to\infty} M(x, x_n, t) = 1 \ (\forall t > 0) \\ &\Longleftrightarrow \forall \lambda \in (0,1), \forall t > 0,\ \exists n_0 \in \mathbb{N} \text{ such that } M(x, x_n, t) > 1 - \lambda \ (\forall n \geq n_0). \end{aligned}$$

Dually,

$$\begin{aligned} x_n \xrightarrow{M^{-1}} x &\Longleftrightarrow \lim_{n\to\infty} M(x_n, x, t) = 1 \ (\forall t > 0) \\ &\Longleftrightarrow \forall \lambda \in (0,1), \forall t > 0,\ \exists n_0 \in \mathbb{N} \text{ such that } M(x_n, x, t) > 1 - \lambda \ (\forall n \geq n_0). \end{aligned}$$

Example 2.7 ([7]). Let (X, d) be a quasi-metric space. Define a map $M_d : X \times X \times (0, +\infty) \longrightarrow [0,1]$ by

$$M_d(x, y, t) = \frac{t}{t + d(x, y)} \ (\forall x, y \in X, t > 0).$$

Then M_d is a fuzzy quasi-metric under both $*_p$ and $*_m$. Furthermore, $(X, M_d, *_p)$ is usually called the standard fuzzy T_0-quasi-metric spaces. Furthermore, it is easy to check that $(M_d)^{-1} = M_{d^{-1}}$ and $(M_d)^s = M_{d^s}$, and that the topology $\tau(d)$ generated by d coincides with that τ_{M_d} generated by M_d.

We list some topological properties for fuzzy quasi-pseudometric spaces, which will be used in the following sections.

Proposition 2.8. *Let $(X, M, *)$ be a fuzzy quasi-pseudometric space. Then the following statements hold.*

(1) *The topology τ_{M^s} is finer than the topologies τ_M and $\tau_{M^{-1}}$, which means that:*
  - *An M-open (resp., M-closed) set is M^s-open (resp., M^s-closed), and the similar statement holds for the topology $\tau_{M^{-1}}$;*
  - *A sequence $\{x_n\}$ in X is M^s-convergent to $x \in X$ if and only if it is both M-convergent and M^{-1}-convergent to x.*

(2) *If M is a fuzzy quasi-metric, then the topology τ_M is T_0.*
(3) *The topology τ_M is T_1 if and only if the condition $M(x, y, t) = 1 (\forall t > 0)$ implies $x = y$.*
(4) *If M is a fuzzy metric, then the topology τ_M is T_2.*

Proof. The proof of (1) is obvious, and that of (2–4) can see [30, Theorem 2.8]. □

3. Sequential completeness of fuzzy quasi-pseudometric spaces

In this section, we focus our eyes on the relations between the corresponding notions of different completeness of sequences in fuzzy quasi-pseudometric spaces. We also provide characterizations to certain sequential completeness by descending sequences of closed sets.

Definition 3.1. Let $(X, M, *)$ be a fuzzy quasi-pseudometric space. We say that a sequence $\{x_n\}$ in X is

(1) left (resp., right) M-Cauchy if for all $t > 0$ and $\lambda \in (0,1)$, there exist $x \in X$ and $n_0 \in \mathbb{N}$ such that $M(x, x_n, t) > 1-\lambda$ (resp., $M(x_n, x, t) > 1-\lambda$) for all $n \geq n_0$;
(2) M-Cauchy if for all $t > 0$ and $\lambda \in (0,1)$, there exists $n_0 \in \mathbb{N}$ such that $M(x_k, x_n, t) > 1-\lambda$ for all $n, k \geq n_0$;
(3) left (resp., right) K-Cauchy if for all $t > 0$ and $\lambda \in (0,1)$, there exists $n_0 \in \mathbb{N}$ such that $M(x_k, x_n, t) > 1-\lambda$ (resp., $M(x_n, x_k, t) > 1-\lambda$) for all $n \geq k \geq n_0$;
(4) weakly left (resp., right) M-Cauchy if for all $t > 0$ and $\lambda \in (0,1)$, there exists $n_0 \in \mathbb{N}$ such that $M(x_{n_0}, x_n, t) > 1-\lambda$ (resp., $M(x_n, x_{n_0}, t) > 1-\lambda$) for all $n \geq n_0$.

To emphasize the fuzzy quasi-pseudometric M, we sometimes say that a sequence is left (right) M-K-Cauchy in Definition 3.1 (3).

Remark 3.2. The names in Definition 3.1 (1) come from two aspects. On one hand, the definition of a d-Cauchy sequence in a quasi-pseudometric space (X, d) was first introduced in [21, Definition 3], which is related to a generalized Banach contraction principle. On the other hand, the distinction between left and right Cauchyness was made in [27, Definition 2] and those notions were used to establish several fixed point theorems. The letter K in the definition of a left K-Cauchy sequence derives from Kelly [17] who was the first to consider a notion of Cauchy sequences in quasi-pseudometric spaces.

Remark 3.3. Let $(X, M, *)$ be a fuzzy quasi-pseudometric space.

(1) The relations among the notions in Definition 3.1 are presented in the following way:

M-Cauchy $\Rightarrow$ left (right) K-Cauchy $\Rightarrow$ weakly left (right) K-Cauchy $\Rightarrow$ left (right) M-Cauchy.

None of the above implications is reversible, which will be shown by the examples below.
(2) A sequence is (weakly) left K-Cauchy with respect to M if and only if it is (weakly) right K-Cauchy with respect to M^{-1}.
(3) A sequence is M-Cauchy if and only if it is both left and right K-Cauch, since M-Cauchy is equivalent to that of M^s-Cauchy.
(4) An M- (resp., M^{-1}-)convergent sequence is left (resp., right) M-Cauchy, while the converse is false (see Example 3.5).

Inspired by the classical setting, the following example shows that a sequence can be left M-Cauchy, right M-Cauchy and convergent without being left or right K-Cauchy, and one another sequence is weakly left K-Cauchy without being left K-Cauchy.

Example 3.4. Let X be the closed unit interval $[0,1]$ and define a map $d : X \times X \longrightarrow [0, \infty)$ by

$$d(x,y) = \begin{cases} 0, & \text{if } x \leq y; \\ 1, & \text{if } x > y. \end{cases}$$

Then (X, d) is a quasi-pseudometric space [22, Example 1]. In the standard fuzzy quasi-pseudometric space $(X, M_d, *_p)$, consider a sequence $\{x_n\}$ in X given by:

$$x_n = \begin{cases} \frac{1}{2} + \frac{1}{2^n}, & \text{if } n \text{ is odd;} \\ \frac{1}{3} + \frac{1}{3^n}, & \text{if } n \text{ is even.} \end{cases}$$

Then $M_d(\frac{1}{3}, x_n, t) = 1$ ($\forall n \geq 1$, $\forall t > 0$). Hence $\{x_n\}$ is left M-Cauchy and M-convergent. Since $M(x_n, 1, t) = 1$ ($\forall n \geq 1$, $\forall t > 0$), we have that $\{x_n\}$ is also right M-Cauchy. But taking $\lambda = \frac{1}{2}$ and $t = 1$, we have $M(x_n, x_m, 1) = \frac{1}{2} = 1 - \lambda$ whenever n is odd and m is even. Thus, $\{x_n\}$ is not even weakly right or left K-Cauchy or M-Cauchy.

Observe another sequence

$$\{y_n\} = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots\right\}.$$

Then it is weakly left K-Cauchy but not left K-Cauchy, since $M_d(y_1, y_n, t) = 1$ ($\forall n \geq 1$, $\forall t > 0$), and $M_d(y_l, y_k, 1) = \frac{1}{2} = 1 - \lambda$ ($\forall k > l > 1$) for $\lambda = \frac{1}{2}$ and $t = 1$.

Further, the following example shows that the notions of left M-Cauchy sequece and right M-Cauchy sequence are distinct.

Example 3.5. Let X be the open unit interval $(0,1)$ and define a map $d : X \times X \longrightarrow [0, \infty)$ by

$$d(x,y) = \begin{cases} x - y, & \text{if } x \geq y; \\ 1, & \text{if } x < y. \end{cases}$$

Then (X, d) is a quasi-pseudometric space [22, Example 2]. In the standard fuzzy quasi-pseudometric space $(X, M_d, *_p)$, we consider the sequence $\{x_n\}$ in X given by $x_n = \frac{1}{n+1}(\forall n \in \mathbb{N})$. Let $k, l \in \mathbb{N}$ with $k > l$. Then for $0 < \frac{1}{k+1} < t$ and $0 < \frac{k-l}{1+k} < \lambda < 1$, we have

$$M_d(x_l, x_k, t) \geq M_d\left(x_l, x_k, \frac{1}{k+1}\right) = \frac{\frac{1}{k+1}}{\frac{1}{k+1} + \left(\frac{1}{l+1} - \frac{1}{k+1}\right)} = \frac{l+1}{k+1} = 1 - \frac{k-l}{1+k} > 1 - \lambda.$$

Hence $\{x_n\}$ is left K-Cauchy sequence and hence left M-Cauchy. However, $\{x_n\}$ is not right M-Cauchy, since for $t = 1$ and $\lambda = \frac{1}{2}$, $M(x_m, x, t) = M(x_m, x, 1) = \frac{1}{2} = 1 - \lambda$ for all $x \in (0,1)$ whenever m is enough large. Similarly, the sequence $\{y_n : n \in \mathbb{N}\}$ defined by $y_n = 1 - \frac{1}{n+1}$ is right M-Cauchy but not left M-Cauchy. We observe that $\{y_n\}$ is in fact right K-Cauchy but not M-Cauchy: If $l > k$, then for $0 < \frac{1}{l+1} < t$ and $0 < \frac{l-k}{1+l} < \lambda < 1$, we have

$$M_d(y_l, y_k, t) \geq M_d\left(y_l, y_k, \frac{1}{l+1}\right) = \frac{\frac{1}{l+1}}{\frac{1}{l+1} + \left(\frac{1}{k+1} - \frac{1}{l+1}\right)} = \frac{k+1}{l+1} = 1 - \frac{l-k}{1+l} > 1 - \lambda.$$

While, if $l < k$, then for $t = 1$ and $\lambda = \frac{1}{2}$, we have $M_d(y_l, y_k, t) = M_d(y_l, y_k, 1) = \frac{1}{2} = 1 - \lambda$.

The following proposition gives a simple but useful property of Cauchy sequence.

Proposition 3.6. *Let $(X, M, *)$ be a fuzzy quasi-pseudometric space and let $\{x_n\}$ be a left K-Cauchy sequence in X. Then the following statements hold.*

(1) *If $\{x_n\}$ has a subsequence M-converging to x, then $\{x_n\}$ is M-convergent to x.*
(2) *If $\{x_n\}$ has a subsequence M^{-1}-converging to x, then $\{x_n\}$ is M^{-1}-convergent to x.*
(3) *If $\{x_n\}$ has a subsequence M^s-converging to x, then $\{x_n\}$ is M^s-convergent to x.*

Proof. (1) Suppose that $\{x_n\}$ is left K-Cauchy and $\{x_{n_k} : k \in \mathbb{N}\}$ is a subsequence of $\{x_n\}$ such that $\lim_{k\to\infty} M(x, x_{n_k}, t) = 1$ for all $t > 0$. For $\lambda \in (0,1)$ and $t > 0$, choose $n_0 \in \mathbb{N}$ such that $M(x_m, x_n, t/2) > 1 - \lambda'$ whenever $n > m \geq n_0$, where $\lambda' \in (0,1)$ satisfies $(1 - \lambda') * (1 - \lambda') > 1 - \lambda$. Let $k_0 \in \mathbb{N}$ such that $n_{k_0} \geq n_0$ and $M(x, x_{n_k}, t/2) > 1 - \lambda'$ for all $k \geq k_0$. Then, for all $n \geq n_{k_0}$,

$$M(x, x_n, t) \geq M\left(x_{n_{k_0}}, x_n, \frac{t}{2}\right) * M\left(x, x_{n_{k_0}}, \frac{t}{2}\right) \geq (1 - \lambda') * (1 - \lambda') > 1 - \lambda.$$

(2) It is similar to (1).

(3) Suppose that there exists a subsequence $\{x_{n_k} : k \in \mathbb{N}\}$ of $\{x_n\}$ such that $x_{n_k} \xrightarrow{M^s} x$. Then by Proposition 2.8 (1), we have $x_{n_k} \xrightarrow{M} x$ and $x_{n_k} \xrightarrow{M^{-1}} x$. Hence, $x_n \xrightarrow{M} x$ and $x_n \xrightarrow{M^{-1}} x$ by (1) and (2). Appealing again to Proposition 2.8 (1), it follows that $x_n \xrightarrow{M^s} x$. □

Corresponding to the seven definitions of Cauchy sequence in a fuzzy quasi-pseudometric space, we have seven notions of completeness.

Definition 3.7. Let $(X, M, *)$ be a fuzzy quasi-pseudometric space. We call $(X, M, *)$
(1) sequentially M-complete if for every M-Cauchy sequence is M-convergent;
(2) sequentially left (resp., right) M-complete if for every left (resp., right) M-Cauchy sequence is M-convergent;
(3) strongly sequentially left (resp., right) K-complete if for every weakly left (resp., right) K-Cauchy sequence is M-convergent;
(4) sequentially left (resp., right) K-complete if for every left (resp., right) K-Cauchy sequence is M-convergent.

Remark 3.8. The implications between these completeness notions can be obtained by reversing the implications between the corresponding notions of Cauchy sequences by Remark 3.3(1), i.e.,

$$\text{sequentially } M\text{-complete} \Longrightarrow \text{strongly sequentially left } K\text{-complete} \Longrightarrow$$
$$\text{sequentially left } K\text{-complete} \Longrightarrow \text{sequentially left } M\text{-complete}.$$

The same implications hold for the corresponding notions of right completeness.

Due to the equivalence

$$\text{left } M\text{-Cauchy} \Longleftrightarrow \text{right } M^{-1}\text{-Cauchy},$$

we obtain nothing new by requiring that a left M-Cauchy sequence is M^{-1}-convergent. For instance, the M^{-1}-convergence of any left M-K-Cauchy sequence is equivalent to the right K-completeness of the space $(X, M^{-1}, *)$. However, it is noteworthy that left M-completeness and right M^{-1}-completeness do not coincide in general, due to the fact that right M^{-1}-completeness means that every left M-Cauchy sequence converges with respect to $\tau_{M^{-1}}$, while left M-completeness means the convergence with respect to τ_M.

The next example gives a fuzzy quasi-pseudometric space which is sequentially left M-complete but neither sequentially left nor right K-complete.

Example 3.9. Let $(X, M_d, *_p)$ be the standard fuzzy quasi-pseudometric space in Example 3.5. Then every M-Cauchy sequence in $(X, M_d, *_p)$ is eventually constant and hence convergent. However, the left K-Cauchy sequence $\left\{\frac{1}{n+1} : n \in \mathbb{N}\right\}$ is not convergent in $(X, M_d, *_p)$. Furthermore, the sequence $\left\{1 - \frac{1}{n+1} : n \in \mathbb{N}\right\}$ does not convergent in $(X, M_d, *_p)$ but is right K-Cauchy.

The next example distinguishes between sequential M-completeness and sequential left K-completeness.

Example 3.10. Defined a map $d : \mathbb{N} \times \mathbb{N} \longrightarrow [0, \infty)$ by:

$$d(m,n) = \begin{cases} 0, & \text{if } m = n; \\ \frac{1}{n}, & \text{if } m > n, m \text{ is even}, n \text{ is odd}; \\ 1, & \text{otherwise}. \end{cases}$$

Then $(\mathbb{N}, d)$ is a quasi-metric space [21, Example 3]. In the standard fuzzy metric space $(\mathbb{N}, M_d, *_p)$, since there is no nontrivial right K-Cauchy sequence, $(\mathbb{N}, M_d, *_p)$ is sequentially right K-complete.

However, $(\mathbb{N}, M_d, *_p)$ is not sequentially right M-complete, since the sequence $\{2,4,6,8,\cdots\}$ is right M-Cauchy but not convergent. For the conjugate fuzzy quasi-metric $(M_d)^{-1}$ of M_d on $\mathbb{N}$, $(\mathbb{N}, (M_d)^{-1}, *_p)$ is sequentially left K-complete but not sequentially left N-complete. We observe that the sequence above is left M^{-1}-Cauchy but not weakly left K-Cauchy in $(\mathbb{N}, (M_d)^{-1}, *_p)$.

Remark 3.8 points out that a strongly sequentially left K-complete fuzzy quasi-pseudometric space is sequentially left K-complete. We will further show that these two notions are actually equivalent.

Theorem 3.11. *For a fuzzy quasi-pseudometric space $(X, M, *)$,*

it is strongly sequentially left K-complete $\Longleftrightarrow$ it is sequentially left K-complete.

Proof. It suffices to shows that a sequentially left K-complete fuzzy quasi-pseudometric space is strongly sequentially left K-complete. Suppose that the space $(X, M, *)$ is sequentially left K-complete and let $\{x_n\}$ be a weakly left K-Cauchy sequence in X. We need to show that $\{x_n\}$ is M-convergent to some $x \in X$.

Let $\lambda \in (0,1)$ and let $n(1)$ be the smallest natural number such that

$$\forall n \geq n(1),\ M(x_{n(1)}, x_n, 1) > 1 - \lambda. \tag{3.1}$$

If $M(x_{n(1)}, x_n, t) = 1$ for all $n \geq n(1)$ and $t > 0$, then $x_n \xrightarrow{M} x_{n(1)}$. Suppose that there exist $m(1) > n(1)$ and $t_1 > 0$ such that $M(x_{n(1)}, x_{m(1)}, t_1) < 1$. Note $M(x_{n(1)}, x_{m(1)}, 1) > 1 - \lambda$ and let $k_2 \in \mathbb{N}$ such that

$$M\left(x_{n(1)}, x_{m(1)}, \frac{1}{k_2}\right) \leq 1 - \lambda, \tag{3.2}$$

and let $n(2)$ be the smallest natural number such that

$$\forall n \geq n(2),\ M\left(x_{n(2)}, x_n, \frac{1}{k_2}\right) > 1 - \lambda. \tag{3.3}$$

By the choice of $n(1)$, we have $n(2) \geq n(1)$ and by (3.2), $n(2) \neq n(1)$ so that $n(2) > n(1)$.

Again, if $M(x_{n(2)}, x_n, t) = 1$ for all $n \geq n(2)$ and $t > 0$, then $x_n \xrightarrow{M} x_{n(2)}$. If not, choose $m(2) > n(2)$ and $t_2 > 0$ such that $M(x_{n(2)}, x_{m(2)}, t_2) < 1$. Note $M\left(x_{n(2)}, x_{m(2)}, \frac{1}{k_2}\right) > 1 - \lambda$ and let $k_3 \in \mathbb{N}$ such that

$$M\left(x_{n(2)}, x_{m(2)}, \frac{1}{k_3}\right) \leq 1 - \lambda, \tag{3.4}$$

and take $n(3)$ to be the smallest natural number such that

$$\forall n \geq n(3),\ M\left(x_{n(3)}, x_n, \frac{1}{k_3}\right) > 1 - \lambda. \tag{3.5}$$

Continuing in this manner, we can get at some step i an element $x_{n(i)}$ such that $M(x_{n(i)}, x_n, t) = 1$ for all $n \geq n(i)$ and $t > \frac{1}{k_i} > 0$, which implies $x_n \xrightarrow{M} x_{n(i)}$. If such an $i \in \mathbb{N}$ does not exist, we find the sequences of natural numbers

$$1 = k_1 < k_2 < \cdots \text{ and } n(1) < n(2) < \cdots$$

such that

$$\forall n \geq n(i),\ M\left(x_{n(i)}, x_n, \frac{1}{k_i}\right) > 1 - \lambda. \tag{3.6}$$

It is easy check that the condition (3.6) implies that the sequence $\{x_{n(i)} : i \in \mathbb{N}\}$ is left K-Cauchy. Hence, by the sequentially left K-completeness of the space $(X, M, *)$, it is M-convergent to some $x \in X$. We want to show that the sequence $\{x_n\}$ is M-convergent to x. Since $\{x_{n(i)} : i \in \mathbb{N}\}$ is M-convergent to x, we have that for $t > 0$ and $\lambda \in (0,1)$, let $i_0 \in \mathbb{N}$ such that

$$\frac{1}{k_{i_0}} < \frac{t}{2} \text{ and } M\left(x, x_{n(i)}, \frac{t}{2}\right) > 1 - \lambda' \ (\forall i \geq i_0),$$

where $\lambda' \in (0,1)$ with $(1-\lambda') * (1-\lambda') > 1 - \lambda$. Then for every $n \geq n(i_0)$,

$$\begin{aligned} M(x, x_n, t) &\geq M\left(x, x_{n(i_0)}, \frac{t}{2}\right) * M\left(x_{n(i_0)}, x_n, \frac{t}{2}\right) \\ &\geq M\left(x, x_{n(i_0)}, \frac{t}{2}\right) * M\left(x_{n(i_0)}, x_n, \frac{1}{k_{i_0}}\right) \\ &\geq (1-\lambda') * (1-\lambda') \\ &> 1 - \lambda. \end{aligned}$$

Therefore, $\{x_n\}$ is M-convergent to x. □

In what follows, we will present Cantor-type characterizations of certain completeness in terms of descending sequences of special sets. For a subset A of a nonempty set X, denote

$$\beta(A) = \inf_{t>0} \inf_{x,y \in A} M(x, y, t).$$

Theorem 3.12. *A fuzzy quasi-pseudometric space $(X, M, *)$ is sequentially M-complete if and only if each decreasing sequence $F_1 \supseteq F_2 \supseteq \cdots$ of nonempty M-closed sets with $\beta(F_n) \to 1$ as $n \to \infty$ has a nonempty intersection, which is a singleton if M is a fuzzy quasi-metric.*

Proof. Necessity. Let $F_1 \supseteq F_2 \supseteq \cdots$ be a sequence of nonempty M-closed sets with $\beta(F_n) \to 1$ as $n \to \infty$. Let $x_n \in F_n$ for each $n \in \mathbb{N}$. For all $\lambda \in (0,1)$, we take $n_0 \in \mathbb{N}$ such that $\beta(F_n) > 1 - \lambda$ for $n \geq n_0$. Then $x_n, x_m \in F_{n_0}$ for all $n, m \geq n_0$. Hence, for all $\lambda \in (0,1)$ and $t > 0$, $M(x_n, x_m, t) > 1 - \lambda$ whenever $n, m \geq n_0$, which implies that $\{x_n\}$ is M-Cauchy. By the sequential M-completeness of $(X, M, *)$, we have that $\{x_n\}$ is M-convergent to some $x \in X$. Since for every $n \in \mathbb{N}$, $x_{n+k} \in F_{n+k} \subseteq F_n$ for all $k \in \mathbb{N}$, letting $k \to \infty$ and taking into account the closedness of the set F_n, it follows that $x \in F_n$ for all $n \in \mathbb{N}$, i.e., $x \in \bigcap_{n=1}^{\infty} F_n$. If y is any point in $\bigcap_{n=1}^{\infty} F_n$, then

$$\inf_{t>0} M(x, y, t) \geq \beta(F_n) \text{ and } \inf_{t>0} M(y, x, t) \geq \beta(F_n) \ (\forall n \in \mathbb{N}).$$

Thus $M(x, y, t) = M(y, x, t) = 1$ for all $t > 0$, so $y = x$ if M is a fuzzy quasi-metric.

Sufficiency. We shall prove it by contradiction. Suppose that there exists an M-Cauchy sequence $\{x_n\}$ in X which is not M-convergent. By Remark 3.3 (1), we have that $\{x_n\}$ is left K-Cauchy. By Proposition 3.6 (1), it is M-convergent if it contains an M-convergent subsequence. Thus, $\{x_n\}$ does not contain M-convergent subsequence. Let $F_n = \{x_k : k \geq n\}$. Then F_n is decreasing and nonempty M-closed for each $n \in \mathbb{N}$. Since $\{x_n\}$ is M-Cauchy, it follows that $\beta(F_n) \to 1$ as $n \to \infty$. Hence, there exists $x \in X$ such that $x \in \bigcap_{n=1}^{\infty} F_n$. Thus, $\inf_{t>0} M(x, x_n, t) \geq \beta(F_n)$ for all $n \in \mathbb{N}$, which implies $\lim_{n\to\infty} M(x, x_n, t) = 1$ for all $t > 0$, i.e., $x_n \xrightarrow{M} x$, a contradiction. Therefore, $(X, M, *)$ is sequentially M-complete. □

The following theorem gives a characterization of sequential right K-completeness using a different terminology.

Theorem 3.13. *A fuzzy quasi-pseudometric space $(X, M, *)$ is sequentially right K-complete if and only if each decreasing sequence $B_{M^{-1}}[x_1, \lambda_1, t_1] \supseteq B_{M^{-1}}[x_2, \lambda_2, t_2] \supseteq \cdots$ of closed balls with*

$$1 > \lambda_1 > \lambda_2 > \cdots > 0 \text{ and } \lim_{n\to\infty} t_n = 0$$

has nonempty intersection, which is a singleton if the topology τ_M is Hausdorff.

Proof. Necessity. Let $(X, M, *)$ be sequentially right K-complete and let $\{B_{M^{-1}}[x_n, \lambda_n, r_n] : n \in \mathbb{N}\}$ be a sequence of closed balls satisfying the requirements of this theorem. We show first that the sequence $\{x_n\}$ is right K-Cauchy. For all $\lambda \in (0,1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $0 < \lambda_n < \lambda < 1$ and $0 < t_n < t$ for all $n \geq n_0$. If $n_0 \leq n < m$, then $x_m \in B_{M^{-1}}[x_n, \lambda_n, t_n]$, which implies

$$M^{-1}(x_n, x_m, t) \geq M^{-1}(x_n, x_m, t_n) > 1 - \lambda_n > 1 - \lambda, \text{ i.e., } M(x_m, x_n, t) > 1 - \lambda.$$

Hence $\{x_n\}$ is right K-Cauchy. It follows that there exists $x \in X$ such that $x_n \xrightarrow{M} x$. For every $k \in \mathbb{N}$, $x_n \in B_{M^{-1}}[x_k, \lambda_k, t_k]$ for all $n \geq k$. Since the ball $B_{M^{-1}}[x_k, \lambda_k, t_k]$ is M-closed, it follow that $x = \lim\limits_{n\to\infty} x_n \in B_{M^{-1}}[x_k, \lambda_k, t_k]$, showing $x \in \bigcap_{k=1}^{\infty} B_{M^{-1}}[x_k, \lambda_k, t_k]$.

If $y \in \bigcap_{n=1}^{\infty} B_{M^{-1}}[x_n, \lambda_n, t_n]$, then for all $n \in \mathbb{N}$, $M^{-1}(x_n, y, t_n) > 1 - \lambda_n$. Hence,

$$M(y, x_n, t) \geq M(y, x_n, t_n) > 1 - \lambda_n > 1 - \lambda,$$

which implies $x_n \xrightarrow{M} y$. If the topology τ_M is Hausdorff, then $y = x$.

Sufficiency. Let $\{x_n\}$ be a right K-Cauchy sequence in $(X, M, *)$. Then for $\lambda_1 \in (0,1)$, there exists $n_1 \in \mathbb{N}$ such that $M(x_m, x_n, \frac{1}{2}) > 1 - \lambda_2$ for all $m > n \geq n_1$, where $\lambda_2 \in (0,1)$ such that $(1 - \lambda_2) * (1 - \lambda_2) > 1 - \lambda_1$. In particular,

$$M^{-1}\left(x_{n_1}, x_n, \frac{1}{2}\right) = M\left(x_n, x_{n_1}, \frac{1}{2}\right) > 1 - \lambda_2 \ (\forall n \geq n_1). \tag{3.7}$$

Consider the closed ball $B_{M^{-1}}[x_{n_1}, \lambda, 1]$.

Let now $n_2 > n_1$ such that $M\left(x_m, x_n, \frac{1}{2^2}\right) > 1 - \lambda_2$ for all $m > n \geq n_2$. In particular,

$$M^{-1}\left(x_{n_2}, x_n, \frac{1}{2^2}\right) = M\left(x_n, x_{n_2}, \frac{1}{2^2}\right) > 1 - \lambda_2 \ (\forall n \geq n_2). \tag{3.8}$$

We claim

$$B_{M^{-1}}[x_{n_2}, \lambda_2, 1/2] \subseteq B_{M^{-1}}[x_{n_1}, \lambda_1, 1].$$

Indeed, if $y \in B_{M^{-1}}[x_{n_2}, \lambda_2, 1/2]$, then $M^{-1}(x_{n_2}, y, 1/2) > 1 - \lambda_2$. Hence, we have

$$M^{-1}(x_{n_1}, y, 1) \geq M^{-1}\left(x_{n_1}, x_{n_2}, \frac{1}{2}\right) * M^{-1}\left(x_{n_2}, y, \frac{1}{2}\right) \geq (1 - \lambda_2) * (1 - \lambda_2) > 1 - \lambda_1.$$

Continuing in this manner, we obtain s sequence $n_1 < n_2 < \cdots$ such that

$$M^{-1}\left(x_{n_k}, x_n, \frac{1}{2^k}\right) = M\left(x_n, x_{n_k}, \frac{1}{2^k}\right) > 1 - \lambda_k \ (\forall n \geq n_k). \tag{3.9}$$

It follows that $M^{-1}\left(x_{n_k}, x_{n_{k+1}}, \frac{1}{2^k}\right) > 1 - \lambda_k$ and

$$B_{M^{-1}}[x_{n_{k+1}}, \lambda_{k+1}, 1/2^k] \subseteq B_{M^{-1}}[x_{n_k}, \lambda_k, 1/2^{(k-1)}] \ (\forall k \in \mathbb{N}).$$

Hence, there exists $x \in X$ such that

$$x \in \bigcap_{k=1}^{\infty} B_{M^{-1}}[x_{n_k}, \lambda_k, 1/2^{(k-1)}].$$

For all $t > \frac{1}{2^{(k-1)}} > 0$ and all $0 < \lambda_k < \lambda < 1$, as $k \to \infty$, we then have

$$M(x, x_{n_k}, t) \geq M\left(x, x_{n_k}, \frac{1}{2^{(k-1)}}\right) = M^{-1}\left(x_{n_k}, x, \frac{1}{2^{(k-1)}}\right) > 1 - \lambda_k > 1 - \lambda,$$

which implies $\lim\limits_{k\to\infty} x_{n_k} = x$. By Proposition 3.6 (2), we can obtain that the right K-Cauchy sequence $\{x_n\}$ is M-convergent to x. $\square$

4. Completeness of filters and of nets

In this section, we shall examine the relations between completeness of sequences, of nets and of filters in fuzzy quasi-metric spaces. For some notions of completeness they can keep coincidence, but they can be different from others.

A filter on a set X is a nonempty family $\mathcal{F}$ of the subsets of X satisfying the following conditions:

(F0) $\varnothing \notin \mathcal{F}$;
(F1) if $F \subseteq G$ and $F \in \mathcal{F}$, then $G \in \mathcal{F}$;
(F2) if $F, G \in \mathcal{F}$, then $F \cap G \in \mathcal{F}$.

A base of a filter $\mathcal{F}$ is a subset $\mathcal{B}$ of $\mathcal{F}$ such that every $F \in \mathcal{F}$ contains a $B \in \mathcal{B}$.

A nonempty family $\mathcal{B}$ of nonempty subsets of X is called a filter base provided that

$$\text{(BF)} \qquad \forall B_1, B_2 \in \mathcal{B},\ \exists B \in \mathcal{B},\ B \subseteq B_1 \cap B_2.$$

Furthermore, a filter base generates a filter $\mathcal{F}_{\mathcal{B}}$ by the following way:

$$\mathcal{F}_{\mathcal{B}} = \{U \subseteq X : \exists B \in \mathcal{B},\ B \subseteq U\}.$$

Let D be a set. A pair $(D, \le)$ is a called a partially ordered set (in short poset) if $\le$ is a partial order on D, i.e., it is reflexive, transitive and antisymmetric. A poset $(D, \le)$ is called directed provided that for all $i_1, i_2 \in D$, there exists $j \in D$ such that $i_1 \le j$ and $i_2 \le j$.

Throughout this paper, we suppose that in the definition a directed set $(D, \le)$, the relation $\le$ is always supposed to be a partial order.

A net in a set X is a map $\phi : D \longrightarrow X$, where $(D, \le)$ is a directed set. The alternative notation $\{x_i\}_{i \in D}$, where $x_i = \phi(i)$ $(\forall i \in D)$, is also used.

Definition 4.1. Let $(X, M, *)$ be a fuzzy quasi-pseudometric space. We say that

(1) a filter $\mathcal{F}$ in $(X, M, *)$ is left K-Cauchy if for all $\lambda \in (0,1)$ and $t > 0$, there exists $F \in \mathcal{F}$ such that $B_M(x, \lambda, t) \in \mathcal{F}$ for all $x \in F$;
(2) a net in $(X, M, *)$ is left K-Cauchy if for all $\lambda \in (0,1)$ and $t > 0$, there exists $i_0 \in D$ such that $M(x_i, x_j, t) > 1 - \lambda$ for all $i, j \in D$ with $i_0 \le i \le j$.

Theorem 4.2. *Let $(X, M, *)$ be a fuzzy quasi-pseudometric space. Then the following statements are equivalent:*

(1) *the space $(X, M, *)$ is sequentially left K-complete;*
(2) *every left K-Cauchy filter in X is M-convergent;*
(3) *every left K-Cauchy net in X is M-convergent.*

Proof. (1) $\Longrightarrow$ (2). If $\mathcal{F}$ is a left K-Cauchy filter in $(X, M, *)$, then for all $n \in \mathbb{N}$ and $\lambda \in (0,1)$, there exists $F_n \in \mathcal{F}$ such that $B_M(x, \lambda, 1/2^n) \in \mathcal{F}$ for all $x \in F_n$. Pick

$$x_1 \in F_1 \text{ and } x_n \in F_n \cap \left(\bigcap_{k=1}^{n-1} B_M\left(x_k, \lambda, 1/2^k\right) \right) \ (\forall n > 1).$$

We claim that the sequence $\{x_n\}$ is left K-Cauchy. In fact, for given $t > 0$, let $k \in \mathbb{N}$ such that $\frac{1}{2^k} < t$. Then, by the choice of x_n, for $k \le m < n$, $x_n \in B_M\left(x_m, \lambda, 1/2^m\right)$, which implies

$$M\left(x_m, x_n, t\right) \ge M\left(x_m, x_n, \frac{1}{2^k}\right) \ge M\left(x_m, x_n, \frac{1}{2^m}\right) > 1 - \lambda.$$

Since $(X, M, *)$ is sequentially left K-Cauchy complete, there exists $x \in X$ such that

$$x_n \xrightarrow{M} x \Longleftrightarrow \lim_{n \to \infty} M(x, x_n, t) \to 1 \ (\forall t > 0). \tag{4.1}$$

We want to show that $x = \lim \mathcal{F}$, or, equivalently, $B_M\left(x, \lambda, 1/2^k\right) \in \mathcal{F}$ for all $k \in \mathbb{N}$ and $\lambda \in (0,1)$. Given $k \in \mathbb{N}$ and $\lambda \in (0,1)$, by (4.1), there exists $n > k$ such that

$$M\left(x, x_n, \frac{1}{2^{k+1}}\right) > 1 - \lambda',$$

where $\lambda' \in (0,1)$ with $(1-\lambda') * (1-\lambda') > 1-\lambda$. Let $y \in B_M(x_n, \lambda', 1/2^n)$. Then $M\left(x_n, y, \frac{1}{2^n}\right) > 1 - \lambda'$, which implies

$$M\left(x_n, y, \frac{1}{2^{k+1}}\right) \geq M\left(x_n, y, \frac{1}{2^n}\right) > 1 - \lambda'.$$

Thus,

$$M\left(x, y, \frac{1}{2^k}\right) \geq M\left(x, x_n, \frac{1}{2^{k+1}}\right) * M\left(x_n, y, \frac{1}{2^{k+1}}\right) \geq (1-\lambda') * (1-\lambda') > 1-\lambda.$$

This shows that $B_M(x_n, \lambda', 1/2^n) \subseteq B_M(x, \lambda, 1/2^k)$. Since $B_M(x_n, \lambda', 1/2^n) \in \mathcal{F}$ for $x_n \in F_n$, we have $B_M(x, \lambda, 1/2^k) \in \mathcal{F}$.

(2) $\Longrightarrow$ (3). If $\{x_i\}_{i \in D}$ is a left K-Cauchy net in X. Put $F_i = \{x_j : j \geq i\}$ $(\forall i \in D)$. By the definitions of nets and filter bases, we have that $\mathcal{F} = \{F_i : i \in D\}$ is a filter base on X. Let $\lambda \in (0,1)$ and $t > 0$ and fix $i_0 \in \mathbb{N}$ with $i_0 \leq i \leq j$. Then $M(x_i, x_j, t) > 1 - \lambda$. Hence $F_i \subseteq B_M(x_i, \lambda, t)$, and so $B_M(x_i, \lambda, t) \in \mathcal{F}$ for every $x_i \in F_{i_0}$. Thus, $\mathcal{F}$ is a left K-Cauchy filter. By (2) there exists $x \in X$ such that $x = \lim \mathcal{F}$. Using the definition of $\mathcal{F}$, it is easy to check that $x = \lim_i x_i$: For all $\lambda \in (0,1)$ and $t > 0$, $B_M(x, \lambda, t) \in \mathcal{F}$, and thus there exists $i_0 \in D$ such that $F_{i_0} \subseteq B_M(x, \lambda, t)$, implying $M(x, x_i, t) > 1 - \lambda$ for every $i \geq i_0$.

(3) It is obvious by the definition. □

However, some results similar to left completeness are valid for right completeness via supplying some conditions.

Definition 4.3. Let $(X, M, *)$ be a fuzzy quasi-pseudometric space. We say that $(X, M, *)$ is

(1) point-symmetric if $\tau_M \subseteq \tau_{M^{-1}}$;
(2) locally symmetric if for all $x \in X$, $\lambda \in (0,1)$ and $t > 0$, there exist $\mu \in (0,1)$ and $r > 0$ such that

$$\bigcup \{B_{M^{-1}}(y, \mu, r) : y \in B_M(x, \mu, r)\} \subseteq B_M(x, \lambda, t)$$

or, equivalently, if

$$\forall z \in X,\ \exists y \in X,\ M(x, y, r) > 1 - \mu \text{ and } M(z, y, r) > 1 - \mu \Longrightarrow M(x, z, t) > 1 - \lambda. \quad (4.2)$$

We first give some basic properties concerning point-symmetry and local symmetry.

Proposition 4.4. *Let $(X, M, *)$ be a fuzzy quasi-pseudometric space. Then the following statements hold.*

(1) *If $(X, M, *)$ is locally symmetric, then it is point-symmetric.*
(2) *If $(X, M, *)$ is point-symmetric if and only if for every sequence $\{x_n\}$ in X,*

$$x_n \xrightarrow{M^{-1}} x \Longrightarrow x_n \xrightarrow{M} x.$$

Proof. (1) If $U \subseteq X$ is open with respect to τ_M, then for every $x \in U$, there exist $\lambda \in (0,1)$ and $t > 0$ such that $B_M(x, \lambda, t) \subseteq U$. Choosing $\mu \in (0,1)$ and $r > 0$ by the local symmetry of $(X, M, *)$, it follows that the $\tau_{M^{-1}}$-open set $\bigcup \{B_{M^{-1}}(y, \mu, r) : y \in B_M(x, \mu, r)\}$ contains x and is contained in U. Thus U is $\tau_{M^{-1}}$-open.

(2) The equivalence follows from the fact that F is a closed subset of $(X, M, *)$ if and only if for every sequence $\{x_n\}$ in F, if $x_n \xrightarrow{M} x$, then $x \in F$ (see [2, Theorem 3.7]). □

Proposition 4.5. *Let $(X, M, *)$ be a fuzzy quasi-metric space. If $(X, M, *)$ is strongly sequentially right K-complete and the induced topology τ_M is T_1, then it is point-symmetric.*

Proof. Let $\{x_n\}$ be a sequence in X which is M^{-1}-convergent to $y \in X$. Define a sequence $\{y_n : n \in \mathbb{N}\}$ by $y_{2k-1} = y$ and $y_{2k} = x_k$ for $k \in \mathbb{N}$. Then $\{y_n : n \in \mathbb{N}\}$ is clearly weakly right K-Cauchy. By the strong sequential right K-completeness of $(X, M, *)$, it follows that $\{y_n : n \in \mathbb{N}\}$ is M-convergent to some point $z \in X$. Since $M(z, y, t) = \lim_{k\to\infty} M(z, y_{2k-1}, t) = 1$ for all $t > 0$, we have $M(z, y, t) = 1$ for all $t > 0$, which implies $z = y$ by Proposition 2.8(3). Since

$$\lim_{k\to\infty} M(y, x_k, t) = \lim_{k\to\infty} M(z, y_{2k}, t) = 1 \ (\forall t > 0),$$

we get $x_n \xrightarrow{M} y$. By Proposition 4.4 (2), we conclude that $(X, M, *)$ is point-symmetric. □

We have seen in Theorem 3.11 that strong sequential left K-completeness and sequential left K-completeness are equivalent notions for every fuzzy quasi-pseudometric space. The following example illustrates that this result is not true for right completeness even with the point-symmetry, i.e., there exists a point-symmetrically sequentially right K-complete fuzzy quasi-metric space which is not strongly sequentially right K-complete.

Example 4.6. Let $X = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ and define a map $d : X \times X \longrightarrow [0, \infty)$ by: $d(x, x) = 0$ for all $x \in X$ and

$$d(x,y) = \begin{cases} \frac{1}{2k} & \text{if } x = 0, y = \frac{1}{2k}; \\ \frac{1}{2k} - \frac{1}{n} & \text{if } x = \frac{1}{n}, y = \frac{1}{2k} \text{ and } n > 2k; \\ 1 & \text{otherwise.} \end{cases} \quad (\forall x \neq y).$$

Then (X, d) is a quasi-pseudometric space (see [1]). Consider the standard fuzzy quasi-pseudo metric space $(X, M_d, *_p)$. Then $\tau_{(M_d)^{-1}}$ is the discrete topology (since it coincides with $\tau_{d^{-1}}$), so that $\tau_{M_d} \subseteq \tau_{(M_d)^{-1}}$, that is the space $(X, M_d, *_p)$ is point-symmetric. A right K-Cauchy sequence in $(X, M_d, *_p)$ is either eventually constant or a subsequence of $\left\{\frac{1}{2k} : k \in \mathbb{N}\right\}$, denoted by $x_i = \frac{1}{2k_i}$ $(i \in \mathbb{N})$. In this case,

$$\lim_{i\to\infty} M_d(0, x_i, t) = \lim_{i\to\infty} \frac{t}{t + \frac{1}{2k_i}} = 1 \ (\forall t > 0).$$

Hence, $(X, M_d, *_p)$ is sequentially right K-complete. But the sequence $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ is weakly right K-Cauchy without being M-convergent. Hence, $(X, M_d, *_p)$ is not strongly sequentially right K-complete.

Further, the right version of Theorem 3.11 can be obtained by supplying certain prerequisite.

Theorem 4.7. *For a locally symmetric fuzzy quasi-pseudometric space $(X, M, *)$,*

it strongly sequentially right K-complete $\Longleftrightarrow$ it is sequentially right K-complete.

Proof. It suffices to prove that the sequential right K-completeness of $(X, M, *)$ implies strong sequential right K-completeness. Let $\{x_n\}$ be a weakly right K-Cauchy sequence in X. We need to show that $\{x_n\}$ is M-convergent to some $x \in X$.

For all $\lambda \in (0, 1)$, let n_1 be the first natural number such that

$$\forall n \geq n_1, \ M\left(x_n, x_{n_1}, \frac{1}{2}\right) > 1 - \lambda. \tag{4.3}$$

If $M(x_n, x_{n_1}, t) = 1$ for all $n \geq n_1$ and $t > 0$, then $x_n \xrightarrow{M^{-1}} x_{n_1}$. Since $(X, M, *)$ is locally symmetric, it follows from Proposition 4.4 (2) that $x_n \xrightarrow{M} x_{n_1}$. Otherwise, let m_1 be the first natural number greater than n_1 such that $M(x_{m_1}, x_{n_1}, t_1) < 1$ for some $t_1 > 0$, and let $k_2 \in \mathbb{N}$ such that

$$M\left(x_{m_1}, x_{n_1}, \frac{1}{2^{k_2-1}}\right) > 1-\lambda, \tag{4.4}$$

Since $M(x_{m_1}, x_{n_1}, 1/2) > 1-\lambda$, we have $k_2 > 1$. Let n_2 be the first natural number such that

$$\forall n \geq n_2,\ M\left(x_n, x_{n_2}, \frac{1}{2^{k_2}}\right) > 1-\lambda. \tag{4.5}$$

If $n_2 < n_1$, then, by the definition of the number n_1, there exists $m \in \mathbb{N}$ with $n_2 < m < n_1$ such that $M(x_m, x_{n_2}, 1/2) \leq 1-\lambda$, resulting in the contradiction

$$1-\lambda \geq M\left(x_m, x_{n_2}, \frac{1}{2}\right) \geq M\left(x_m, x_{n_2}, \frac{1}{2^{k_2}}\right) > 1-\lambda.$$

If $n_1 \leq n_2 < m_1$, then $M(x_{n_2}, x_{n_1}, t) = 1$ for all $t > 0$. Thus

$$\begin{aligned} M\left(x_{m_1}, x_{n_1}, \frac{1}{2^{k_2}}\right) &= \lim_{t_1+t_2=\frac{1}{2^{k_2}}} M\left(x_{m_1}, x_{n_1}, t_1+t_2\right) \\ &\geq \lim_{t_1+t_2=\frac{1}{2^{k_2}}} M\left(x_{m_1}, x_{n_2}, t_1\right) * M\left(x_{n_2}, x_{n_1}, t_2\right) \\ &= \lim_{t_1+t_2=\frac{1}{2^{k_2}}} M\left(x_{m_1}, x_{n_2}, t_1\right) * 1 \\ &= M\left(x_{m_1}, x_{n_2}, \frac{1}{2^{k_2}}\right), \end{aligned}$$

a contraction, again. Consequently, $n_2 \geq m_1 > n_1$ and $M(x_{n_2}, x_{n_1}, 1/2) > 1-\lambda$ for all $\lambda \in (0,1)$.

Continuing in this way, we can get at some step i an element x_{n_i} such that $M(x_n, x_{n_i}, t) = 1$ for all $n \geq n_i$ and $t > 0$, implying $x_n \xrightarrow{M^{-1}} x_{n_i}$. If such an $i \in \mathbb{N}$ does not exist, there exist the increasing sequences of natural numbers

$$n_1 < n_2 < \cdots \text{ and } k_1 < k_2 < \cdots$$

such that

$$\forall i \in \mathbb{N},\ \forall n \geq n_i,\ M\left(x_n, x_{n_i}, \frac{1}{2^{k_i}}\right) > 1-\lambda,$$

which implies

$$\forall i \in \mathbb{N},\ M\left(x_{n_{i+1}}, x_{n_i}, \frac{1}{2^{k_i}}\right) > 1-\lambda. \tag{4.6}$$

It is easy check that the condition (4.6) implies that the sequence $\{x_{n_i} : i \in \mathbb{N}\}$ is right K-Cauchy. By the sequentially right K-completeness of $(X, M, *)$, there exists $x \in X$ such that $x_{n_i} \xrightarrow{M} x$, i.e., $\lim_{i\to\infty} M(x, x_{n_i}, t) = 1$ for all $t > 0$.

Finally, we show that the sequence $\{x_n\}$ is M-convergent to x. For $\lambda \in (0,1)$ and $t > 0$, choose $\mu \in (0,1)$ and $r > 0$ according to (4.2).

Let $i_0 \in \mathbb{N}$ such that

$$\forall i \geq i_0,\ M(x, x_{n_i}, r) > 1-\mu,$$

and let $j \geq i_0$ such that $\frac{1}{2^{k_j}} < r$. Then for $n > n_j$,

$$M(x, x_{n_j}, r) > 1-\mu \text{ and } M(x_n, x_{n_j}, r) \geq M\left(x_n, x_{n_j}, \frac{1}{2^{k_j}}\right) > 1-\mu.$$

Hence $M(x, x_n, t) > 1-\lambda$ for all $n > n_j$. Therefore, $\{x_n\}$ is M-convergent to x. □

Definition 4.8. Let $(X, M, *)$ be a fuzzy quasi-pseudometric space. We say that

(1) a filter $\mathcal{F}$ in X is right K-Cauchy if for all $\lambda \in (0,1)$ and $t > 0$, there exists $F \in \mathcal{F}$ such that $B_{M^{-1}}(x, \lambda, t) \in \mathcal{F}$ for all $x \in F$;

(2) a net in X is right K-Cauchy if for all $\lambda \in (0,1)$ and $t > 0$, there exists $i_0 \in D$ such that $M(x_j, x_i, t) > 1 - \lambda$ for all $i, j \in D$ with $i_0 \le i \le j$.
(3) the space $(X, M, *)$ is right K-complete of filter if every right K-Cauchy filter in X is M-convergent to some $x \in X$;

Lemma 4.9. *Let $(X, M, *)$ be a fuzzy quasi-pseudometric space. If $(X, M, *)$ is right K-complete of filters, then every right K-Cauchy net in X is M-convergent. In particular, every right K-complete of filters fuzzy quasi-pseudometric space is sequentially right K-complete.*

Proof. If $\{x_i\}_{i \in D}$ is right K-Cauchy net in X, then $F_i = \{x_j : j \in D, i \le j\}$ $(i \in D)$ is the base of a filter $\mathcal{F}$ in X. For $\lambda \in (0,1)$ and $t > 0$, let $i_0 \in D$ such that $M(x_j, x_i, t) > 1 - \lambda$, i.e., $M^{-1}(x_i, x_j, t) > 1 - \lambda$, for all $i, j \in D$ with $i_0 \le i \le j$. Then for every $n \ge n_0$, $F_n \subseteq B_{M^{-1}}(x_n, \lambda, t)$, implying $B_{M^{-1}}(x, \lambda, t) \in \mathcal{F}$ for every $i \ge i_0$. Hence, the filter $\mathcal{F}$ is right K-Cauchy. Since $(X, M, *)$ is sequentially right K-complete, we have that $\mathcal{F}$ is convergent to some $x \in X$. It is easy check the net $\{x_i\}_{i \in D}$ is M-convergent to x: For all $\lambda \in (0,1)$ and $t > 0$, $B_{M^{-1}}(x, \lambda, t) \in \mathcal{F}$, and so there exists $i_0 \in D$ such that $F_{i_0} \subseteq B_{M^{-1}}(x, \lambda, t)$, implying $M^{-1}(x, x_i, t) = M(x_i, x, t) > 1 - \lambda$ for all $i \ge i_0$. □

Inspired by Davis [5], we will propose the following definition on fuzzy quasi-pseudometric spaces, which will be useful to obtain an analog of Theorem 4.2 for right K-completeness.

Definition 4.10. Let $(X, M, *)$ be a fuzzy quasi-pseudometric space. We call $(X, M, *)$ R_1 provided that if for all $x, y \in X$, $\mathrm{cl}_M\{x\} \ne \mathrm{cl}_M\{y\}$, then there exist two disjoint M-open sets U, V such that $x \in U$ and $y \in V$.

Theorem 4.11. *For an R_1 fuzzy quasi-pseudometric space,*

it is right K-complete of filters $\Longleftrightarrow$ it is sequentially right K-complete.

Proof. By Lemma 4.9, we only need to prove that a sequentially right K-completeness implies right K-completeness by filters. Let $\mathcal{F}$ be a right K-Cauchy filter on X. Then for each $n \in \mathbb{N}$ and $\lambda \in (0,1)$, there exists $F_n \in \mathcal{F}$ such that $B_{M^{-1}}(x, \lambda, 1/2^n)$ for all $x \in F_n$. Let $x_1 \in F_1$ and

$$x_n \in F_n \cap \bigcap_{k=1}^{n-1} B_{M^{-1}}(x_k, \lambda, 1/2^k) \ (\forall n > 1).$$

It follows that the so-constructed sequence $\{x_n\}$ is right K-Cauchy, so it is M-convergent to some $x \in X$. We shall show that the filter $\mathcal{F}$ converges with respect to τ_M to x. Supposing the contrary, there exists $m \in \mathbb{N}$ such that $B_M(x, \lambda, 1/2^m) \notin \mathcal{F}$. We construct a new sequence $\{y_n : n \in \mathbb{N}\}$ in the following way: Take

$$y_1 \in F_1 \cap B_{M^{-1}}(x_1, \lambda, 1/2) \setminus B_M(x, \lambda, 1/2^m)$$

and

$$y_n \in F_n \cap \ B_{M^{-1}}(x_n, \lambda, 1/2^n) \cap \bigcap_{k=1}^{n-1} B_{M^{-1}}(y_k, \lambda, 1/2^k) \setminus B_M(x, \lambda, 1/2^m) \ (\forall n > 1).$$

Since $y_n \in \bigcap_{k=1}^{n-1} B_{M^{-1}}(y_k, \lambda, 1/2^k)$ $(\forall n > 1)$ implies that $\{y_n\}$ is also a right K-Cauchy sequence, so it is M-convergent to some $y \in X$.

If $\mathrm{cl}_M\{x\} = \mathrm{cl}_M\{y\}$, then $x \in \mathrm{cl}_M\{y\}$, which implies $M(x, y, t) = 1$ for all $t > 0$. Since

$$\begin{aligned} M(x, y_n, t) &= \lim_{t_1+t_2=t} M(x, y_n, t_1 + t_2) \\ &\ge \lim_{t_1+t_2=t} M(x, y, t_1) * M(y, y_n, t_2) \\ &= \lim_{t_1+t_2=t} 1 * M(y, y_n, t_2) \\ &= M(y, y_n, t) \end{aligned}$$

and $y_n \xrightarrow{M} y$, it follows that $\lim_{n\to\infty} M(x, y_n, t) \to 1$ for all $t > 0$. Thus $y_n \xrightarrow{M} x$, which contradicts with $y_n \notin B_M(x, \lambda, 1/2^m)$ for all $n \in \mathbb{N}$. Hence $\mathrm{cl}_M\{x\} \neq \mathrm{cl}_M\{y\}$. By Condition R_1, there exist $\lambda \in (0, 1)$ and $t > 0$ such that $B_M(x, \lambda, t) \cap B_M(y, \lambda, t) = \varnothing$. Since $y_n \in B_{M^{-1}}(x_n, \lambda, 1/2^n)$, it follows that for all $t > \frac{1}{2^n} > 0$,

$$M(y_n, x_n, t) \geq M\left(y_n, x_n, \frac{1}{2^n}\right) \to 1, \text{ as } n \to \infty.$$

Then for all $t > 0$,

$$M(y, x_n, t) \geq M\left(y, y_n, \frac{t}{2}\right) * M\left(y_n, x_n, \frac{t}{2}\right) \to 1, \text{ as } n \to \infty.$$

Consequently, $x_n \xrightarrow{M} x$ and $x_n \xrightarrow{M} y$, implying that $x_n \in B_M(x, \lambda, t) \cap B_M(y, \lambda, t)$ for sufficiently large n, a contradiction. □

In the sequel, we are interested in the equivalence between right K-completeness of sequences and of nets. In ordinary quasi-metric spaces, Stoltenberg has shown these notions are not equivalent in general [26, Example 2.4]. To achieve the effective goal, accordingly, we shall introduce a kind of more general right Cauchy nets in fuzzy quasi-metric spaces, which follows from the ideas of Stoltenberg [26] and Gregori and Ferrer [9]. Let us begin with some basic notions and results.

Let $(P, \leq)$ be a poset. An element $j \in P$ is called maximal if there is no $i \in P \setminus \{j\}$ with $j \leq i$, or, equivalently, $j \leq i$ implies $i = j$ for all $i \in P$;

Proposition 4.12 ([4]). *Let $(P, \leq)$ be a poset. Then the following statements hold.*

(1) *Every maximal element j of P is a maximum for P, i.e., $i \leq j$ for all $i \in P$.*
(2) *If j is a maximal element and $j' \in P$ satisfies $j \leq j'$, then j' is also a maximal element.*
(3) (Uniqueness of the maximal element) *If j is a maximal element, then $j' = j$ for any maximal element j' of P.*

Next, we analyze the relations between maximal elements in posets and the convergence of nets in fuzzy quasi-metric spaces.

Proposition 4.13. *Let $(X, M, *)$ be a fuzzy quasi-metric space, let $(D, \leq)$ be a directed set and let $\{x_i\}_{i\in D}$ be a net in X.*

(1) *If $(D, \leq)$ has a maximal element j, then the net $\{x_i\}_{i\in D}$ is M-convergent to x_j.*
(2) *If the net $\{x_i\}_{i\in D}$ is M-convergent to $x \in X$, then $M(x, x_j, t) = 1$ for all $t > 0$ and all maximal element j in D. If the topology τ_M is T_1, then $x_j = x$.*
(3) *If the net $\{x_i\}_{i\in D}$ is M-convergent to x_j and $x_{j'}$, where j, j' are maximal elements of D, then $x_j = x_{j'}$.*
(4) *If D has maximal elements and there exists $x \in X$ such that $x_j = x$ for every maximal element in D, then the net $\{x_i\}_{i\in D}$ is M-convergent to x.*

Proof. (1) For all $\lambda \in (0, 1)$ and $t > 0$, take $i_0 = j$. Then $i \geq j$ implies $i = j$, and so

$$M(x_j, x_i, t) = M(x_j, x_j, t) = 1 > 1 - \lambda.$$

This shows that $\{x_i\}_{i\in D}$ is M-convergent to x_j.

(2) Since the net $\{x_i\}_{i\in D}$ is M-convergent to $x \in X$, we have that for all $\lambda \in (0, 1)$ and $t > 0$, there exists $i_0 \in D$ such that $M(x, x_i, t) > 1 - \lambda$ for all $i \geq i_0$. By Proposition 4.12(2), we have $j \geq i_0$ for every maximal j, and so $M(x, x_j, t) > 1 - \lambda$ for all $\lambda \in (0, 1)$, which implies $M(x, x_j, t) = 1$ for all $t > 0$. If the topology τ_M is T_1, then $x_j = x$ by Proposition 2.8(3).

(3) By (2), we have $M(x_j, x_{j'}, t) = 1$ and $M(x_{j'}, x_j, t) = 1$ for all $t > 0$. Hence $x_j = x_{j'}$.

(4) Let $x \in X$ such that $x_j = x$ for every maximal element j in D and let j be a fixed maximal element in D. For all $\lambda \in (0, 1)$ and $t > 0$, put $i_0 = j$. Then by Proposition

4.12(3), any $i \in D$ such that $i \geq j$ is also a maximal element, and thus $x_i = x$. Hence $M(x, x_i, t) = 1 > 1 - \lambda$. This shows that the net $\{x_i\}_{i \in D}$ is M-convergent to x. □

Definition 4.14. Let $(X, M, *)$ be a fuzzy quasi-metric space. We say that a net $\{x_i\}_{i \in D}$ in $(X, M, *)$ is M-GF-Cauchy if the net $\{x_i\}_{i \in D}$ satisfies the following condition:

$$\forall \lambda \in (0,1), \forall t > 0,\ \exists i_0 \in D \text{ such that } M(x_i, x_j, t) > 1 - \lambda \ (\forall i, j \geq i_0 \text{ with } i \nleq j). \tag{4.7}$$

Further, we say that a fuzzy quasi-metric space $(X, M, *)$ is M-GF-complete if every M-GF-Cauchy net is M-convergent.

Theorem 4.15. *For a fuzzy quasi-metric space $(X, M, *)$ with the topology τ_M being T_1,*

it is sequentially right K-complete $\Longleftrightarrow$ every M-GF-Cauchy net is M-convergent.

Proof. It suffices to prove that the left-to-right implication holds. Suppose that a net $\{x_i\}_{i \in D}$ in X is M-GF-Cauchy. For all $\lambda \in (0,1)$, let $\{i_k : i_k \leq i_{k+1}, k \in \mathbb{N}\}$ be a sequence of indices in D such that $M(x_i, x_j, 1/2^k) > 1 - \lambda$ for all $i, j \geq i_k$ with $i \nleq j$. We show that we can further suppose $i_{k+1} \nleq i_k$.

Indeed, the fact that D has no maximal elements implies that for every $i \in D$, there exists $i' \in D$ such that

$$i \leq i' \text{ and } i' \nleq i. \tag{4.8}$$

Let $i'_1 \in D$ such that (4.7) holds for $t = 1/2$, i.e., $M(x_i, x_j, 1/2) > 1 - \lambda$ for all $i, j \geq i'_1$ with $i \nleq j$. Then we pick i_1 such that $i'_1 \leq i_1$ and $i_1 \nleq i'_1$. Let $i'_2 \geq i_1$ such that (4.7) holds for $t = 2^{-2}$ and let $i_2 \in D$ satisfy $i'_2 \leq i_2$ and $i_2 \nleq i'_2$. Then $i_1 \leq i_2$ and $i_2 \nleq i_1$, since $i_2 \leq i_1 \leq i'_2$ contradicts to the choice of i_2. By induction, we obtain a sequence $\{i_k : k \in \mathbb{N}\}$ in D satisfying $i_k \leq i_{k+1}$ and $i_{k+1} \nleq i_k$ such that (4.7) us satisfied with $t = 2^{-k}$ for every i_k and every $\lambda \in (0,1)$. The condition $M(i_k, x_{i_{k+1}}, 1/2^k) > 1 - \lambda$ $(\forall k \in \mathbb{N})$ implies that the sequence $\{x_{i_k} : k \in \mathbb{N}\}$ is right K-Cauchy. Hence, there exists $x \in X$ such that

$$\lim_{k \to \infty} M(x, x_{i_k}, t) = 1 \ (\forall t > 0).$$

We want to prove that $\{x_i\}_{i \in D}$ is M-convergent to x. We distinguish two cases.

Case I. $\exists j_0 \in D$, $\exists k_0 \in \mathbb{N}$, $\forall k \geq k_0$, $i_k \leq j_0$.

Let $i \geq j_0$. By (4.7) there exists $i' \in D$ such that $i \leq i'$ and $i' \nleq i$, which implies $M(x_{i'}, x_i, 1/2^k) > 1 - \lambda$ for all $k \geq k_0$, and so $M(x_{i'}, x_i, t) = 1$ for all $t > 0$. By T_1 of the topology τ_M, we get $x_{i'} = x_i$.

We also have $i' \nleq j_0$ since $i' \leq j_0$ can imply $i' \leq i$, in contradiction to the choice of i'. Thus, $M(x_{i'}, x_{j_0}, 1/2^k) > 1 - \lambda$ for all $k \geq k_0$. So $M(x_{i'}, x_{j_0}, t) = 1$ for all $t > 0$. Thus $x_{i'} = x_{j_0}$ by T_1 of τ_M.

Therefore, $x_i = x_{j_0}$ for all $i \geq j_0$. This shows that the net $\{x_i\}_{i \in D}$ is M-convergent to x_{j_0}.

Case II. $\forall j \in D$, $\forall k \in \mathbb{N}$, $\exists k' \geq k$, $i_{k'} \nleq j$.

For all $t > 0$ and $\lambda \in (0,1)$, choose $k_0 \in \mathbb{N}$ and $\lambda' \in (0,1)$ such that $\frac{1}{2^{k_0}} < \frac{t}{2}$ and $(1 - \lambda') * (1 - \lambda') > 1 - \lambda$. Then by the convergence of $\{x_{i_k} : k \in \mathbb{N}\}$, we have

$$M\left(x, x_{i_k}, \frac{t}{2}\right) > 1 - \lambda' \ (\forall k \geq k_0).$$

Let $i \in D$ such that $i \geq i_{k_0}$. By the hypothesis of Case II, there exists $k \geq k_0$ such that $i_k \nleq i$. The conditions $k \geq k_0$, $i_{k_0} \leq i$, $i_{k_0} \leq i_k$ and $i_k \nleq i$ imply that

$$M\left(x, x_{i_k}, \frac{t}{2}\right) > 1 - \lambda' \text{ and } M\left(x_{i_k}, x_i, \frac{t}{2}\right) > 1 - \lambda'.$$

Hence,

$$M(x, x_i, t) \geq M\left(x, x_{i_k}, \frac{t}{2}\right) * M\left(x_{i_k}, x_i, \frac{t}{2}\right) \geq (1 - \lambda') * (1 - \lambda') > 1 - \lambda \ (\forall i \geq i_{k_0}).$$

This shows that the net $\{x_i\}_{i \in D}$ is M-convergent to x. □

5. Conclusions

In this paper, we studied the relationship between various notions of sequential completeness and the corresponding notions of completeness of nets or filters in fuzzy quasi-pseudometric spaces. In particular, we showed that the right completeness by filters coincide with the sequential right completeness under the certain condition, and proposed two new types of right K-Cauchy nets in fuzzy quasi-metric spaces for which the corresponding completeness is equivalent to the sequential completeness.

Metric, uniformity and general topology are three closely related mathematical structures. In probabilistic quasi-metric spaces, Yue and Fang [32] presented the relationships between the bicompleteness of the induced classical quasi-uniform space and that of probabilistic quasi-metric space. In the future, we will study the completess of the associated quasi-uniform spaces by fuzzy quasi-metric spaces. In addition, fuzzy metric spaces could be applied in the context of fuzzy convex spaces [31, 35] and fuzzy convergence spaces [33, 34].

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