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Double and Type (3,0) Minkowski Pythagorean Hodograph Curves

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Abstract
In present paper, Double Minkowski Pythagorean Hodograph (DMPH) curves and type (3,0) Minkowski Pythagorean Hodograph (MPH) curves are studied. Firstly, we obtained the conditions for a MPH curve to be a DMPH curve. Then, we examined these conditions in split quaternion form. Finally, a special class of seventh degree MPH curves is characterized and illustrative examples are given.

1. Introduction

Polynomials are symbolic objects that are frequently used, especially in computer science and computational algebra. Consisting of polynomial components, polynomial curves are one of the curves studied extensively in computational geometry. These curves have application areas such as computer aided geometric design, robotics, navigation, and motion control, therefore they maintain their importance today. Pythagorean hodograph curves, simply PH curves, are polynomial curves that provide the equality called the Pythagorean condition. This condition is satisfied by the hodograph of these curves and a distinguishing property for them among the polynomial curves. For planar PH curves, this condition can be expressed using the conformal map $\mathbb{C} \to \mathbb{C}$ defined by $z \to z^2$ and taking z as a complex polynomial [4]. For spatial PH curves, this condition can be given using quaternion polynomials. The quaternion formulation gives a very elegant and concise description of this structure which contributes to the development of basic algorithms to construct and analyze the PH curves [5]. Alternatively, using complex polynomials, the construction of spatial PH curves can be given with the Hopf map $\mathbb{C}^2 \to \mathbb{R}^3$. This transformation associates points $P \in \mathbb{R}^3$ with complex number pairs α, β such that $P = H(\alpha, \beta)$. Taking α and β as complex polynomials, the Pythagorean condition can be obtained [6].

In computer aided design and manufacturing, PH curves play an important role, as rational representations of shapes are important in fields such as robotics, animation, computer graphic design, and motion control. Considering the applications mentioned above, orthonormal frames are needed to describe the direction of the particle moving along a path. One of the most commonly used orthonormal frames is the Frenet frame. However, this frame is not very suitable for practical applications because it is not defined at points where the second derivative of the curve is zero and because the normal plane vectors rotate unnecessarily about the tangent. To overcome this problem, rotation minimizing orthonormal frames (RMF) are used [7]. Using this, RRMF-PH curves of type (n, m) is defined by Dospra [2]. In this study, some special type (n, m) curves are examined. For further information on PH curves and applications, see [3], [9-11].

In this paper, we describe DMPH curves as a new concept that will be helpful for Lorentzian geometry and its physical applications in Minkowski 3-space. We express the conditions provided by such curves using split quaternion polynomials. Also, we study MPH curves of type (3,0) which are a class of MPH curves with degree 7. We give illustrative examples for both MPH curves of type (3,0) and DMPH curves that support the constructed theories that will find applications.

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2. Preliminaries

The set

$$\widetilde{\mathbb{H}} = \{\varepsilon = \varepsilon_0 + \varepsilon_1 \boldsymbol{i} + \varepsilon_2 \boldsymbol{j} + \varepsilon_3 \boldsymbol{k} : \varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3 \in \mathbb{R}, \boldsymbol{i}^2 = \boldsymbol{j}^2 = 1, \boldsymbol{k}^2 = -1, \boldsymbol{ijk} = 1\}$$

which is defined in $(-,+,+,-)$ signed $\mathbb{R}_2^4$ semi-Euclidean space is called the ring of split quaternions. We present the classification of split quaternions according to their semi-Euclidean scalar product with themselves in $\mathbb{R}_2^4$. Let $\varepsilon = \varepsilon_0 + \varepsilon_1 \boldsymbol{i} + \varepsilon_2 \boldsymbol{j} + \varepsilon_3 \boldsymbol{k} \in \widetilde{\mathbb{H}}$. In this case, if

i. $\langle \varepsilon, \varepsilon \rangle_{\mathbb{R}_2^4} = -\varepsilon_0^2 + \varepsilon_1^2 + \varepsilon_2^2 - \varepsilon_3^2 > 0$,
ii. $\langle \varepsilon, \varepsilon \rangle_{\mathbb{R}_2^4} = -\varepsilon_0^2 + \varepsilon_1^2 + \varepsilon_2^2 - \varepsilon_3^2 < 0$,
iii. $\langle \varepsilon, \varepsilon \rangle_{\mathbb{R}_2^4} = -\varepsilon_0^2 + \varepsilon_1^2 + \varepsilon_2^2 - \varepsilon_3^2 = 0$,

then ε is called spacelike, timelike or lightlike split quaternion, respectively [8].

Let $\omega(t) = (\alpha(t), \beta(t), \gamma(t))$ be a PH curve in $\mathbb{R}^3$. If the components of the hodograph of ω are relatively prime, there exist polynomials $k(t), l(t), m(t), q(t)$ such that

$$\begin{aligned}
\alpha'(t) &= k^2(t) + l^2(t) - m^2(t) - q^2(t) \\
\beta'(t) &= 2[k(t)q(t) + l(t)l(t)] \\
\gamma'(t) &= 2[l(t)q(t) - k(t)m(t)] \\
\sigma(t) &= k^2(t) + l^2(t) + m^2(t) + q^2(t).
\end{aligned}$$

where,

$$[\alpha'(t)]^2 + [\beta'(t)]^2 + [\gamma'(t)]^2 = \sigma^2(t).$$

This kind of hodographs are called primitive hodographs [5].

Spatial PH curves can be generated by quaternion polynomials. Let $\boldsymbol{K}(t) = k(t) + l(t)\boldsymbol{i} + m(t)\boldsymbol{j} + q(t)\boldsymbol{k}$ be a quaternion polynomial. The quaternion product

$$\begin{aligned}
\omega'(t) &= \boldsymbol{K}(t)\boldsymbol{i}\boldsymbol{K}^*(t) \\
&= [k^2(t) + l^2(t) - m^2(t) - q^2(t)]\, \boldsymbol{i} \\
&\quad + 2[k(t)q(t) + l(t)m(t)]\, \boldsymbol{j} \\
&\quad + 2[l(t)q(t) - k(t)m(t)]\, \boldsymbol{k}
\end{aligned}$$

gives the hodograph of the PH curve, so generates the PH curve [1].

3. DMPH Curves and Type (3,0) Curves

In this section, we give the definition of DMPH curve. We construct DMPH conditions in Minkowski-Hopf map form and split quaternion form. Then, we give illustrative examples. Also, we characterize type (3,0) MPH curves and give an example.

First of all, we present some basic concepts on MPH curves. Since all null curves in $\mathbb{R}_1^3$ are MPH curves and there is no timelike MPH curve, we consider regular spacelike MPH curves.

The characterization of planar MPH curves can be given with hyperbolic polynomials. If $\omega(t) = (\alpha(t), \beta(t))$ is a MPH curve in $\mathbb{R}_1^2$, the hodograph of ω is expressed with the hyperbolic polynomial $\gamma(t) = k(t) + l(t)\boldsymbol{e}$ such that $\omega'(t) = \gamma^2(t)$. On the otherhand, spatial MPH curves are characterized by split quaternion polynomials. If $\omega(t) = (\alpha(t), \beta(t), \gamma(t))$ is a MPH curve in $\mathbb{R}_1^3$, the hodograph of ω is expressed with the split quaternion polynomial $\boldsymbol{T}(t) = k(t) + l(t)\boldsymbol{i} + m(t)\boldsymbol{j} + q(t)\boldsymbol{k}$ such that $\omega'(t) = \boldsymbol{T}(t)\boldsymbol{i}\boldsymbol{T}^*(t)$ [9].

Definition.3.1. For a regular polynomial curve $\omega(t)$ in $\mathbb{R}_1^3$ Minkowski space, if both $\|\omega'(t)\|_L$ and $\|\omega'(t) \times_L \omega''(t)\|_L$ are polynomials of t, then $\omega(t)$ is called a Double Minkowski Pythagorean Hodograph (DMPH) curve [12].

Theorem.3.1. Let $\omega(t)$ be a regular MPH curve given in Minkowski-Hopf map form with hyperbolic polynomials $f_1(t)$ and $f_2(t)$. $\omega(t)$ with timelike normal is a DMPH curve iff the proportionality polynomial of $f_1(t)$ and $f_2(t)$ defines a planar MPH curve.

Proof. The DMPH condition given in [12] is expressed as

$$f_1(t)f_2'(t) - f_1'(t)f_2(t) = \delta(t)\boldsymbol{\mu}^2(t) \tag{3.1}$$

where $\delta(t)$ is a real polynomial and $\mu(t) = a(t) + b(t)\boldsymbol{e}$ is a hyperbolic polynomial such that the polynomials $a(t)$ and $b(t)$ are relatively prime.

Identifying the set of hyperbolic numbers H with $\mathbb{R}_1^2$, since the right side of the equality (3.1) defines a planar MPH curve, we can consider the proportionality polynomial of $f_1(t)$ and $f_2(t)$ as the hodograph of a planar MPH curve.

Minkowski-Hopf map forms of DMPH conditions can be derived by direct calculations. Differentiating the equalities

$$\alpha'(t) = f_1(t)\overline{f_1}(t) - f_2(t)\overline{f_2}(t), \quad \beta'(t) - \gamma'(t)\boldsymbol{e} = 2f_1(t)\overline{f_2}(t)$$

we get

$$\alpha''(t) = f_1'(t)\overline{f_1}(t) + f_1(t)\overline{f_1}'(t) - f_2'(t)\overline{f_2}(t) - f_2(t)\overline{f_2}'(t)$$

and

$$\beta''(t) - \gamma''(t)\boldsymbol{e} = 2\left(f_1'(t)\overline{f_2}(t) + f_1(t)\overline{f_2}'(t)\right).$$

Then, we can write

$$\beta'(t)\gamma''(t) - \beta''(t)\gamma'(t) = \frac{1}{2}\boldsymbol{e}[(\beta'(t) - \gamma'(t)\boldsymbol{e})(\beta''(t) + \gamma''(t)\boldsymbol{e}) + (\beta'(t) + \gamma'(t)\boldsymbol{e})(\beta''(t) - \gamma''(t)\boldsymbol{e})$$

and

$$(\beta'(t)x''(t) - \beta''(t)\gamma'(t)) + (\gamma'(t)\alpha''(t) - \gamma''(t)\alpha'(t))\boldsymbol{e} = \alpha''(t)(\beta'(t) + \gamma'(t)\boldsymbol{e}) - \alpha'(t)(\beta''(t) + \gamma''(t)\boldsymbol{e}).$$

Denoting

$$\eta(t) = f_1(t)f_2'(t) - f'_1(t)f_2(t),$$

we find

$$\beta'(t)\gamma''(t) - \beta''(t)\gamma'(t) = 2\boldsymbol{e}(\overline{f_1}(t)\overline{f_2}(t)\eta(t) - f_1(t)f_2(t)\bar{\eta}(t))$$

and

$$(\beta'(t)\alpha''(t) - \beta''(t)\gamma'(t)) + (\gamma'(t)\alpha''(t) - \gamma''(t)\alpha'(t))\boldsymbol{e} = -2\left(\overline{f_1}^2(t)\eta(t) + f_2^2(t)\bar{\eta}(t)\right).$$

Thus, if N is timelike,

$$\|\omega' \times_L \omega''\|_L^2 = \left(\beta'(t)\gamma''(t) - \beta''(t)\gamma'(t)\right)^2 + \left(\gamma'(t)\alpha''(t) - \gamma''(t)\alpha'(t)\right)^2 - \left(\beta'(t)\alpha''(t) - \beta''(t)\alpha'(t)\right)^2$$

$$= -4\sigma^2(t)|\eta(t)|$$

we obtain

$$\rho(t) = -4|\eta(t)| = 4|\boldsymbol{e}\eta(t)|.$$

If N is spacelike, we obtain

$$\rho(t) = 4|\eta(t)|.$$

Example.3.1. Consider the curve

$$\omega(t) = \left(t, \frac{2}{3}t^3, -t^2\right)$$

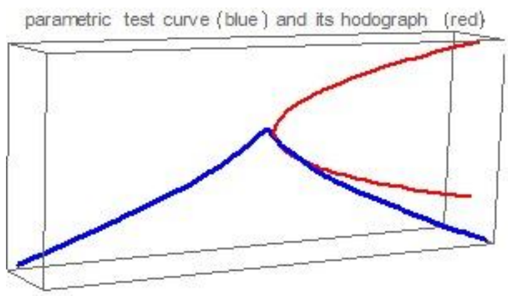
in $\mathbb{R}_1^3$ Minkowski space. The curve $\omega(t)$ is a DMPH curve and Frenet vectors and curvatures of $\omega(t)$ are

$$T(t) = \frac{1}{2t^2 - 1}(1, 2t^2, -2t),$$
$$N(t) = \frac{2}{(2t^2 - 1)^2}(-2t(t^2 - 1), 2t^3 - t, 1 - 2t^4),$$
$$B(t) = \frac{2}{2t^2 - 1}(t^2, 1, -t)$$

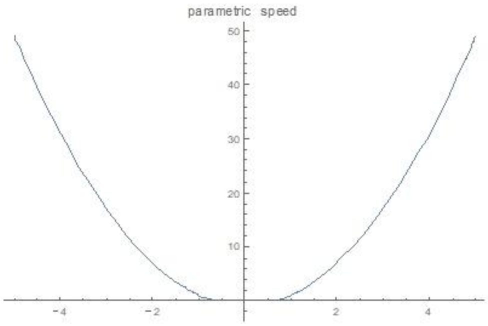
and

$$\kappa(t) = \frac{2}{(2t^2 - 1)^2}, \tau(t) = \frac{2}{(2t^2 - 1)^2}.$$

Since $\langle\omega'(t), \omega'(t)\rangle_L = (2t^2 - 1)^2$, $\omega(t)$ is a spacelike curve for $t \neq \pm\frac{1}{\sqrt{2}}$. Hence, for all $t \in \mathbb{R} \setminus (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $\|\omega'(t)\|_L = 2t^2 - 1$, so $\omega(t)$ is a MPH curve. On the other hand, since $\langle\omega'(t) \times_L \omega''(t), \omega'(t) \times_L \omega''(t)\rangle_L = 4(2t^2 - 1)^2$ for $t \neq \pm\frac{1}{\sqrt{2}}$, $\omega'(t) \times_L \omega''(t)$ and so $B(t)$ is spacelike and $N(t)$ is timelike. Hence, for all $t \in \mathbb{R} \setminus (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $\|\omega'(t) \times_L \omega''(t)\|_L = 2(2t^2 - 1)$, so $\omega(t)$ is a DMPH curve.



(1.a)



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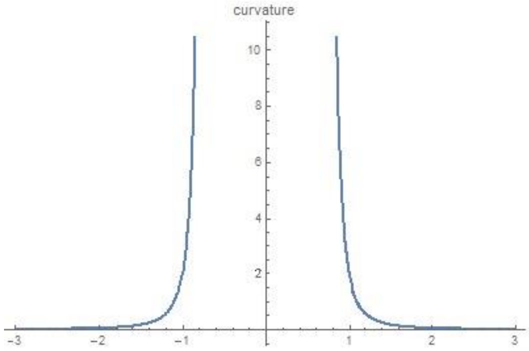


(1.c)

Figure 1. DMPH curve interpolator

Example.3.2. Consider the curve

$$\omega(t)=\left(\frac{3}{2}\left(\frac{t^3}{3}-t\right),\frac{3}{2}t^2,\frac{1}{2}\left(\frac{t^3}{3}+t\right)\right)$$

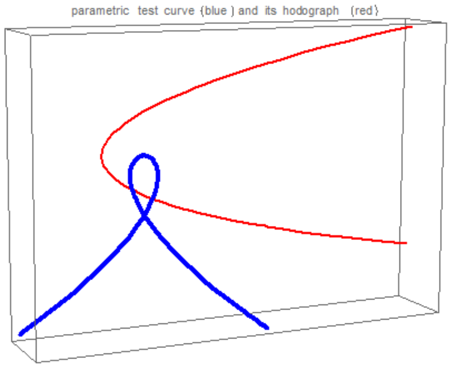
in $\mathbb{R}_1^3$ Minkowski space. The curve $\omega(t)$ is a DMPH curve and Frenet vectors and curvatures of $\omega(t)$ are

$$T(t)=\frac{1}{2\sqrt{2}(t^2+1)}(3t^2-3,6t,t^2+1),$$
$$N(t)=\left(\frac{2t}{t^2+1},-\frac{t^2+1}{t^2-1},0\right),$$
$$B(t)=\frac{1}{2\sqrt{2}(t^2+1)}(1-t^2,-2t,-3(t^2+1)),$$

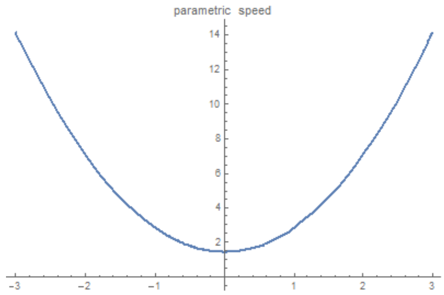
and

$$\kappa(t)=\frac{3}{2(t^2+1)^2},\tau(t)=\frac{1}{2(t^2+1)^2}.$$

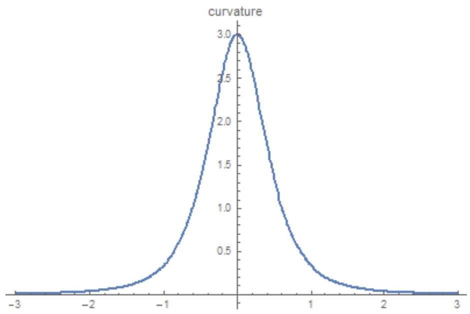
Since $\langle\omega'(t),\omega'(t)\rangle_L=2(t^2+1)^2$, $\omega(t)$ is a spacelike curve for every $t\in\mathbb{R}$. Hence, for every $t\in\mathbb{R}$, $\|\omega'(t)\|_L=\sqrt{2}(t^2+1)$, so $\omega(t)$ is a MPH curve. On the other hand, since $\langle\omega'(t)\times_L\omega''(t),\omega'(t)\times_L\omega''(t)\rangle_L=-18(t^2+1)^2$ for all $t\in\mathbb{R}$, $\omega'(t)\times_L\omega''(t)$ and so $B(t)$ is timelike and $N(t)$ is spacelike. Hence, for all $t\in\mathbb{R}$, $\|\omega'(t)\times_L\omega''(t)\|_L=3\sqrt{2}(t^2+1)$, so $\omega(t)$ is a DMPH curve.



(2.a)



(2.b)



(2.c)

Figure 2. DMPH curve interpolator

Let $\omega(t)$ be a seventh degree MPH curve constructed by a cubic split quaternion polynomial $\boldsymbol{T}(t)=k(t)+l(t)\boldsymbol{i}+m(t)\boldsymbol{j}+q(t)\boldsymbol{k}$ in normal form. For $i=0,1,2$, $k_i,l_i,m_i,q_i\in\mathbb{R}$ and we can write

$$k(t)=t^3+k_2t^2+k_1t+k_0, l(t)=l_2t^2+l_1t+l_0,$$
$$m(t)=m_2t^2+m_1t+m_0, q(t)=q_2t^2+q_1t+q_0.$$

We suppose that $|\boldsymbol{T}(t)| \neq 0$, since we study non-null MPH curves. A MPH curve of degree 7 is of type (3,0) iff

$$l_1 = l_2 = 0,$$
$$3l_0 = m_2q_1 - m_1q_2,$$
$$3m_0 = m_1k_2 - m_2k_1,$$
$$3q_0 = k_2q_1 - k_1q_2.$$

If $\alpha(t)$ is a non-planar MPH space curve of type (3,0), then

$$l_1 = l_2 = 0, l_0 = \frac{1}{3}(m_2q_1 - m_1q_2), k_2 = 3\frac{m_2q_0 - m_0q_2}{m_2q_1 - m_1q_2}, k_1 = 3\frac{m_1q_0 - m_0q_1}{m_2q_1 - m_1q_2}, [12].$$

Example.3.3. Let $k_0 = -2, m_0 = -1, m_1 = -2, m_2 = 1, q_0 = -3, q_1 = 4, q_2 = 1$. In this case, since

$$l_0 = \frac{1}{3}(m_2q_1 - m_1q_2) = 2,$$
$$k_1 = 3\frac{m_1q_0 - m_0q_1}{m_2q_1 - m_1q_2} = 5,$$
$$k_2 = 3\frac{m_2q_0 - m_0q_2}{m_2q_1 - m_1q_2} = -1$$

we find

$$k(t) = t^3 - t^2 + 5t - 2,$$
$$l(t) = 2,$$
$$m(t) = t^2 - 2t - 1,$$
$$q(t) = t^2 + 4t - 3.$$

Since

$$\alpha'(t) = k^2(t) - l^2(t) + m^2(t) - q^2(t) = t^6 - 2t^5 - 3t^4 - 2t^3 + 3t^2 - 2t,$$
$$\beta'(t) = 2[k(t)q(t) - l(t)m(t)] = 2t^5 + 2t^4 - 8t^3 - 10t^2 + 2t - 2,$$
$$\gamma'(t) = 2[k(t)m(t) - l(t)q(t)] = 2t^5 - 4t^4 - 8t,$$
$$|\boldsymbol{T}(t)| = k^2(t) - l^2(t) - m^2(t) + q^2(t) = t^6 - 2t^5 - 3t^4 + 10t^3 + 5t^2 + 2t - 2$$

and

$$[\alpha'(t)]^2 + [\beta'(t)]^2 - [\gamma'(t)]^2 = |\boldsymbol{T}(t)|^2$$

the curve

$$\omega(t) = (\frac{1}{7}t^7 - \frac{1}{3}t^6 + \frac{11}{5}t^5 - \frac{13}{2}t^4 + 7t^3 + 4t^2 - 8t, \frac{1}{3}t^6 + \frac{6}{5}t^5 - t^4 + \frac{38}{3}t^3 - 19t^2 + 16t, \frac{1}{3}t^6 - \frac{6}{5}t^5 + 3t^4 - \frac{26}{3}t^3 - 9t^2 + 16t)$$

is a non-planar MPH space curve of type (3,0) such that $\omega(0) = (0,0,0)$. For all $t \in \mathbb{R}$ such that $|\boldsymbol{T}(t)| \neq 0$, Euler-Rodrigues frame of $\omega(t)$ is obtained as

$$\boldsymbol{e}_1(t) = \Omega(t^6 - 2t^5 + 11t^4 - 26t^3 + 21t^2 + 8t - 8, 2t^5 + 6t^4 - 4t^3 + 38t^2 - 38t + 16, 2t^5 - 6t^4 + 12t^3 - 26t^2 - 18t + 16),$$
$$\boldsymbol{e}_2(t) = \Omega(-2t^5 - 6t^4 + 4t^3 - 46t^2 + 54t - 8, t^6 - 2t^5 + 9t^4 - 18t^3 + 17t^2 - 2, -2t^4 - 8t^3 + 28t^2 - 24t + 2),$$
$$\boldsymbol{e}_3(t) = \Omega(2t^5 - 6t^4 + 12t^3 - 18t^2 + 14t - 8, 2t^4 - 20t^2 - 16t + 14, t^6 - 2t^5 + 13t^4 - 10t^3 + 41t^2 - 40t + 18).$$

where $\Omega = (t^6 - 2t^5 + 11t^4 - 2t^3 + 37t^2 - 48t + 8)^{-1}$. Since $\omega(t)$ is of type (3,0), its ERF is a RRMF.

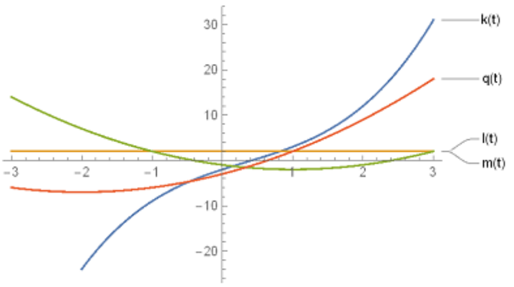


Figure 3. The condition of ERF to be RRMF

4. DMPH Curves and Type (3,0) Curves

In this study, DMPH curves are discussed as a new model of MPH curves. This model is based on rational forms of Frenet frame and curvatures of MPH curves. Considering the importance of rational representations of curves in applications, this study will also be beneficial in practice. It is obvious that by associating DMPH curves with split quaternion polynomials and Minkowski-Hopf map, it will contribute to the theory of interpolation and approximation in Minkowski 3-space, as well as physical applications. DMPH curve conditions are investigated according to the normal vector of the curve. Thus, two examples are given for timelike and spacelike cases of the normal vector.

On the other hand, type (3,0) MPH curves are examined in this work. This special type of seventh degree MPH curves is characterized by a cubic split quaternion polynomial which is in normal form. The conditions satisfied by non-planar type (3,0) curves are given and an example is constructed by means of these properties. Type (3,0) MPH curves have a rotation minimizing ERF, so ERF is more useful than the Frenet frame in applications of these curves.

Contributions of the Authors

All authors contributed equally to this manuscript.

Conflict of Interest Statement

There is no conflict of interest between the authors.

Statement of Research and Publication Ethics

The study is complied with research and publication ethics.

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