Fundamentals of Contemporary Mathematical Sciences
doi:10.54974/fcmathsci.1083724
(**2023**) **4**(1) 1 – 8
Research Article

Exact Solutions of Nonlinear Time Fractional Schrödinger Equation with Beta-Derivative

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Received: 06 March 2022 **Accepted:** 19 October 2022

Abstract: This article consists of Improved Bernoulli Sub-Equation Function Method (IBSEFM) to get the new solutions of nonlinear fractional Schrödinger equation described by beta-derivative. Foremost, it is dealt with derivative of Atangana. Secondly, basic properties of the IBSEFM are given. Finally, the proposed method has been applicated to the considered equation to get its new solutions. Moreover, the graphs of the obtained solutions are plotted via Mathematica. It is inferred from the results that IBSEFM is effectual technique for new solutions of nonlinear equations containing conformable derivatives.

Keywords: Atangana derivative, IBSEFM, Schrödinger equation

1. Introduction

Fractional equations are useful tool to determine numerous nonlinear phenomena of physics such as chaotic systems, heat transmission, diffusion, acoustic waves, viscoelasticity, plasma waves [12–17]. Lots of fractional operators have been defined, for instance: Riemann-Liouville, Caputo derivative [19], Caputo-Fabrizio [9], Jumarie's modified Riemann-Liouville [13], Atangana-Baleanu [4]. By the aid of these derivative operators, lots of techniques have been advanced which supply analytical solutions of fractional equations such as generalized Kudryashov [11], extended direct algebraic [20], IBSEFM [5, 6], modified trial equation method [18].

In [14] the definiton of conformable derivative is given and then using this derivative exact solutions of the time-heat differential equation have been investigated in [10]. In addition to this, a new definition of fractional derivative called beta-derivative is obtained in [4]. Several analytical methods are improved to get the exact solutions of fractional equations with beta-conformable time derivative [22–24].

The aim of this study is to get the exact solutions of nonlinear time fractional Schrödinger

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2020 *AMS Mathematics Subject Classification*: 35C08, 34K20, 32W50

equation with beta-derivative using IBSEFM. Before the solution process we will give the basic properties of Atangana's conformable derivative and fundamental steps of proposed method in the rest of the paper.

2. Beta Derivative and It's Specifications

This section contains some essential concepts of beta derivative that have been utilized in this work.

Let $f(t)$ be a function defined for all non-negative t. Then, β-derivative of $f(t)$ of order β is given by in [1, 4] as

$$D^{\beta}\left(f(t)\right) = \frac{d^{\beta}f(t)}{dt^{\beta}} = \lim_{\varepsilon\to 0}\frac{f\left(t+\varepsilon\left(t+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\right)-f(t)}{\varepsilon},$$

where $0<\beta\leq 1$. In fractional calculus, the β-derivative is known as the generalization of classical derivative and it's characteristics properties have been given in [1, 4]. Suppose that $u(t)$ and $v(t)$ are β-differentiable functions for all $t>0$ and $\beta\in(0,1]$. Then

$i)$ $D^{\beta}\left(af(t)+bg(t)\right) = aD^{\beta}\left(f(t)\right)+bD^{\beta}\left(g(t)\right)\ (\forall\ a,b\in\mathbb{R})$,

$ii)$ $D^{\beta}\left(f(t)g(t)\right) = g(t)D^{\beta}\left(f(t)\right)+f(t)D^{\beta}\left(g(t)\right)$,

$iii)$ $D^{\beta}\left(\frac{f(t)}{g(t)}\right) = \frac{g(t)D^{\beta}(f(t))-f(t)D^{\beta}(g(t))}{(g(t))^2}$,

$iv)$ $D^{\beta}\left(f(t)\right) = \left(t+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\frac{df(t)}{dt}$.

It should be noted that these properties provide us an easy way to convert a nonlinear partial differential equation with β-derivative to a nonlinear ordinary differential equation of integer-order. There are many works with β-derivative in literature [2, 3].

3. Description of The Proposed Method

In this part, the fundamental properties of IBSEFM is given ([6–8]). There are five main steps of the IBSEFM below the following:

Step 1: Let us consider following equation with beta derivative for a function according to the two variables space x and time t;

$$P(u_0^A, D_t^{\beta}u, u_x, u_x x, \ldots) = 0, \tag{1}$$

here P involves $u(x,t)$ and partial derivatives. The goal is to exchange (1) to nonlinear ordinary differential equation with a suitable wave transformation as

$$u(x,t) = V(\eta),\quad \eta = mx-\ \frac{\gamma}{\beta}\left(t+\frac{1}{\Gamma\left(\beta\right)}\right)^{\beta}, \tag{2}$$

m and γ are arbitrary constants. Using (2), (1) turns into the ordinary differential equation in the form

$$N(V, V', V'', \ldots) = 0, \tag{3}$$

where N is the function of $V, V', V'', \ldots$ and its derivatives with respect to η. Integrating (3) term to term, we acquire integration constants which may be determined then.

Step 2: We hypothesize that the solution of (3) may be presented below;

$$V(\eta) = \frac{\sum_{i=0}^{n} a_i Q^i(\eta)}{\sum_{j=0}^{m} b_j Q^j(\eta)} = \frac{a_0 + a_1 Q(\eta) + a_2 Q^2(\eta) + \ldots + a_n Q^n(\eta)}{b_0 + b_1 Q(\eta) + b_2 Q^2(\eta) + \ldots + b_m Q^m(\eta)}, \tag{4}$$

where $a_0, a_1, \ldots, a_n$ and $b_0, b_1, \ldots, b_m$ are coefficients which will be determined later. $m \neq 0, n \neq 0$ are chosen arbitrary according to the balance principle and considering the form of Bernoulli differential equation below the following;

$$Q'(\eta) = \sigma Q(\eta) + d Q^M(\eta), \; d \neq 0, \sigma \neq 0, \; M \in \mathbb{R} \setminus \{0, 1, 2\}, \tag{5}$$

here $Q(\eta)$ is a polynomial.

Step 3: The positive integer m, n, M (are different from zero) are found respect to the balance principle that is both nonlinear term and the highest order derivative term of (3). Substituting (4) and (5) into (3) an equation of polynomial $\Omega(Q)$ of Q is acquired below the following;

$$\Omega(Q(\eta)) = \alpha_s Q(\eta)^s + \ldots + \alpha_1 Q(\eta) + \alpha_0 = 0,$$

where α_i are coefficients that will be determined later.

Step 4: The coefficients of $\Omega(Q(\eta))$ which will give us an algebraic equations systems;

$$\alpha_i = 0, i = 0, \ldots, s.$$

Step 5: When we solve (5), we get the following two cases with respect to σ and d,

$$Q(\eta) = \left[\frac{-d e^{\sigma(\epsilon - 1)\eta} + \epsilon\sigma}{\sigma e^{\sigma(\epsilon - 1)\eta}}\right]^{\frac{1}{1-\epsilon}}, \quad d \neq \sigma, \tag{6}$$

$$Q(\eta) = \left[\frac{(\epsilon - 1) + (\epsilon + 1)\tanh\left(\sigma(1-\epsilon)\frac{\eta}{2}\right)}{1 - \tanh\left(\sigma(1-\epsilon)\frac{\eta}{2}\right)}\right], \; d = \sigma, \; \epsilon \in \mathbb{R}. \tag{7}$$

Using a complete discrimination system for polynomial of $Q(\eta)$, exact solutions of (1) are get via Wolfram Mathematica and categorize the exact solutions of (1). To achieve better results, 2D and 3D graphs of exact solutions might be plotted taking proper values of parameters.

4. Mathematical Analysis of The Model

Let us consider the nonlinear Schrödinger equation in β-derivative sense

$$i{}_0^A D_t^\beta u + pu_{xx} + q\left|u\right|^2 u = 0, \quad 0 < \beta \leq 1 \tag{8}$$

and apply the transformation

$$u(x,t) = e^{i\theta}U(\xi), \; \theta = \tau x + \frac{\lambda}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^\beta, \quad \xi = x - \frac{2r\lambda}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^\beta. \tag{9}$$

Here τ, λ and r are constants, using the basic properties of β-derivative and substituting (9) into (8), we get the following equation containing the real and imaginary part;

$$i\left[-2r\lambda\frac{dU}{d\xi} + 2p\tau\frac{dU}{d\xi}\right] + p\frac{d^2U}{d\xi^2} - \left(\lambda + p\tau^2\right)U + qU^3 = 0. \tag{10}$$

From the imaginary part of (10), $r = \frac{p\tau}{\lambda}$. Moreover, the real part of (10) is

$$pU'' - \left(\lambda + p\tau^2\right)U + qU^3 = 0. \tag{11}$$

When we reconsider (11) for balance principle between U'' and U^3, we get the relationship as follow;

$$M = n - m + 1. \tag{12}$$

(12) shows us the different cases of the solutions of (11) and we can obtain some analytical solutions. According to the balance, we consider $M = 3, m = 1, n = 3$ for (12) and the following equations hold:

$$U(\xi) = \frac{a_0 + a_1Q(\xi) + a_2Q^2(\xi) + a_3Q^3(\xi)}{b_0 + b_1Q(\xi)} \equiv \frac{\Upsilon(\xi)}{\Psi(\xi)}, \tag{13}$$

$$U'(\xi) = \frac{\Upsilon'(\xi)\Psi(\xi) - \Upsilon(\xi)\Psi'(\xi)}{\Psi^2(\xi)} \tag{14}$$

and

$$U''(\xi) = \frac{\Upsilon'(\xi)\Psi(\xi) - \Upsilon(\xi)\Psi'(\xi)}{\Psi^2(\xi)} - \frac{\left[\Upsilon(\xi)\Psi'(\xi)\right]'\Psi^2(\xi) - 2\Upsilon(\xi)[\Psi'(\xi)]^2\Psi(\xi)}{\Psi^4(\xi)}, \tag{15}$$

where $Q' = \sigma Q + dQ^3$, $a_3 \neq 0$, $b_1 \neq 0$, $\sigma \neq 0$, $d \neq 0$. Using (13)-(15) in (11), we get from coefficients of polynomial of Q as follow;

$$Q^0 : qa_0^3 - \lambda a_0b_0^2 - p\tau^2a_0b_0^2 = 0,$$

$$Q^1 : 3qa_0^2a_1 - \lambda a_1b_0^2 + p\sigma^2a_1b_0^2 - p\tau^2a_1b_0^2 - 2\lambda a_0b_0b_1 - p\sigma^2a_0b_0b_1 - 2p\tau^2a_0b_0b_1 = 0,$$

$$\vdots$$

$$Q^7 : 3qa_3a_2^2 + 3qa_1a_3^2 + 15d^2pa_3b_0^2 + 9d^2pa_2b_0b_1 + 12dp\sigma a_3b_1^2 = 0,$$

$$Q^8 : 3qa_2a_3^2 + 21d^2pa_3b_0b_1 + 3d^2pa_2b_1^2 = 0,$$

$$Q^9 : qa_3^3 + 8d^2pa_3b_1^2 = 0.$$

Solving above the equation system of Q via Mathematica, the coefficients are obtained for $\sigma \neq d$:

Family1.

$$a_0 = -\frac{i\sqrt{2}\sqrt{p}\sigma b_0}{\sqrt{q}},\ a_1 = -\frac{i\sqrt{2}\sqrt{p}\sigma b_1}{\sqrt{q}}, a_2 = -\frac{2i\sqrt{2}d\sqrt{p}b_0}{\sqrt{q}}, a_3 = -\frac{2i\sqrt{2}d\sqrt{p}b_1}{\sqrt{q}},\ \tau = -\frac{\sqrt{-\lambda - 2p\sigma^2}}{\sqrt{p}}.$$

Substituting these coefficients along with (7) in (13), we obtain the following solution of (8) as follows;

$$q_1(x,t) = \frac{-1}{2}\exp\left\{-\frac{x\sqrt{-\lambda - 2p\sigma^2}}{\sqrt{p}} + \frac{\lambda}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta}\right\}\exp\left\{-2x\sigma + \frac{4rl\sigma\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta}}{\beta}\epsilon - \frac{d}{\sigma}\right\}.$$

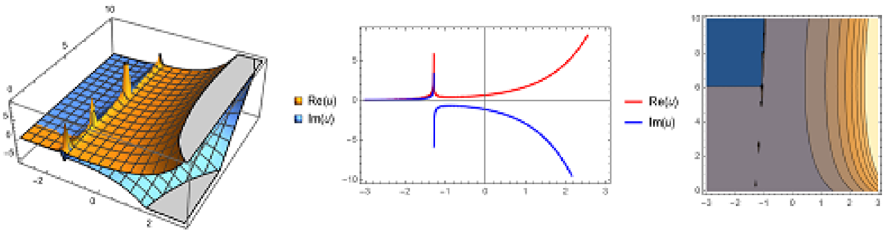

Figure 1: 3D-plots of $q_1(x,t)$ for the values $\beta = 0.5;\ d = 0.4;\ r = 0.1;\ \epsilon = 0.2;\ \lambda = 0.3;\ \sigma = 0.5;\ p = 0.3;\ t = 0.4;\ -3 < x < 3,\ 0 < t < 10$, 2D-plots and contoursurfaces

Family2. For $\sigma \neq d$,

$$a_0 = -\frac{i\sqrt{-\lambda - p\tau^2}b_0}{\sqrt{q}},\ a_1 = -\frac{i\sqrt{-\lambda - p\tau^2}b_1}{\sqrt{q}},\ a_2 = \frac{2i\sqrt{2}d\sqrt{p}b_0}{\sqrt{q}}, a_3 = \frac{2i\sqrt{2}d\sqrt{p}b_1}{\sqrt{q}},\ \sigma = \frac{\sqrt{-\lambda - p\tau^2}}{\sqrt{2}\sqrt{p}}.$$

Substituting these coefficients along with (7) in (13), we obtain the following solution of (8) as follows;

$$q_2(x,t) = \frac{\exp\left\{ix\tau + \frac{i\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right\}\left(\lambda+p\tau^2\right)\left(2d^2\exp\left\{-\frac{2\sqrt{2}\sqrt{-\lambda-p\tau^2}\left(x-\frac{2r\lambda\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}}{\beta}\right)}{\sqrt{p}}\right\}p+\epsilon^2\left(\lambda+p\tau^2\right)\right)}{\sqrt{\exp\left\{-2x\sigma+\frac{4rl\sigma\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}}{\beta}\epsilon-\frac{d}{\sigma}\right\}}}.$$

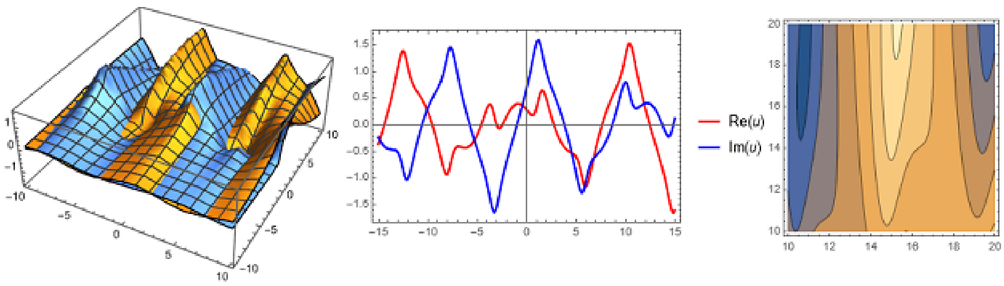

Figure 2: 3D- plots of $q_2(x,t)$ for the values $\beta = 0.5$; $d = 0.4$; $r = 0.1$; $\epsilon = 0.2$; $\lambda = 0.3$; $\sigma = 0.5$; $p = 0.3$; $t = 0.4$; $-10 < x < 10$, $-10 < t < 10$, 2D-plots and contoursurfaces

We can understand the characteristics of the solutions from the figures that for a few parameter values, the displayed numerical analysis acknowledges that the solutions are periodic wave shapes in exponential classes. According to the figures, one can see that the formats of exact solutions in two and three dimensional surfaces are similar to the physical meaning of results.

5. Conclusion

In this paper, the IBSEFM is applied for fractional Schrödinger equation in β-derivative. Using wave transformation the considered equation has been converted into the ordinary differential equation which can be solved according to the IBSEFM. By means of this method, exact solutions are obtained. Figures of all solutions according to the suitable parameters are plotted by showing the main characteristic physical properties of the solutions with the help of Wolfram Mathematica. It seems from the results that the more steps are developed and the better approximations are obtained. It is inferred from the conclusions that IBSEFM is simple, effective and powerful. Thus, in mathematical physics it is applicable to solve other nonlinear differential equations.

Declaration of Ethical Standards

The author declares that the materials and methods used in their study do not require ethical committee and/or legal special permission.

Conflict of Interest

The author declares no conflicts of interest.

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