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Research Article

A Multiplicative Dual Nil Q-Clean Rings

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Abstract
In this paper our goal to thoroughly determine the rings in which each non-unit element is a product of a nilpotent and a quasi-idempotent.

Keywords: Nilpotent element, nil-clean ring, quasi-idempotent element, nil q-rings.

Çarpımsal Çift Nil Q-Temiz Halkalar

Öz
Bu çalışmada amacımız, her bir terslenir olmayan elemanı bir nilpotent ve bir yarı-eşkare elamanın çarpımı şeklinde yazılan halkaları tam olarak belirlemektir.

Anahtar Kelimeler: Nilpotent eleman, nil-temiz halka, yarı-eşkare eleman, nil q-halkalar.

1. Introduction

All rings considered in this work are associative and with non-zero identity element. $J(R)$, $E(R)$, $N(R)$, $U(R)$ and $UC(R)$ denote the Jacobson radical, the set of idempotent elements, the set of nilpotent elements, the set of unit elements and the set of central units of R respectively.

In this paper, we introduce the notion of dual nil q-clean rings via quasi-idempotent elements. A ring R is a dual nil q-clean rings if each non-unit of R is a product of a nilpotent and a quasi-idempotent. Also, we provide that a ring in which every quasi-idempotent is central, is dual nil-q-clean if and only if it is a local ring with Jacobson radical nil.

We write $\mathbb{M}_n(R)$ for the $n \times n$ matrix ring over R. For an element a in a ring R, $a^{\perp}$ (resp. ${}^{\perp}a$) denotes the right (resp. left) annihilator of a in R. Also, we freely use the terminology and basic notions of Anderson and Fuller [1].

1. Preliminaries

A ring R is called *clean* if each element of its can be written as the sum of a unit and an idempotent. Clean rings were introduced by W. K. Nicholson [6]. A ring R is called *nil-clean* if each element of its can be written as the sum of a unit and a nilpotent. Nil-clean rings were introduced by A. J. Diesl [3].

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It is known that $a \in R$ is *strongly regular* if and only if $a^2 = ua = au$ for a unit u of R. In [9] Tang, Su and Yuan introduced a generalization of the idempotent element which is also a special kind of the strongly regular element. An element q of a ring R is a *quasi-idempotent*, if $q^2 = uq$ for some central unit u of R. For a ring R, the set of quasi-idempotent elements is denoted by $QE(R)$. In [9], the authors also introduced *quasi-clean rings* as ring in which every element can be written as the sum of a unit and a quasi-idempotent.

All these notions have natural multiplicative duals. A ring is unit-regular if every element can be written as the product of a unit element and an idempotent. Clean rings can be regarded as an additive analogy of unit-regular rings. Thus, unit-regular elements and unit- regular rings are multiplicative duals of clean elements and clean rings, respectively.

Nil-clean rings have been extensively studied by many authors. (see [2], [4], [5], [7], [8]). Recently, the notation of multiplicative dual of nil-clean rings was introduced by Zhou [10]. An element a of ring R is a dual nil-clean, if $a = be$ where b is a nilpotent and e is an idempotent. A ring is called dual nil-clean if every non-unit element is dual nil-clean.

2. Main Theorem and Proof

Definition 2.1 A non-unit element a *in* R is called *dual nil q-clean* if $a = bq$ *where* $b \in N(R)$ *and* $q \in QE(R)$. A ring R is called *dual nil-q-clean* if every non-unit a in R is dual nil q-clean.

Lemma 2.2 Let R be dual nil-q-clean ring. If $a^{\perp} = 0$ or ${}^{\perp}a = 0$ then $a \in U(R)$.

Proof. Assume $a^{\perp} = 0$ and $a \notin U(R)$. Since R is a dual nil q-clean ring, $a = bq$ where $b \in N(R)$ and $q \in QE(R)$ such that $q^2 = uq$ where $u \in UC(R)$. Then $aq = bq^2 = buq = bqu = au$ and $aq - au = a(q - u) = 0$. So, $q - u \in a^{\perp}$ and hence $q = u$. So $u^{-1}a = b$ is nilpotent. Choose $n \geq 1$ such that $(u^{-1}a)^n \neq 0$ but $(u^{-1}a)^{n+1} = 0$. Hence $(u^{-1}a)^{n+1} = au^{-1}(u^{-1}a)^n = 0$ and $a(u^{-1}a)^n = 0$. So $0 \neq (u^{-1}a)^n \in a^{\perp}$, a contradiction.

Assume ${}^{\perp}a = 0$ and $a \notin U(R)$ and write $a = bq$ where $b \in N(R)$ and $q \in QE(R)$ such that $q^2 = uq$ where $u \in UC(R)$. Then $b \neq 0$. Let us $b^{n+1} = 0$ but $b^n \neq 0$. Then $b^n a = b^{n+1} q = 0$, so $0 \neq b^n \in^{\perp} a$, a contradiction.

Lemma 2.3 Dual nil-clean rings are dual nil q-clean.

Proof. It is clear since every idempotent is quasi-idempotent.

Theorem 2.4 If R is either a local ring with $J(R)$ nil or a 2×2 matrix ring over a division ring then R is dual nil-q-clean.

Proof. This obvious by [10, Theorem 2.3] and Lemma 2.3.

Theorem 2.5 Let R be a ring with every quasi-idempotent is central. R is a dual nil q-clean ring if and only if R is a local ring with $J(R)$ nil.

Proof. $(\Rightarrow:)$ Assume R be a ring with every quasi-idempotent is central. Let $a \in R$ be a non-unit and let $x \in aR$. Since every quasi-idempotent is central in R, x is a non-unit. Write $x = bq$ where $b \in N(R)$ and $q \in QE(R)$ such that $q^2 = uq$ where $u \in UC(R)$. So, $x^n = b^n u^{n-1} q$ for all $n \geqslant 1$. As b is nilpotent, x is nilpotent. Thus aR is nil and hence $a \in J(R)$.It follows that R is local with $J(R)$ nil.

$(: \Leftarrow)$ It is clear from Lemma 2.4.

Corollary 2.6 Let $n \geqslant 2$ be a fixed integer and R be a ring with every quasi-idempotent is central. Then the following are equivalent:

(1) For each non-unit $a \in R$, $a = bq$ where $b^n = 0$ and $q \in QE(R)$ such that $q^2 = uq$ where $u \in UC(R)$.

(2) R is a local ring with $j^n = 0$ for all $j \in J(R)$.

Proof. (1) $\Rightarrow$ (2) By Theorem 2.5, R is a local ring. For$j \in J(R)$, $j = bq$ where $b^n = 0$ and $q \in QE(R)$ such that $q^2 = uq$ where $u \in UC(R)$. As R is a local ring, $q = u$ or $q = 0$. Then $j^n = b^n u^n$ and $(u^n)^{-1} j^n = b^n$. So, $(u^n)^{-1} j^n = 0$ and it follows $j^n = 0$

(2) $\Rightarrow$ (1) It is clear from Theorem 2.5.

Corollary 2.7 Let $n \geqslant 2$ be a fixed integer and R be a ring. If R either is a local ring with $j^n = 0$ for all $j \in J(R)$ or the 2×2 matrix ring over a division ring then for each non-unit $a \in R$, $a = bq$ where $b^n = 0$ and $q \in QE(R)$ such that $q^2 = uq$ where $u \in UC(R)$.

Proof. This obvious by [10, Corollary 2.4] and Theorem 2.4.

By Theorem 2.5, for a ring R with every quasi-idempotent is central, each element of R is product of a nilpotent and a quasi-idempotent if and only if R is product of a quasi-idempotent and a nilpotent. The following example, we show that an element a in a ring R such that $a = bq$ where b is nilpotent and q is quasi-idempotent but $a \neq sc$ for any nilpotent c and any quasi-idempotent s in R.

Example 2.8 Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 4\mathbb{Z} & \mathbb{Z} \end{pmatrix}$ and $A = \begin{pmatrix} 4 & 2 \\ 0 & 0 \end{pmatrix}$. Then $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 4 & 2 \\ -8 & -4 \end{pmatrix}$ where $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is a quasi-idempotent and $\begin{pmatrix} 4 & 2 \\ -8 & -4 \end{pmatrix}$ is a nilpotent. Assume that $A = BQ$ where $B \in R$ is nilpotent and Q is a quasi-idempotent such that $Q^2 = UQ$ where U is central unit in R. We can choose $Q = \begin{pmatrix} a & b \\ c & 1-a \end{pmatrix}$ where $bc = a - a^2$ since $Q^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} Q$. Thus $UA = AQ = \begin{pmatrix} 4a + 2c & 4b + 2 - 2a \\ 0 & 0 \end{pmatrix}$ where $U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and it follows $4a + 2c = 4$ and $4b + 2 - 2a = 2$. Hence $a = 2b$ and $c = 2 - 4b$. As $c \in 4\mathbb{Z}$, we conclude that $2 = c + 4b$ divided by 4. This is a contradiction.

3. Conclusion

In this paper, we introduced, the concept of dual nil q-clean rings through quasi-idempotent elements. We proved that a ring in which every quasi-idempotent is central is a dual nil-q-clean ring if and only if it is a local ring with the Jacobson radical nil. Also, let $n \geqslant 2$ be a fixed integer and R be a ring with every quasi-idempotent is central. We showed that, R is a local ring with $j^n = 0$ for all $j \in J(R)$ if and only if every non-unit element of R can be written product of a nilpotent element and a quasi-idempotent.

Ethics in Publishing

There are no ethical issues regarding the publication of this study.

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