Konuralp Journal of Mathematics, 10 (1) (2022) 73-78

Konuralp Journal of Mathematics

Research Paper

Journal Homepage: *www.dergipark.gov.tr/konuralpjournalmath*

e-ISSN: 2147-625X

Dual Simpson Type Inequalities for Functions Whose Absolute Value of the First Derivatives are Preinvex

T. Chiheb[1], **B. Meftah**[1*] **and A. Dih**[1]

[1]*Département des Mathématiques, Faculté des mathématiques, de l'informatique et des sciences de la matière, Université 8 mai 1945 Guelma, Algeria.*

[*]*Corresponding author*

Abstract

In this paper, we prove a new integral identity. Basing on this identity, we establish some new dual Simpson-type inequalities for functions whose absolute value of the first derivatives are preinvex. Applications are also given.

Keywords: *Dual Simpson inequality; Newton-Cotes quadrature; preinvex functions; Hölder inequality.*

2010 Mathematics Subject Classification: *26D10; 26D15; 26A51.*

1. Introduction

Definition 1.1. *[14] A function $f : I \to \mathbb{R}$ is said to be convex, if the inequality*

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y)$$

holds for all $x,y \in I$ and all $t \in [0,1]$.

The concept of convexity plays an important and very central role in many areas, such as economics [23], finances [11], optimization [10], and game theory [6].

Due to its diverse applications this concept has been extended and generalized in several directions. The significant one is that introduced by Hanson [2], called invex functions.

Definition 1.2. *[21] A set $K \subseteq \mathbb{R}^n$ is said an invex with respect to the bifunction $\eta : K \times K \to \mathbb{R}^n$, if for all $x,y \in K$ and $t \in [0,1]$, we have*

$$x+t\eta(y,x) \in K.$$

Weir and Mond [21], gave the concept of preinvex functions which is special case of invexity.

Definition 1.3. *[21] A function $f : K \to \mathbb{R}$ is said to be preinvex with respect to η, if the inequality*

$$f(x+t\eta(y,x)) \leq (1-t)f(x)+tf(y)$$

holds for all $x,y \in K$ and all $t \in [0,1]$.

Pini [15], Noor [12, 13], Yang and Li [22] and Weir [21], have studied the basic properties of preinvex functions and their roles in optimization, variational inequalities and equilibrium problems.

It is well known that the concept of convexity has a close relationship in the development of the theory of inequalities, which is an important tool in the study of the error estimates of quadrature formulas. Indeed in several problems in applied mathematics as well as in engineering sciences requires the evaluation of integrals by adapting some quadrature.

For some some papers related to inequalities and their applications to some special means and numerical quadrature rules, we refer readers to [1, 2, 3, 7, 8, 9, 16, 17, 18, 19, 20] and references therein.

The following inequality is known as the dual Simpson's formula see [4], which can be stated as follows:

$$\left|\frac{1}{3}\left(2f\left(\frac{3a+b}{4}\right)-f\left(\frac{a+b}{2}\right)+2f\left(\frac{a+3b}{4}\right)\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \leq \frac{7(b-a)^4}{23040}\left\|f^{(4)}\right\|_\infty, \tag{1.1}$$

Email addresses: **tchiheb@yahoo.fr (Tarek Chiheb), badrimeftah@yahoo.fr (Badreddine Meftah), ameldih210716@gmail.com (Amel Dih)**

where f is four-times continuously differentiable function on (a,b), and $\left\|f^{(4)}\right\|_{\infty}=\sup_{x\in(a,b)}\left|f^{(4)}(x)\right|$.

In this paper, we prove a new integral identity as a partial result. Based on this identity, we first establish the error estimate of the so-called dual Simpson-type inequality for functions whose absolute value of first derivatives are preinvex. Secondly, we discuss the case where the absolute value of the first derivatives at certain power are preinvex by using the well-known Hölder and power mean inequalities combined with the discrete power mean inequality in order to find some simple estimates. We end the paper with some applications to special means.

2. Main results

In order to prove our results, we need the following lemma:

Lemma 2.1. *Let $f:[a,a+\eta(b,a)]\subset\mathbb{R}\to\mathbb{R}$ be a differentiable function on $[a,a+\eta(b,a)]$ with $\eta(b,a)>0$, and $f'\in L^1[a,a+\eta(b,a)]$, then the following equality holds:*

$$\begin{aligned}
&\tfrac{1}{3}\left(2f\left(\tfrac{4a+\eta(b,a)}{4}\right)-f\left(\tfrac{2a+\eta(b,a)}{2}\right)+2f\left(\tfrac{4a+3\eta(b,a)}{4}\right)\right)-\tfrac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(u)\,du\\
&=\tfrac{\eta(b,a)}{16}\left(\int_0^1(1-t)f'\left(a+\tfrac{1-t}{4}\eta(b,a)\right)dt+\int_0^1\left(t-\tfrac{5}{3}\right)f'\left(a+\tfrac{1+t}{4}\eta(b,a)\right)dt\right.\\
&\quad\left.+\int_0^1\left(t+\tfrac{2}{3}\right)f'\left(a+\tfrac{2+t}{4}\eta(b,a)\right)dt+\int_0^1(t-1)f'\left(a+\tfrac{3+t}{4}\eta(b,a)\right)dt\right).
\end{aligned}$$

Proof. Let take

$$\begin{aligned}
I_1&=\int_0^1(1-t)f'\left(a+\tfrac{1-t}{4}\eta(b,a)\right)dt,\\
I_2&=\int_0^1\left(t-\tfrac{5}{3}\right)f'\left(a+\tfrac{1+t}{4}\eta(b,a)\right)dt,\\
I_3&=\int_0^1\left(t+\tfrac{2}{3}\right)f'\left(a+\tfrac{2+t}{4}\eta(b,a)\right)dt,
\end{aligned}$$

and

$$I_4=\int_0^1(t-1)f'\left(a+\tfrac{3+t}{4}\eta(b,a)\right)dt.$$

Integrating by parts I_1

$$\begin{aligned}
I_1&=-\tfrac{4}{\eta(b,a)}(1-t)f\left(a+\tfrac{1-t}{4}\eta(b,a)\right)\Big|_{t=0}^{t=1}-\tfrac{4}{\eta(b,a)}\int_0^1 f\left(a+\tfrac{1-t}{4}\eta(b,a)\right)dt\\
&=\tfrac{4}{\eta(b,a)}f\left(\tfrac{4a+\eta(b,a)}{4}\right)-\tfrac{4}{\eta(b,a)}\int_0^1 f\left(a+\tfrac{1-t}{4}\eta(b,a)\right)dt\\
&=\tfrac{4}{\eta(b,a)}f\left(\tfrac{4a+\eta(b,a)}{4}\right)-\tfrac{16}{(\eta(b,a))^2}\int_a^{\frac{4a+\eta(b,a)}{4}}f(u)\,du.
\end{aligned}\tag{2.1}$$

Similarly, we have

$$\begin{aligned}
I_2&=\tfrac{4}{\eta(b,a)}\left(t-\tfrac{5}{3}\right)f\left(a+\tfrac{1+t}{4}\eta(b,a)\right)\Big|_{t=0}^{t=1}-\tfrac{4}{\eta(b,a)}\int_0^1 f\left(a+\tfrac{1+t}{4}\eta(b,a)\right)dt\\
&=-\tfrac{8}{3\eta(b,a)}f\left(\tfrac{2a+\eta(b,a)}{2}\right)+\tfrac{20}{3\eta(b,a)}f\left(\tfrac{4a+\eta(b,a)}{4}\right)-\tfrac{4}{\eta(b,a)}\int_0^1 f\left(a+\tfrac{1+t}{4}\eta(b,a)\right)dt\\
&=\tfrac{20}{3\eta(b,a)}f\left(\tfrac{4a+\eta(b,a)}{4}\right)-\tfrac{8}{3\eta(b,a)}f\left(\tfrac{2a+\eta(b,a)}{2}\right)-\tfrac{16}{(\eta(b,a))^2}\int_{\frac{4a+\eta(b,a)}{4}}^{\frac{2a+\eta(b,a)}{2}}f(u)\,du,
\end{aligned}\tag{2.2}$$

$$I_3=\tfrac{4}{\eta(b,a)}\left(t+\tfrac{2}{3}\right)f\left(a+\tfrac{2+t}{4}\eta(b,a)\right)\Big|_{t=0}^{t=1}-\tfrac{4}{\eta(b,a)}\int_0^1 f\left(a+\tfrac{2+t}{4}\eta(b,a)\right)dt$$
$$=\tfrac{20}{3\eta(b,a)}f\left(\tfrac{4a+3\eta(b,a)}{4}\right)-\tfrac{8}{3\eta(b,a)}f\left(\tfrac{2a+\eta(b,a)}{2}\right)-\tfrac{4}{\eta(b,a)}\int_0^1 f\left(a+\tfrac{2+t}{4}\eta(b,a)\right)dt$$
$$=\tfrac{20}{3\eta(b,a)}f\left(\tfrac{4a+3\eta(b,a)}{4}\right)-\tfrac{8}{3\eta(b,a)}f\left(\tfrac{2a+\eta(b,a)}{2}\right)-\tfrac{16}{(\eta(b,a))^2}\int_{\frac{2a+\eta(b,a)}{2}}^{\frac{4a+3\eta(b,a)}{4}} f(u)\,du, \tag{2.3}$$

and

$$I_4=\tfrac{4}{\eta(b,a)}(t-1)f\left(a+\tfrac{3+t}{4}\eta(b,a)\right)\Big|_{t=0}^{t=1}-\tfrac{4}{\eta(b,a)}\int_0^1 f\left(a+\tfrac{3+t}{4}\eta(b,a)\right)dt$$
$$=\tfrac{4}{\eta(b,a)}f\left(\tfrac{4a+3\eta(b,a)}{4}\right)-\tfrac{4}{\eta(b,a)}\int_0^1 f\left(a+\tfrac{3+t}{4}\eta(b,a)\right)dt$$
$$=\tfrac{4}{\eta(b,a)}f\left(\tfrac{4a+3\eta(b,a)}{4}\right)-\tfrac{16}{(\eta(b,a))^2}\int_{\frac{4a+3\eta(b,a)}{4}}^{a+\eta(b,a)} f(u)\,du. \tag{2.4}$$

Summing (2.1)-(2.4), and then multiplying the result by $\frac{\eta(b,a)}{16}$ we get the desired result. □

Theorem 2.2. *Let $f:[a,a+\eta(b,a)]\to\mathbb{R}$ be a differentiable function on $[a,a+\eta(b,a)]$ such that $f'\in L^1[a,a+\eta(b,a)]$ with $\eta(b,a)>0$. If $|f'|$ is preinvex, then we have:*

$$\left|\tfrac{1}{3}\left(2f\left(\tfrac{4a+\eta(b,a)}{4}\right)-f\left(\tfrac{2a+\eta(b,a)}{2}\right)+2f\left(\tfrac{4a+3\eta(b,a)}{4}\right)\right)-\tfrac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(u)\,du\right|\le\tfrac{5\eta(b,a)}{48}\left(|f'(a)|+|f'(b)|\right).$$

Proof. From Lemma 2.1, properties of modulus, and the preinvexity of $|f'|$, we have

$$\left|\tfrac{1}{3}\left(2f\left(\tfrac{4a+\eta(b,a)}{4}\right)-f\left(\tfrac{2a+\eta(b,a)}{2}\right)+2f\left(\tfrac{4a+3\eta(b,a)}{4}\right)\right)-\tfrac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(u)\,du\right|$$
$$\le\tfrac{\eta(b,a)}{16}\Bigg(\int_0^1(1-t)\left|f'\left(a+\tfrac{1-t}{4}\eta(b,a)\right)\right|dt+\int_0^1\left(\tfrac{5}{3}-t\right)\left|f'\left(a+\tfrac{1+t}{4}\eta(b,a)\right)\right|dt$$
$$+\int_0^1\left(t+\tfrac{2}{3}\right)\left|f'\left(a+\tfrac{2+t}{4}\eta(b,a)\right)\right|dt+\int_0^1(1-t)\left|f'\left(a+\tfrac{3+t}{4}\eta(b,a)\right)\right|dt\Bigg)$$
$$\le\tfrac{\eta(b,a)}{16}\Bigg(\int_0^1(1-t)\left(\tfrac{3+t}{4}|f'(a)|+\tfrac{1-t}{4}|f'(b)|\right)dt+\int_0^1\left(\tfrac{5}{3}-t\right)\left(\tfrac{3-t}{4}|f'(a)|+\tfrac{1+t}{4}|f'(b)|\right)dt$$
$$+\int_0^1\left(t+\tfrac{2}{3}\right)\left(\tfrac{2-t}{4}|f'(a)|+\tfrac{2+t}{4}|f'(b)|\right)dt+\int_0^1(1-t)\left(\tfrac{1-t}{4}|f'(a)|+\tfrac{3+t}{4}|f'(b)|\right)dt\Bigg)$$
$$=\tfrac{\eta(b,a)}{16}\Bigg(|f'(a)|\left(\int_0^1(1-t)\left(\tfrac{3+t}{4}\right)dt+\int_0^1\left(\tfrac{5}{3}-t\right)\left(\tfrac{3-t}{4}\right)dt+\int_0^1\left(t+\tfrac{2}{3}\right)\left(\tfrac{2-t}{4}\right)dt+\int_0^1(1-t)\left(\tfrac{1-t}{4}\right)dt\right)$$
$$+|f'(b)|\left(\int_0^1(1-t)\left(\tfrac{1-t}{4}\right)dt+\int_0^1\left(\tfrac{5}{3}-t\right)\left(\tfrac{1+t}{4}\right)dt+\int_0^1\left(t+\tfrac{2}{3}\right)\left(\tfrac{2+t}{4}\right)dt+\int_0^1(1-t)\left(\tfrac{3+t}{4}\right)dt\right)\Bigg)$$
$$=\tfrac{5\eta(b,a)}{48}\left(|f'(a)|+|f'(b)|\right),$$

where we have used the fact that

$$\int_0^1(1-t)\left(\tfrac{3+t}{4}\right)dt=\tfrac{5}{12},$$

$$\int_0^1\left(\tfrac{5}{3}-t\right)\left(\tfrac{3-t}{4}\right)dt=\int_0^1\left(t+\tfrac{2}{3}\right)\left(\tfrac{2+t}{4}\right)dt=\tfrac{3}{4},$$

$$\int_0^1 \left(t+\tfrac{2}{3}\right)\left(\tfrac{2-t}{4}\right)dt = \int_0^1 \left(\tfrac{5}{3}-t\right)\left(\tfrac{1+t}{4}\right)dt = \tfrac{5}{12},$$

and

$$\int_0^1 (1-t)\left(\tfrac{1-t}{4}\right)dt = \tfrac{1}{12}. \tag{2.5}$$

The proof is completed. □

Corollary 1. *In Theorem 2.2 taking $\eta(b,a) = b-a$, then we obtain:*

$$\left|\tfrac{1}{3}\left(2f\left(\tfrac{3a+b}{4}\right) - f\left(\tfrac{a+b}{2}\right) + 2f\left(\tfrac{a+3b}{4}\right)\right) - \tfrac{1}{b-a}\int_a^b f(u)\,du\right| \leq \tfrac{5(b-a)}{48}\left(|f'(a)| + |f'(b)|\right).$$

Theorem 2.3. *Let $f : [a, a+\eta(b,a)] \to \mathbb{R}$ be a differentiable function on $[a, a+\eta(b,a)]$ such that $f' \in L^1[a, a+\eta(b,a)]$ with $\eta(b,a) > 0$. If $|f'|^\zeta$ is preinvex where $\zeta > 1$ with $\frac{1}{\delta} + \frac{1}{\zeta} = 1$, then we have:*

$$\left|\tfrac{1}{3}\left(2f\left(\tfrac{4a+\eta(b,a)}{4}\right) - f\left(\tfrac{2a+\eta(b,a)}{2}\right) + 2f\left(\tfrac{4a+3\eta(b,a)}{4}\right)\right) - \tfrac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(u)\,du\right|$$
$$\leq \tfrac{\eta(b,a)}{8(\delta+1)^{\frac{1}{\delta}}}\left(1 + \left(\tfrac{5^{\delta+1}-2^{\delta+1}}{3^{\delta+1}}\right)^{\frac{1}{\delta}}\right)\left(\tfrac{|f'(a)|^\zeta + |f'(b)|^\zeta}{2}\right)^{\frac{1}{\zeta}}.$$

Proof. From Lemma 2.1, properties of modulus, Hölder's inequality, and preinvexity $|f'|^\zeta$, we have

$$\left|\tfrac{1}{3}\left(2f\left(\tfrac{4a+\eta(b,a)}{4}\right) - f\left(\tfrac{2a+\eta(b,a)}{2}\right) + 2f\left(\tfrac{4a+3\eta(b,a)}{4}\right)\right) - \tfrac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(u)\,du\right|$$
$$\leq \tfrac{\eta(b,a)}{16}\left(\left(\int_0^1 (1-t)^\delta dt\right)^{\frac{1}{\delta}}\left(\int_0^1 \left|f'\left(a+\tfrac{1-t}{4}\eta(b,a)\right)\right|^\zeta dt\right)^{\frac{1}{\zeta}} + \left(\int_0^1 \left(\tfrac{5}{3}-t\right)^\delta dt\right)^{\frac{1}{\delta}}\left(\int_0^1 \left|f'\left(a+\tfrac{1+t}{4}\eta(b,a)\right)\right|^\zeta dt\right)^{\frac{1}{\zeta}}\right.$$
$$\left.+\left(\int_0^1 \left(t+\tfrac{2}{3}\right)^\delta dt\right)^{\frac{1}{\delta}}\left(\int_0^1 \left|f'\left(a+\tfrac{2+t}{4}\eta(b,a)\right)\right|^\zeta dt\right)^{\frac{1}{\zeta}} + \left(\int_0^1 (1-t)^\delta dt\right)^{\frac{1}{\delta}}\left(\int_0^1 \left|f'\left(a+\tfrac{3+t}{4}\eta(b,a)\right)\right|^\zeta dt\right)^{\frac{1}{\zeta}}\right)$$
$$\leq \tfrac{\eta(b,a)}{16(\delta+1)^{\frac{1}{\delta}}}\left(\left(\int_0^1 \left(\tfrac{3+t}{4}|f'(a)|^\zeta + \tfrac{1-t}{4}|f'(b)|^\zeta\right)dt\right)^{\frac{1}{\zeta}} + \left(\tfrac{5^{\delta+1}-2^{\delta+1}}{3^{\delta+1}}\right)^{\frac{1}{\delta}}\left(\int_0^1 \left(\tfrac{3-t}{4}|f'(a)|^\zeta + \tfrac{1+t}{4}|f'(b)|^\zeta\right)dt\right)^{\frac{1}{\zeta}}\right.$$
$$\left.+\left(\tfrac{5^{\delta+1}-2^{\delta+1}}{3^{\delta+1}}\right)^{\frac{1}{\delta}}\left(\int_0^1 \left(\tfrac{2-t}{4}|f'(a)|^\zeta + \tfrac{2+t}{4}|f'(b)|^\zeta\right)dt\right)^{\frac{1}{\zeta}} + \left(\int_0^1 \left(\tfrac{1-t}{4}|f'(a)|^\zeta + \tfrac{3+t}{4}|f'(b)|^\zeta\right)dt\right)^{\frac{1}{\zeta}}\right)$$
$$= \tfrac{\eta(b,a)}{16(\delta+1)^{\frac{1}{\delta}}}\left(\left(|f'(a)|^\zeta\int_0^1 \tfrac{3+t}{4}dt + |f'(b)|^\zeta\int_0^1 \tfrac{1-t}{4}dt\right)^{\frac{1}{\zeta}} + \left(\tfrac{5^{\delta+1}-2^{\delta+1}}{3^{\delta+1}}\right)^{\frac{1}{\delta}}\left(|f'(a)|^\zeta\int_0^1 \tfrac{3-t}{4}dt + |f'(b)|^\zeta\int_0^1 \tfrac{1+t}{4}dt\right)^{\frac{1}{\zeta}}\right.$$
$$\left.+\left(\tfrac{5^{\delta+1}-2^{\delta+1}}{3^{\delta+1}}\right)^{\frac{1}{\delta}}\left(|f'(a)|^\zeta\int_0^1 \tfrac{2-t}{4}dt + |f'(b)|^\zeta\int_0^1 \tfrac{2+t}{4}dt\right)^{\frac{1}{\zeta}} + \left(|f'(a)|^\zeta\int_0^1 \tfrac{1-t}{4}dt + |f'(b)|^\zeta\int_0^1 \tfrac{3+t}{4}dt\right)^{\frac{1}{\zeta}}\right)$$
$$= \tfrac{\eta(b,a)}{16(\delta+1)^{\frac{1}{\delta}}}\left(\left(\tfrac{7|f'(a)|^\zeta + |f'(b)|^\zeta}{8}\right)^{\frac{1}{\zeta}} + \left(\tfrac{|f'(a)|^\zeta + 7|f'(b)|^\zeta}{8}\right)^{\frac{1}{\zeta}} + \left(\tfrac{5^{\delta+1}-2^{\delta+1}}{3^{\delta+1}}\right)^{\frac{1}{\delta}}\left(\left(\tfrac{5|f'(a)|^\zeta + 3|f'(b)|^\zeta}{8}\right)^{\frac{1}{\zeta}} + \left(\tfrac{3|f'(a)|^\zeta + 5|f'(b)|^\zeta}{8}\right)^{\frac{1}{\zeta}}\right)\right). \tag{2.6}$$

Using the discrete power mean inequality it follows

$$\left(\tfrac{7|f'(a)|^\zeta + |f'(b)|^\zeta}{8}\right)^{\frac{1}{\zeta}} + \left(\tfrac{|f'(a)|^\zeta + 7|f'(b)|^\zeta}{8}\right)^{\frac{1}{\zeta}} \leq 2^{1-\frac{1}{\zeta}}\left(|f'(a)|^\zeta + |f'(b)|^\zeta\right)^{\frac{1}{\zeta}}, \tag{2.7}$$

and

$$\left(\tfrac{5|f'(a)|^\zeta + 3|f'(b)|^\zeta}{8}\right)^{\frac{1}{\zeta}} + \left(\tfrac{3|f'(a)|^\zeta + 5|f'(b)|^\zeta}{8}\right)^{\frac{1}{\zeta}} \leq 2^{1-\frac{1}{\zeta}}\left(|f'(a)|^\zeta + |f'(b)|^\zeta\right)^{\frac{1}{\zeta}}. \tag{2.8}$$

Substituting (2.7) and (2.8) in (2.6), we obtain

$$\left|\frac{1}{3}\left(2f\left(\frac{4a+\eta(b,a)}{4}\right)-f\left(\frac{2a+\eta(b,a)}{2}\right)+2f\left(\frac{4a+3\eta(b,a)}{4}\right)\right)-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(u)\,du\right|\leq\frac{\eta(b,a)}{8(\delta+1)^{\frac{1}{\delta}}}\left(1+\left(\frac{5^{\delta+1}-2^{\delta+1}}{3^{\delta+1}}\right)^{\frac{1}{\delta}}\right)\left(\frac{|f'(a)|^{\zeta}+|f'(b)|^{\zeta}}{2}\right)^{\frac{1}{\zeta}},$$

which is the desired result. The proof is completed. □

Corollary 2. *In Theorem 2.3 taking $\eta(b,a)=b-a$, then we obtain:*

$$\left|\frac{1}{3}\left(2f\left(\frac{3a+b}{4}\right)-f\left(\frac{a+b}{2}\right)+2f\left(\frac{a+3b}{4}\right)\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right|\leq\frac{(b-a)}{8(\delta+1)^{\frac{1}{\delta}}}\left(1+\left(\frac{5^{\delta+1}-2^{\delta+1}}{3^{\delta+1}}\right)^{\frac{1}{\delta}}\right)\left(\frac{|f'(a)|^{\zeta}+|f'(b)|^{\zeta}}{2}\right)^{\frac{1}{\zeta}}.$$

Theorem 2.4. *Let $f:[a,a+\eta(b,a)]\to\mathbb{R}$ be a differentiable function on $[a,a+\eta(b,a)]$ such that $f'\in L^1[a,a+\eta(b,a)]$ with $\eta(b,a)>0$. If $|f'|^{\zeta}$ is preinvex where $\zeta\geq 1$, then we have:*

$$\left|\frac{1}{3}\left(2f\left(\frac{4a+\eta(b,a)}{4}\right)-f\left(\frac{2a+\eta(b,a)}{2}\right)+2f\left(\frac{4a+3\eta(b,a)}{4}\right)\right)-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(u)\,du\right|\leq\frac{5\eta(b,a)}{24}\left(\frac{|f'(a)|^{\zeta}+|f'(b)|^{\zeta}}{2}\right)^{\frac{1}{\zeta}}.$$

Proof. From Lemma 2.1, properties of modulus, power mean inequality, and the preinvexity of $|f'|^{\zeta}$, we have

$$\begin{aligned}
&\left|\frac{1}{3}\left(2f\left(\frac{4a+\eta(b,a)}{4}\right)-f\left(\frac{2a+\eta(b,a)}{2}\right)+2f\left(\frac{4a+3\eta(b,a)}{4}\right)\right)-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(u)\,du\right|\\
&\leq\frac{\eta(b,a)}{16}\left(\left(\int_0^1(1-t)\,dt\right)^{1-\frac{1}{\zeta}}\left(\int_0^1(1-t)\left|f'\left(a+\frac{1-t}{4}\eta(b,a)\right)\right|^{\zeta}dt\right)^{\frac{1}{\zeta}}+\left(\int_0^1\left(\frac{5}{3}-t\right)dt\right)^{1-\frac{1}{\zeta}}\left(\int_0^1\left(\frac{5}{3}-t\right)\left|f'\left(a+\frac{1+t}{4}\eta(b,a)\right)\right|^{\zeta}dt\right)^{\frac{1}{\zeta}}\right.\\
&\quad\left.+\left(\int_0^1\left(t+\frac{2}{3}\right)dt\right)^{1-\frac{1}{\zeta}}\left(\int_0^1\left(t+\frac{2}{3}\right)\left|f'\left(a+\frac{2+t}{4}\eta(b,a)\right)\right|^{\zeta}dt\right)^{\frac{1}{\zeta}}+\left(\int_0^1(1-t)\,dt\right)^{1-\frac{1}{\zeta}}\left(\int_0^1(1-t)\left|f'\left(a+\frac{3+t}{4}\eta(b,a)\right)\right|^{\zeta}dt\right)^{\frac{1}{\zeta}}\right)\\
&\leq\frac{\eta(b,a)}{16}\left(\left(\frac{1}{2}\right)^{1-\frac{1}{\zeta}}\left(\int_0^1(1-t)\left(\frac{3+t}{4}|f'(a)|^{\zeta}+\frac{1-t}{4}|f'(b)|^{\zeta}\right)dt\right)^{\frac{1}{\zeta}}+\left(\frac{7}{6}\right)^{1-\frac{1}{\zeta}}\left(\int_0^1\left(\frac{5}{3}-t\right)\left(\frac{3-t}{4}|f'(a)|^{\zeta}+\frac{1+t}{4}|f'(b)|^{\zeta}\right)dt\right)^{\frac{1}{\zeta}}\right.\\
&\quad\left.+\left(\frac{7}{6}\right)^{1-\frac{1}{\zeta}}\left(\int_0^1\left(t+\frac{2}{3}\right)\left(\frac{2-t}{4}|f'(a)|^{\zeta}+\frac{2+t}{4}|f'(b)|^{\zeta}\right)dt\right)^{\frac{1}{\zeta}}+\left(\frac{1}{2}\right)^{1-\frac{1}{\zeta}}\left(\int_0^1(1-t)\left(\frac{1-t}{4}|f'(a)|^{\zeta}+\frac{3+t}{4}|f'(b)|^{\zeta}\right)dt\right)^{\frac{1}{\zeta}}\right)\\
&=\frac{\eta(b,a)}{16}\left(\left(\frac{1}{2}\right)^{1-\frac{1}{\zeta}}\left(|f'(a)|^{\zeta}\int_0^1(1-t)\left(\frac{3+t}{4}\right)dt+|f'(b)|^{\zeta}\int_0^1(1-t)\left(\frac{1-t}{4}\right)dt\right)^{\frac{1}{\zeta}}\right.\\
&\quad+\left(\frac{7}{6}\right)^{1-\frac{1}{\zeta}}\left(|f'(a)|^{\zeta}\int_0^1\left(\frac{5}{3}-t\right)\left(\frac{3-t}{4}\right)dt+|f'(b)|^{\zeta}\int_0^1\left(\frac{5}{3}-t\right)\left(\frac{1+t}{4}\right)dt\right)^{\frac{1}{\zeta}}\\
&\quad+\left(\frac{7}{6}\right)^{1-\frac{1}{\zeta}}\left(|f'(a)|^{\zeta}\int_0^1\left(t+\frac{2}{3}\right)\left(\frac{2-t}{4}\right)dt+|f'(b)|^{\zeta}\int_0^1\left(t+\frac{2}{3}\right)\left(\frac{2+t}{4}\right)dt\right)^{\frac{1}{\zeta}}\\
&\quad\left.+\left(\frac{1}{2}\right)^{1-\frac{1}{\zeta}}\left(|f'(a)|^{\zeta}\int_0^1(1-t)\left(\frac{1-t}{4}\right)dt+|f'(b)|^{\zeta}\int_0^1(1-t)\left(\frac{3+t}{4}\right)dt\right)^{\frac{1}{\zeta}}\right)\\
&=\frac{\eta(b,a)}{32}\left(\left(\left(\frac{5|f'(a)|^{\zeta}+|f'(b)|^{\zeta}}{6}\right)^{\frac{1}{\zeta}}+\left(\frac{|f'(a)|^{\zeta}+5|f'(b)|^{\zeta}}{6}\right)^{\frac{1}{\zeta}}\right)+\frac{7}{3}\left(\left(\frac{9|f'(a)|^{\zeta}+5|f'(b)|^{\zeta}}{14}\right)^{\frac{1}{\zeta}}+\left(\frac{5|f'(a)|^{\zeta}+9|f'(b)|^{\zeta}}{14}\right)^{\frac{1}{\zeta}}\right)\right),
\end{aligned}\tag{2.9}$$

where we have used (2.5).
Now from the discrete power mean inequality i.e. for any $u,v\geq 0$ and $0\leq\varepsilon\leq 1$, we have $u^{\varepsilon}+v^{\varepsilon}\leq 2^{1-\varepsilon}(u+v)^{\varepsilon}$

$$\left(\frac{5|f'(a)|^{\zeta}+|f'(b)|^{\zeta}}{6}\right)^{\frac{1}{\zeta}}+\left(\frac{|f'(a)|^{\zeta}+5|f'(b)|^{\zeta}}{6}\right)^{\frac{1}{\zeta}}\leq 2^{1-\frac{1}{\zeta}}\left(|f'(a)|^{\zeta}+|f'(b)|^{\zeta}\right)^{\frac{1}{\zeta}},\tag{2.10}$$

and

$$\left(\frac{9|f'(a)|^{\zeta}+5|f'(b)|^{\zeta}}{14}\right)^{\frac{1}{\zeta}}+\left(\frac{5|f'(a)|^{\zeta}+9|f'(b)|^{\zeta}}{14}\right)^{\frac{1}{\zeta}}\leq 2^{1-\frac{1}{\zeta}}\left(|f'(a)|^{\zeta}+|f'(b)|^{\zeta}\right)^{\frac{1}{\zeta}}.\tag{2.11}$$

Substituting (2.10) and (2.11) in (2.9), we obtain

$$\left|\frac{1}{3}\left(2f\left(\frac{4a+\eta(b,a)}{4}\right)-f\left(\frac{2a+\eta(b,a)}{2}\right)+2f\left(\frac{4a+3\eta(b,a)}{4}\right)\right)-\frac{1}{\eta(b,a)}\int\limits_{a}^{a+\eta(b,a)}f(u)\,du\right|\leq\frac{5\eta(b,a)}{24}\left(\frac{|f'(a)|^{\zeta}+|f'(b)|^{\zeta}}{2}\right)^{\frac{1}{\zeta}},$$

which is the desired result. The proof is achieved. □

Corollary 3. *In Theorem 2.4 taking $\eta(b,a)=b-a$, then we obtain:*

$$\left|\frac{1}{3}\left(2f\left(\frac{3a+b}{4}\right)-f\left(\frac{a+b}{2}\right)+2f\left(\frac{a+3b}{4}\right)\right)-\frac{1}{b-a}\int\limits_{a}^{b}f(u)\,du\right|\leq\frac{5(b-a)}{24}\left(\frac{|f'(a)|^{\zeta}+|f'(b)|^{\zeta}}{2}\right)^{\frac{1}{\zeta}}.$$

3. Applications

For arbitrary real numbers $a,a_1,a_2,...,a_n,b$ we have:
The Arithmetic mean: $A(a_1,a_2,...,a_n)=\frac{a_1+a_2+...+a_n}{n}$.
The Geometric mean: $G(a,b)=\sqrt{ab}$, $a,b>0$.
The p-Logarithmic mean: $L_p(a,b)=\left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}$, $a,b>0,a\neq b$ and $p\in\mathbb{R}\backslash\{-1,0\}$.

Proposition 1. *Let $a,b\in\mathbb{R}$ with $0<a<b$, then we have*

$$\left|2A^3(a,a,a,b)+2A^3(a,b,b,b)-A^3(a,b)-3L_3^3(a,b)\right|\leq\frac{15(b-a)}{16}\left(a^2+b^2\right).$$

Proof. The assertion follows from Theorem 2.2 with $\eta(b,a)=b-a$, applied to the function $f(x)=x^3$. □

Proposition 2. *Let $a,b\in\mathbb{R}$ with $0<a<b$, and $\zeta>1$, then we have*

$$\left|2A^3\left(a,a,a,a+\sqrt{ab}\right)+2A^3\left(a,a+\sqrt{ab},a+\sqrt{ab},a+\sqrt{ab}\right)-A^3\left(a,a+\sqrt{ab}\right)-3L_3^3\left(a,a+\sqrt{ab}\right)\right|$$
$$\leq\frac{9\sqrt{ab}}{8(\delta+1)^{\frac{1}{\delta}}}\left(1+\left(\frac{5^{\delta+1}-2^{\delta+1}}{3^{\delta+1}}\right)^{\frac{1}{\delta}}\right)\left(\frac{a^{2\zeta}+b^{2\zeta}}{2}\right)^{\frac{1}{\zeta}}.$$

Proof. The assertion follows from Theorem 2.3 with $\eta(b,a)=G(a,b)$, applied to the function $f(x)=x^3$. □

4. Acknowledgements

We thank the anonymous reviewers for their careful reading of our manuscript and their many insightful comments and suggestions.

References

[1] H. Budak, F. Usta and M. Z. Sarikaya, New upper bounds of Ostrowski type integral inequalities utilizing Taylor expansion. Hacet. J. Math. Stat. 47 (2018), no. 3, 567–578.
[2] H. Budak, F. Usta and M. Z. Sarikaya, Refinements of the Hermite-Hadamard inequality for co-ordinated convex mappings. J. Appl. Anal. 25 (2019), no. 1, 73–81.
[3] H. Budak, F. Usta, M. Z. Sarikaya and M. E. Ozdemir, On generalization of midpoint type inequalities with generalized fractional integral operators. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM 113 (2019), no. 2, 769–790.
[4] Lj. Dedić, M. Matić and J. Pečarić, On dual Euler-Simpson formulae. Bull. Belg. Math. Soc. Simon Stevin 8 (2001), no. 3, 479–504.
[5] M. A. Hanson, On sufficiency of the Kuhn-Tucker conditions. J. Math. Anal. Appl. 80 (1981), no. 2, 545–550.
[6] P. T. Harker, Generalized Nash games and quasi-variational inequalities. European journal of Operational research 54 (1991), no.1, 81-94.
[7] B. Meftah, Two dimensional Čebyšev type inequalities for functions whose second derivatives are co-ordinated (h_1,h_2)-preinvex. Konuralp J. Math. 6 (2018), no. 1, 76-83.
[8] B. Meftah, Fractional Ostrowski type inequalities for functions whose certain power of modulus of the first derivatives are prequasiinvex via power mean inequality. J. Appl. Anal. 25 (2019), no 1, 83-90.
[9] B. Meftah and C. Marrouche, Some new Hermite-Hadamard type inequalities for n-times log-convex functions. Jordan J. Math. Stat. 14(2021), no. 4, 651-669.
[10] B. B. Mordukhovich and J. V. Outrata, Coderivative analysis of quasi-variational inequalities with applications to stability and optimization. SIAM J. Optim. 18 (2007), no. 2, 389–412.
[11] A. Nagurney, Finance and variational inequalities. Quant. Finance 1 (2001), no. 3, 309–317.
[12] M. A. Noor, Variational-like inequalities. Optimization 30 (1994), no. 4, 323–330.
[13] M. A. Noor, Invex equilibrium problems. J. Math. Anal. Appl. 302 (2005), no. 2, 463–475.
[14] J. Pečarić, F. Proschan and Y. L. Tong, Convex functions, partial orderings, and statistical applications. Mathematics in Science and Engineering, 187. Academic Press, Inc., Boston, MA, 1992.
[15] R. Pini, Invexity and generalized convexity. Optimization 22 (1991), no. 4, 513–525.
[16] F. Usta, H. Budak, M. Z. Sarikaya and E. Set, On generalization of trapezoid type inequalities for s-convex functions with generalized fractional integral operators. Filomat 32 (2018), no. 6, 2153–2171.
[17] F. Usta and M. Z. Sarikaya, On generalization conformable fractional integral inequalities. Filomat 32 (2018), no. 16, 5519–5526.
[18] F. Usta and M. Z. Sarikaya, On bivariate retarded integral inequalities and their applications. Facta Univ. Ser. Math. Inform. 34 (2019), no. 3, 553–561.
[19] F. Usta, H. Budak and M. Z. Sarikaya, Montgomery identities and Ostrowski type inequalities for fractional integral operators. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM 113 (2019), no. 2, 1059–1080.
[20] F. Usta, H. Budak and M. Z. Sarikaya, Some new Chebyshev type inequalities utilizing generalized fractional integral operators. AIMS Math. 5 (2020), no. 2, 1147–1161.
[21] T. Weir and B. Mond, Pre-invex functions in multiple objective optimization. J. Math. Anal. Appl. 136 (1988), no. 1, 29–38.
[22] X. -M. Yang and D. Li, On properties of preinvex functions. J. Math. Anal. Appl. 256 (2001), no. 1, 229–241.
[23] G. X. -Z.Yuan, G. Isac, K. -K. Tan and J. Yu, The study of minimax inequalities, abstract economics and applications to variational inequalities and Nash equilibria. Acta Appl. Math. 54 (1998), no. 2, 135–166.