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A NONLINEAR TRANSFORMATION BETWEEN SPACE CURVES DEFINED BY CURVATURE-TORSION RELATIONS IN 3-DIMENSIONAL EUCLIDEAN SPACE

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Abstract. In this paper, we define a nonlinear transformation between space curves which preserves the ratio of τ/κ of the given curve in $3-$dimensional Euclidean space $\mathbb{E}^3$. We investigate invariant and associated curves of this transformation by the help of curvature and torsion functions of the base curve. Moreover, we define a new curve (family) so-called *quasi-slant helix,* and we obtain some characterizations in terms of the curvatures of this curve. Finally, we examine some curves in the kinematics, and give the pictures of some special curves and their images with respect to the transformation.

1. Introduction

In differential geometry, the curvature function $\kappa(s)$which describes the measure of the deviation from the line, and the torsion function $\tau(s)$ which describes the measure of the deviation from the plane, are known as the *natural* or *intrinsic* equations of a curve. It is a well-known fact that these functions are unique for all space curves (fundamental theorem of space curves) [21]. A lot of special curves have been characterized in terms of the curvature functions in Euclidean space, see [3, 5–9, 11–16, 19–22]. Perhaps the most well-known of these is the helix curve. A curve of constant slope or helix is defined by the property that its tangent vector field makes a constant angle with a fixed direction. The fixed direction is called the axis of the helical curve. A classical result stated by M.A. Lancret in 1802 and first proved by B. de Saint Venant in 1845 is: "A necessary and sufficient condition in order to a curve be a helix is that the ratio of curvature to torsion be constant" [21].

Slant helices, which were previously studied by some mathematicians but were firstly characterized by Izumiya and Takeuchi [9] in 2004, can be considered as a generalization of the helix curve. Similar to tangent vectors of the helices, the

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normal vector field of the slant helices makes a constant angle with a fixed direction. The intrinsic equations and tangent vector field of slant helices expressed by Menninger, see [13]. In [6], slant helices with nonzero geodesic curvature of normal indicatrix ($\sigma(s) \neq 0$) are called as *proper slant helix* and the intrinsic equations of these curves calculated in the light of [11]. Moreover, in [22], the authors defined the alternative frame to these curves, and obtained the characterization of some special slant helices with the help of the same frame.

The intrinsic equations of some pair of curves that have been widely studied in the past, such as Bertrand and Mannheim curve pairs, continue to be studied by using different frames in different spaces, see [2, 8, 12, 15, 16].

In kinematics, a curve called as "osculating helix" associate to each space curve in Euclidean space. Osculating helix can be expressed with the help of two functions so-called *radius* (r) and *pitch function* (p). These functions are given by

$$r(s) = \kappa(s)/(\kappa^2(s) + \tau^2(s)),$$

and

$$p(s) = \tau(s)/(\kappa^2(s) + \tau^2(s)).$$

see [2]. Notice that, these functions can be defined for any space curve, and the radius function of Mannheim curve is constant. In contrast to Mannheim curves, the curves whose curvatures are given by $\tau(s)/\left(\kappa^2(s) + \tau^2(s)\right) = \mu$, $\mu \in \mathbb{R}$, are not encountered in the literature frequently. These curves called as *constant pitch curve* in [2] . Selig and Carricato [20] expressed that any Frenet-Serret motion is persistent on a space curve in Euclidean space if and only if the pitch function of the curve is constant. In addition, they stated the curvature functions of constant pitch curve in trigonometric and rational form. For Frenet-Serret motion on the slant helix, see [10].

In this study, we define a nonlinear transformation between space curves with the help of radius and pitch functions (Definition 2). We give the algebraic structure of this transformation by Theorem 2. Moreover, we show that the only slant helix which remains invariant under this transformation is a constant precession curve (Theorem 3). In Section 4.2, we investigate evolute of the image curve of this transformation, and we define a *new curve* (family) by Definition 3. In the last section, we illustrate some of curves and their images, under the mentioned transformation.

2. Preliminaries

Let $\mathbb{E}^3$ be the 3−dimensional Euclidean space equipped with the inner product $\boldsymbol{u} \cdot \boldsymbol{v} = u_1v_1 + u_2v_2 + u_3v_3$, where $\boldsymbol{u} = (u_1, u_2, u_3)$ and $\boldsymbol{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$. The norm of $\boldsymbol{u}$ is given by $|\boldsymbol{u}| = \sqrt{\boldsymbol{u} \cdot \boldsymbol{u}}$ and the vector product is given by

$$\boldsymbol{u} \times \boldsymbol{v} = \mathbf{det} \begin{pmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix},$$

where $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ is the standard basis of $\mathbb{E}^3$.

Let $\alpha : I \to \mathbb{E}^3$ be a differentiable curve parameterized by an arbitrary parameter t and $\alpha'(t) \times \alpha''(t) \neq 0$, where $\alpha'(t) = (d\alpha/dt)(t)$ and I is an open interval. At each point of the curve, there exists a moving frame $\{\boldsymbol{t}(t), \boldsymbol{n}(t), \boldsymbol{b}(t)\}$ associated to the curve which is defined by

$$\boldsymbol{t}(t) = \frac{\alpha'(t)}{|\alpha'(t)|}, \qquad \boldsymbol{n}(t) = \boldsymbol{b}(t) \times \boldsymbol{t}(t), \qquad \boldsymbol{b}(t) = \frac{\alpha'(t) \times \alpha''(t)}{|\alpha'(t) \times \alpha''(t)|},$$

where $\boldsymbol{t}$ is the tangent vector, $\boldsymbol{n}$ is the principal normal vector and $\boldsymbol{b}$ is the binormal vector of the curve α. The Frenet-Serret formula of this curve is given by

$$\begin{pmatrix} \boldsymbol{t}'(t) \\ \boldsymbol{n}'(t) \\ \boldsymbol{b}'(t) \end{pmatrix} = \begin{pmatrix} 0 & |\alpha'(t)|\,\kappa(t) & 0 \\ -|\alpha'(t)|\,\kappa(t) & 0 & |\alpha'(t)|\,\tau(t) \\ 0 & -|\alpha'(t)|\,\tau(t) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{t}(t) \\ \boldsymbol{n}(t) \\ \boldsymbol{b}(t) \end{pmatrix},$$

where

$$\kappa(t) = \frac{|\alpha'(t) \times \alpha''(t)|}{|\alpha'(t)|^3}, \qquad \tau(t) = \frac{(\alpha'(t) \times \alpha''(t)) \cdot \alpha'''(t)}{|\alpha'(t) \times \alpha''(t)|^2}.$$

Moreover, we say that α is non-degenerate, or α satisfies the non-degenerate condition if $\alpha'(t) \times \alpha''(t) \neq 0$ for all $t \in I$. If α is a unit speed curve, that is, $|\alpha'(s)| = 1$ for all s, then the tangent vector, the principal normal vector, and the binormal vector are given by

$$\boldsymbol{t}(s) = \alpha'(s), \qquad \boldsymbol{n}(s) = \frac{\alpha''(s)}{|\alpha''(s)|}, \qquad \boldsymbol{b}(s) = \boldsymbol{t}(s) \times \boldsymbol{n}(s).$$

Then $\{\boldsymbol{t}(s), \boldsymbol{n}(s), \boldsymbol{b}(s)\}$ is a moving frame of $\alpha(s)$ and we have the Frenet-Serret formula:

$$\begin{pmatrix} \boldsymbol{t}'(s) \\ \boldsymbol{n}'(s) \\ \boldsymbol{b}'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{t}(s) \\ \boldsymbol{n}(s) \\ \boldsymbol{b}(s) \end{pmatrix},$$

where

$$\kappa(s) = |\alpha''(s)|, \qquad \tau(s) = \frac{\det(\alpha'(s), \alpha''(s), \alpha'''(s))}{\kappa^2(s)}.$$

Let $\alpha : I \to \mathbb{E}^3$ be a regular curve (i.e., $\alpha'(s) \neq 0$). The vector $\boldsymbol{w}(s) = \tau(s)\boldsymbol{t}(s) + \kappa(s)\boldsymbol{b}(s)$ is called the *Darboux vector* of α. The normalization of the Darboux vector is defined by

$$\widetilde{\boldsymbol{w}} = \frac{\tau(s)\boldsymbol{t}(s) + \kappa(s)\boldsymbol{b}(s)}{\sqrt{\tau^2(s) + \kappa^2(s)}},$$

which is called the *Darboux indicatrix* of α.

A regular curve $\alpha : I \to \mathbb{E}^3$ with $\kappa(s) \neq 0$ is called a *Mannheim curve* if its principal normal lines are binormal lines of another curve $\widehat{\alpha}$ at corresponding points. In this case, $\widehat{\alpha}$ is called a *Mannheim mate* of α. For a space curve α, it is a Mannheim curve if and only if there exists nonzero constant λ such that

$\kappa(s) = \lambda\left(\kappa^2(s) + \tau^2(s)\right)$ for any $s \in I$. It is clear that circular helices are Mannheim curves, see [23].

The concept of the slant helix is firstly introduced by Izumiya and Takeuchi [9]. They characterized the slant helices by following proposition:

Proposition 1. *Let γ be a unit speed curve with $\kappa(s) \neq 0$. Then γ is a slant helix if and only if*

$$\sigma(s) = \left(\frac{\kappa^2}{\left(\kappa^2 + \tau^2\right)^{3/2}}\left(\frac{\tau}{\kappa}\right)'\right)(s)$$

is a constant function where, σ is geodesic curvature of the normal indicatrix of the curve.

Moreover, in recent years, some mathematicians characterized the slant helices in terms of curvatures of the curve, see [1, 6, 13]. In [6], the proper slant helices characterized by following:

Theorem 1. *A unit–speed Frenet curve $\alpha(s): I \to \mathbb{E}^3$ with Frenet–Serret apparatus $\{\kappa, \tau, \boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$ is a proper slant helix if and only if $(\tau/\kappa)(s) = f(s)/\sqrt{1 - f^2(s)}$, where $f(s) = c\int \kappa ds$ with nonzero constant c.*

Also, the following result given in [6]:

Corollary 1. *A unit speed Frenet curve $\alpha(s): I \to \mathbb{E}^3$ with curvature $\kappa = 1$ is a Salkowski curve if and only if its torsion is of the form*

$$\tau(s) = \frac{cs}{\sqrt{1 - c^2 s^2}},$$

where c is a nonzero constant.

Now we make a brief introduction to the concepts that related to the kinematics, in the light of [2] and [20].

2.1. **Osculating helix.**

Definition 1. *For any given regular arc-length parameterized space curve α with non-vanishing curvature and torsion; there exits a circular helix $\overline{\alpha}$ such that accompanying corresponding to each points of $\alpha(s_0)$ with the same curvature and torsion at that point, i.e. $\overline{\kappa}(s_0) = \kappa(s_0)$ and $\overline{\tau}(s_0) = \tau(s_0)$. The circular helix $\overline{\alpha}$ is called the osculating helix to the curve α and has the same Frenet frame as α at $\alpha(s_0)$.*

Notice that, the order of contact of $\overline{\alpha}$ and α at any point is at least two, see [21]. Moreover, the axis of the osculating helix intersects the principal normal $\boldsymbol{n}(s_0)$ of α orthogonally, at a distance of $r(s_0)$ from the corresponding point $\alpha(s_0)$. Therefore, the axis of the osculating helix of $\overline{\alpha}(s_0)$ corresponding to the point $\alpha(s_0)$ is given by

$$\boldsymbol{a}(s_0) = \alpha(s_0) + r(s_0)\,\boldsymbol{n}(s_0), \tag{1}$$

where r is the radius of the osculating helix at $\alpha(s_0)$. Here

$$r(s) = \kappa(s)/\left(\kappa^2(s) + \tau^2(s)\right),$$

where $\kappa(s)$ and $\tau(s)$ are the curvature, and the torsion function of α, respectively. The axis $\boldsymbol{a}$ is can be called as axial curve. We note that this axis corresponds to the base curve of fixed axode of any Frenet-Serret motion along α, see [20].

It can be observed from Eq. (1) that r is constant and $\boldsymbol{a}$ is the Mannheim mate of α when α is a Mannheim curve. Furthermore, the following proposition can be given:

Proposition 2. *The tangent vector field of* $\boldsymbol{a}$ *is parallel to the Darboux vector at the corresponding point of the curve* α *if and only if* α *is a Mannheim curve.*

Proof. Let the tangent vector field of $\boldsymbol{a}$ be parallel to the Darboux vector of α, i.e. $\boldsymbol{a}'(s) = \lambda(\tau\,\boldsymbol{t} + \kappa\boldsymbol{b})(s)$, where λ is constant and $\boldsymbol{a}'(s) = (d\boldsymbol{a}/ds)(s)$. From Eq. (1) we have

$$\boldsymbol{a}'(s) = (1 - r(s)\kappa(s))\,\boldsymbol{t}(s) + r'(s)\boldsymbol{n}(s) + r(s)\,\tau(s)\,\boldsymbol{b}(s).$$

Therefore,

$$\lambda(\tau\,\boldsymbol{t} + \kappa\boldsymbol{b})(s) = (1 - r(s)\,\kappa(s))\,\boldsymbol{t}(s) + r'(s)\,\boldsymbol{n}(s) + r(s)\,\tau(s)\,\boldsymbol{b}(s). \qquad (2)$$

From Eq. (2) we obtain $r'(s) = 0$ and $\frac{\tau(s)}{1 - r(s)\kappa(s)} = \frac{\kappa(s)}{r(s)\tau(s)}$. Hence,

$$r(s) = \kappa(s)/\left(\kappa^2(s) + \tau^2(s)\right)$$

is constant and α is a Mannheim curve. Conversely, if α is a Mannheim curve, then r is constant, and $\boldsymbol{a}'(s) = (1 - r\kappa(s))\,\boldsymbol{t}(s) + r\tau(s)\,\boldsymbol{b}(s)$. Since

$$r = \kappa(s)/\left(\kappa^2(s) + \tau^2(s)\right),$$

we obtain,

$$\frac{1 - r\kappa(s)}{r\tau(s)} = \frac{1 - \frac{\kappa^2(s)}{\kappa^2(s)+\tau^2(s)}}{\frac{\kappa(s)\tau(s)}{\kappa^2(s)+\tau^2(s)}} = \frac{\tau(s)}{\kappa(s)},$$

which requires the tangent vector field of $\boldsymbol{a}$ parallel to the Darboux vector of α. □

Any circular helix is determined by its pitch function $p(s)$ and the radius function $r(s)$ of the cylinder on which it lies where, $p(s) = \tau(s)/\left(\kappa^2(s) + \tau^2(s)\right)$ and $r(s) = \kappa(s)/\left(\kappa^2(s) + \tau^2(s)\right)$. Osculating helix is also determined by the functions $p(s)$ and $r(s)$. By the help of this idea, the pitch and radius functions of any space curve can be defined. Notice that, for circular helix, both radius and pitch function are constant, and Mannheim curves have constant radius function. The pitch function confront in the kinematics, in particular, in persistent rigid body motions. It is well-known that, Frenet-Serret motions are persistent along space curve $\alpha(s) : I \to \mathbb{E}^n$ if and only if the pitch function of the curve is constant for all $s \in I$, see [20].

Remark 1. *The pitch function of space curves corresponds to the distribution parameter of the ruled surfaces whose rulings are the normal vector of the base curve.*

Curves with constant pitch function are called as *constant pitch curve* in [2]. Constant pitch curves can be seen as one dimensional realisations of persistent sub-manifolds. The curvature and torsion functions of this curve can be given by

$$\kappa(\theta) = \frac{1}{2p}\cos\theta; \qquad \tau(\theta) = \frac{1}{2p}(1+\sin\theta),$$

where θ is the parameter and $p = \tau(\theta) / \left(\kappa^2(\theta) + \tau^2(\theta)\right)$ is constant. The curvature and torsion functions can also be parameterized by the rational functions,

$$\kappa(s) = \frac{1-s^2}{2p(1+s^2)}; \qquad \tau(s) = \frac{(1+s)^2}{2p(1+s^2)}.$$

see [20].

2.2. **Fixed axode.** In [20], the fixed axode of a Frenet-Serret motion is given by

$$\boldsymbol{a}(s,\lambda) = \alpha(s) + \frac{\kappa(s)}{\kappa^2(s)+\tau^2(s)}\boldsymbol{n}(s) + \lambda\boldsymbol{w}(s), \tag{3}$$

where $\boldsymbol{w}$ is the Darboux vector and $\boldsymbol{n}$ is the principal normal of α. It is obvious that $\boldsymbol{a}$ is a ruled surface with base curve $\alpha(s) + \left(\kappa(s)/\left(\kappa^2(s)+\tau^2(s)\right)\right)\boldsymbol{n}(s)$ and rulings $\boldsymbol{w}(s)$. As stated in [20], $\boldsymbol{a}(s,\lambda)$ is not developable in general. On the other hand, we give the elementary proof of the following proposition:

Proposition 3. *Assume that γ is a unit speed curve and $\overline{\boldsymbol{w}}$ is the unit Darboux vector of γ. The ruled surface*

$$\overline{\boldsymbol{a}}(s,\lambda) = \gamma(s) + \frac{\kappa(s)}{\kappa^2(s)+\tau^2(s)}\boldsymbol{n}(s) + \lambda\overline{\boldsymbol{w}}(s)$$

is developable if and only if γ is a Mannheim curve.

Proof. Let $\overline{\boldsymbol{a}}$ be a developable surface. Any ruled surface in Euclidean space is developable if the distribution parameter of this surface vanishes. Moreover, the distribution parameter of ruled surface $\overline{\boldsymbol{a}}(s,\lambda)$ is given by

$$P = \frac{\det\left(\Delta'(s), \overline{\boldsymbol{w}}(s), \overline{\boldsymbol{w}}'(s)\right)}{\overline{\boldsymbol{w}}'(s)\cdot\overline{\boldsymbol{w}}'(s)},$$

where $\Delta(s) = \gamma(s) + r(s)\,\boldsymbol{n}(s)$ with $r(s) = \kappa(s)/\left(\kappa^2(s)+\tau^2(s)\right)$. Since γ is a unit speed curve and $\overline{\boldsymbol{w}}(s) = \frac{1}{\sqrt{\kappa^2(s)+\tau^2(s)}}\left(\tau(s)\,\boldsymbol{t}(s) + \kappa(s)\,\boldsymbol{b}(s)\right)$, we have $\Delta'(s) = \left(1 - r(s)\,\kappa(s)\right)\boldsymbol{t}(s) + r'(s)\,\boldsymbol{n}(s) + r(s)\,\tau(s)\,\boldsymbol{b}(s)$ and

$$\overline{\boldsymbol{w}}'(s) = \left(\frac{\tau(s)}{\sqrt{\kappa^2(s)+\tau^2(s)}}\right)'\boldsymbol{t}(s) + \left(\frac{\kappa(s)}{\sqrt{\kappa^2(s)+\tau^2(s)}}\right)'\boldsymbol{b}(s).$$

By straightforward calculations we obtain

$$\Delta'(s) \times \overline{\boldsymbol{w}}(s) = \frac{1}{\sqrt{\kappa^2(s)+\tau^2(s)}} \left\{ r'(s)\,\kappa(s)\,\boldsymbol{t}(s) + \left[r(s)\left(\kappa^2(s)+\tau^2(s)\right) - \kappa(s)\right]\boldsymbol{n}(s) - r'(s)\,\tau(s)\,\boldsymbol{b}(s) \right\}.$$

Therefore we have

$$(\Delta'(s) \times \overline{\boldsymbol{w}}(s)) \cdot \overline{\boldsymbol{w}}'(s) = \frac{r'(s)}{\sqrt{\kappa^2(s)+\tau^2(s)}} \left(\kappa(s) \left(\frac{\tau(s)}{\sqrt{\kappa^2(s)+\tau^2(s)}} \right)' - \tau(s) \left(\frac{\kappa(s)}{\sqrt{\kappa^2(s)+\tau^2(s)}} \right)' \right).$$

Since $\det(\Delta'(s), \overline{\boldsymbol{w}}(s), \overline{\boldsymbol{w}}'(s)) = (\Delta'(s) \times \overline{\boldsymbol{w}}(s)) \cdot \overline{\boldsymbol{w}}'(s)$, the distribution parameter of $\overline{\boldsymbol{a}}$ is as follows:

$$P = \frac{(\Delta'(s) \times \overline{\boldsymbol{w}}(s)) \cdot \overline{\boldsymbol{w}}'(s)}{\overline{\boldsymbol{w}}'(s) \cdot \overline{\boldsymbol{w}}'(s)} = \frac{\frac{r'(s)}{\sqrt{\kappa^2(s)+\tau^2(s)}} \left(\kappa(s) \left(\frac{\tau(s)}{\sqrt{\kappa^2(s)+\tau^2(s)}} \right)' - \tau(s) \left(\frac{\kappa(s)}{\sqrt{\kappa^2(s)+\tau^2(s)}} \right)' \right)}{\left\{ \left(\frac{\kappa(s)}{\sqrt{\kappa^2(s)+\tau^2(s)}} \right)' \right\}^2 + \left\{ \left(\frac{\tau(s)}{\sqrt{\kappa^2(s)+\tau^2(s)}} \right)' \right\}^2}. \tag{4}$$

From Eq. (4), only $r'(s) = 0$ gives the solution. Hence, $r = \kappa(s)/(\kappa^2(s)+\tau^2(s))$ is constant and γ is a Mannheim curve. Conversely, $r = \kappa(s)/\left(\kappa^2(s)+\tau^2(s)\right)$ is constant when γ is a Mannheim curve. Since $\Delta'(s) = (1 - r\kappa(s))\,\boldsymbol{t}(s) + r\tau(s)\,\boldsymbol{b}(s)$ and $\overline{\boldsymbol{w}}(s) = \frac{1}{\sqrt{\kappa^2(s)+\tau^2(s)}}(\tau(s)\,\boldsymbol{t}(s) + \kappa(s)\,\boldsymbol{b}(s))$,

$$\begin{aligned} \Delta'(s) \times \overline{\boldsymbol{w}}(s) &= [(1 - r\kappa(s))\,\boldsymbol{t}(s) + r\tau(s)\,\boldsymbol{b}(s)] \times \frac{1}{\sqrt{\kappa^2(s)+\tau^2(s)}}(\tau(s)\,\boldsymbol{t}(s) + \kappa(s)\,\boldsymbol{b}(s)) \\ &= \frac{1}{\sqrt{\kappa^2(s)+\tau^2(s)}}\left[(1 - r\kappa(s))\,\kappa(s)\,(-\boldsymbol{n}(s)) + r\tau^2(s)\,\boldsymbol{n}(s)\right] \end{aligned}$$

which requires $\Delta'(s) \times \overline{\boldsymbol{w}}(s) = 0$ and $P = 0$. Therefore, $\overline{\boldsymbol{a}}$ is developable. □

3. Geometric Properties of Transformation

3.1. **Fundamentals.** In the light of the ideas at previous section, we introduce a nonlinear transformation between space curves which preserves ratio of curvature and torsion of a given curve.

Definition 2. *Let α and $\overline{\alpha} : I \to \mathbb{E}^3$ be non-degenerate curves with curvature pair (κ, τ) and $(\overline{\kappa}, \overline{\tau})$, respectively. The map $\mathfrak{s} : \alpha \to \overline{\alpha}$, $(\kappa, \tau) \to (\overline{\kappa}, \overline{\tau})$ is called as slope preserving transformation ($\mathcal{SPT}$) such that the curvatures of $\overline{\alpha}$ are given by*

$$\overline{\kappa}(s) = \frac{\kappa(s)}{\kappa^2(s)+\tau^2(s)}; \quad \overline{\tau}(s) = \frac{\tau(s)}{\kappa^2(s)+\tau^2(s)},$$

where s is the arc-length parameter of α. Here α and $\overline{\alpha}$ are called as base curve and image curve of $\mathfrak{s}$, respectively and $(\alpha, \overline{\alpha})$ is called as $\mathcal{SPT}$ pair.

Definition of $\mathcal{SPT}$ requires $\overline{\tau}/\overline{\kappa} = \tau/\kappa$, and this explains why we called the transformation of $\mathfrak{s}$ as $\mathcal{SPT}$. Notice that, the curvature and torsion of the image curve correspond to the radius and pitch functions of the osculating helix of the base curve, respectively. It is easy to see that $\mathfrak{s}$ fixes $\left(\kappa(s)/\sqrt{\kappa^2(s)+\tau^2(s)}, \tau(s)/\sqrt{\kappa^2(s)+\tau^2(s)}\right)$. Also, the *Darboux* vectors of the $\mathcal{SPT}$ *pair* satisfy $|\boldsymbol{w}(s)|\,|\overline{\boldsymbol{w}}(s)| = 1$ i.e.,

$$\left(\kappa^2(s)+\tau^2(s)\right)\left(\overline{\kappa}^2(s)+\overline{\tau}^2(s)\right) = 1.$$

Remark 2. *The image curve has constant curvature (resp. constant torsion) when the base curve is Mannheim curve (resp. constant pitch curve).*

Remark 3. *It is also possible to define $\mathcal{SPT}$ between planar curves. Since $\tau = 0$ for planar curves, $\mathfrak{s}(\kappa(s), 0) = (1/\kappa(s), 0)$, where $\kappa(s) \neq 0$ for all $s \in I$. Therefore, $\mathcal{SPT}$ maps curvature κ to the radius of curvature of planar curves. On the other hand, because of the curvatures vanish, $\mathcal{SPT}$ is not defined for the lines. Hence, $\mathfrak{s}$ is not linear but some curves and some of geometric properties remain invariant under $\mathcal{SPT}$ as we show in the next.*

Now, we obtain a relation between the osculating helix of the image curve and the curvatures of the base curve.

Let $\mathfrak{s} : \alpha \to \overline{\alpha}$ be an $\mathcal{SPT}$, and $(\kappa_\alpha, \tau_\alpha)$ and $(\kappa_{\overline{\alpha}}, \tau_{\overline{\alpha}})$ be the curvature pairs of α and $\overline{\alpha}$ respectively. Let γ be an osculating helix of the image curve $\overline{\alpha}$. In this case, the *pitch function* $p_\gamma^{\overline{\alpha}}$ and *radius function* $r_\gamma^{\overline{\alpha}}$ of γ are given by

$$p_\gamma^{\overline{\alpha}} = \frac{\tau_{\overline{\alpha}}}{\kappa_{\overline{\alpha}}^2 + {\tau_{\overline{\alpha}}}^2} = \frac{\frac{\tau_\alpha}{\kappa_\alpha^2+\tau_\alpha^2}}{\left(\frac{\kappa_\alpha}{\kappa_\alpha^2+\tau_\alpha^2}\right)^2 + \left(\frac{\tau_\alpha}{\kappa_\alpha^2+\tau_\alpha^2}\right)^2} = \tau_\alpha,$$

and

$$r_\gamma^{\overline{\alpha}} = \frac{\kappa_{\overline{\alpha}}}{\kappa_{\overline{\alpha}}^2 + {\tau_{\overline{\alpha}}}^2} = \frac{\frac{\kappa_\alpha}{\kappa_\alpha^2+\tau_\alpha^2}}{\left(\frac{\kappa_\alpha}{\kappa_\alpha^2+\tau_\alpha^2}\right)^2 + \left(\frac{\tau_\alpha}{\kappa_\alpha^2+\tau_\alpha^2}\right)^2} = \kappa_\alpha.$$

Hence, the radius function of osculating helix of the image curve is constant if and only if either the base curve has constant curvature or the image curve is Mannheim curve. On the other hand, the pitch function of osculating helix of the image curve is constant if and only if either the base curve has constant torsion or the image curve is constant pitch curve. Also, the radius function or the pitch function of osculating helix of the *base curve* remain invariant under $\mathcal{SPT}$ if and only if $|\boldsymbol{w}(s)| = 1$, where $\boldsymbol{w}$ is the Darboux vector of the base curve.

The following theorem characterizes the algebraic structure of $\mathcal{SPT}$ with respect to composition of functions.

Theorem 2. *Let α and $\overline{\alpha} : I \to \mathbb{E}^3$ be non-degenerate curves, and $\mathfrak{s} : \alpha \to \overline{\alpha}$ be an $\mathcal{SPT}$. Then the followings are hold:*

(1) $\mathfrak{s}$ is $1-1$ and $\mathfrak{s}^{-1}$ is an $\mathcal{SPT}$.

(2) $\mathfrak{s}\left(\lambda\kappa\left(s\right), \lambda\tau\left(s\right)\right) = \dfrac{1}{\lambda}\mathfrak{s}\left(\kappa\left(s\right), \tau\left(s\right)\right)$, where λ is nonzero constant.

(3) $\mathcal{S} = \langle\mathfrak{s}\rangle$ is a cyclic group of order 2 with respect to composition of functions.

Proof. Let $(\alpha, \overline{\alpha})$ and $\left(\beta, \overline{\beta}\right)$ be an $\mathcal{SPT}$ *pair,* and $\kappa_{\overline{\alpha}} = \kappa_{\overline{\beta}}$, $\tau_{\overline{\alpha}} = \tau_{\overline{\beta}}$. Since $\mathcal{SPT}$ preserves ratio of curvatures of the base curve, we have $\tau_\beta/\tau_\alpha = \kappa_\beta/\kappa_\alpha$. The assumption and the identity of $\left|\boldsymbol{w}\left(s\right)\right|\left|\overline{\boldsymbol{w}}\left(s\right)\right| = 1$ requires

$$\frac{\kappa_{\overline{\alpha}}^2 + \tau_{\overline{\alpha}}^2}{\kappa_{\overline{\beta}}^2 + \tau_{\overline{\beta}}^2} = \frac{\tau_{\overline{\alpha}}/\tau_\alpha}{\tau_{\overline{\beta}}/\tau_\beta} = \frac{\tau_\beta}{\tau_\alpha} = \frac{\kappa_\beta}{\kappa_\alpha} = 1.$$

Hence, we have $\kappa_\alpha = \kappa_\beta$, $\tau_\alpha = \tau_\beta$ and $\mathfrak{s}$ is $1-1$. Moreover, for all non-degenerate image curves, up to fundamental theorem of the local theory of the curves, there exists only one base curve. Therefore, $\mathfrak{s}$ is surjective. Now fix $\mathfrak{s}^{-1} = \mathfrak{s}^\star$ and $\mathfrak{s}^\star\left(\kappa, \tau\right) = \left(\kappa^\star, \tau^\star\right)$. Since $\mathfrak{s}\circ\mathfrak{s}^\star = I$, we have $\left(\mathfrak{s}\circ\mathfrak{s}^\star\right)\left(\kappa, \tau\right) = \mathfrak{s}(\kappa^\star, \tau^\star) = \left(\kappa, \tau\right)$, where "$\circ$" represents the composition of functions. Since

$$\mathfrak{s}\left(\kappa^\star, \tau^\star\right) = \left(\frac{\kappa^\star}{\left(\kappa^\star\right)^2 + \left(\tau^\star\right)^2}, \frac{\tau^\star}{\left(\kappa^\star\right)^2 + \left(\tau^\star\right)^2}\right),$$

we obtain

$$\kappa = \frac{\kappa^\star}{\left(\kappa^\star\right)^2 + \left(\tau^\star\right)^2}\ ;\ \tau = \frac{\tau^\star}{\left(\kappa^\star\right)^2 + \left(\tau^\star\right)^2}. \tag{5}$$

It follows from Eq. (5) that, $\left|\boldsymbol{w}\right|\left|\boldsymbol{w}^\star\right| = 1$. So, we have $\kappa^\star = \kappa/\left(\kappa^2 + \tau^2\right)$ and $\tau^\star = \tau/\left(\kappa^2 + \tau^2\right)$. Hence, $\mathfrak{s}^\star\left(\kappa, \tau\right) = \left(\kappa^\star, \tau^\star\right) = \left(\kappa/\left(\kappa^2 + \tau^2\right), \tau/\left(\kappa^2 + \tau^2\right)\right)$ which proves that $\mathfrak{s}^\star = \mathfrak{s}^{-1}$ is also $\mathcal{SPT}$. The rest of the proof is obvious. □

Remark 4. *In Theorem 2, we observe that*

$$s^2\ \left(\kappa, \tau\right) = s(s\left(\kappa, \tau\right)) = s\left(\frac{\kappa}{\kappa^2 + \tau^2}, \frac{\tau}{\kappa^2 + \tau^2}\right) = \left(\kappa, \tau\right)$$

which requires $s^2 = I$*.*

3.2. **Invariants of $\mathcal{SPT}$.** Let us recall the definition of $\mathcal{SPT}$, $\mathfrak{s} : \left(\kappa, \tau\right) \to \left(\overline{\kappa}, \overline{\tau}\right)$. The identity of $\overline{\tau}/\overline{\kappa} = \tau/\kappa$ requires that the base curve remain invariant under the $\mathcal{SPT}$ when the base curve characterized by ratio of curvature and torsion, and arc-length parameters of the curves are common. Such as, helices and rectifying curves transform to helices and rectifying curves under the $\mathcal{SPT}$, respectively. Furthermore, the base curve remains invariant when the curvatures of the curve satisfy $\kappa^2\left(s\right) + \tau^2\left(s\right) = 1$. For example, the curve with curvature $\kappa\left(s\right) = \sin\varphi\left(s\right)$ and torsion $\tau\left(s\right) = \cos\varphi\left(s\right)$ remains invariant under the $\mathcal{SPT}$. Besides, constant precession curve has the curvatures of $\kappa(s) = w\sin\left(\mu s\right)$ and $\tau(s) = w\cos\left(\mu s\right)$, where $w > 0$ and μ are constant. Under the $\mathcal{SPT}$, these curvatures transform to $\overline{\kappa} = (1/w)\sin\left(\mu s\right)$ and $\overline{\tau} = (1/w)\cos\left(\mu s\right)$ which requires the image curve is also constant precession curve. Hence, constant precession curve remains invariant under the $\mathcal{SPT}$.

Let $(\alpha, \overline{\alpha})$ be an $\mathcal{SPT}$ pair. Respectively the curvature and the torsion of the tangent indicatrix of α is given by $\kappa_{\boldsymbol{\alpha}}^{\boldsymbol{t}}(s) = \sqrt{1 + f^2(s)}$ and $\tau_{\boldsymbol{\alpha}}^{\boldsymbol{t}}(s) = \sigma(s)\sqrt{1 + f^2(s)}$, where $f(s) = (\tau/\kappa)(s)$ and $\sigma(s)$ is the geodesic curvature of the normal indicatrix of α. Since $\mathcal{SPT}$ preserves the ratio of τ/κ, we obtain $\kappa_{\overline{\alpha}}^{\overline{\boldsymbol{t}}}(s) = \kappa_{\alpha}^{\boldsymbol{t}}(s)$. Therefore, curvature of the tangent indicatrix remains invariant under the $\mathcal{SPT}$. On the other hand, $\tau_{\boldsymbol{\alpha}}^{\boldsymbol{t}}(s) = \sigma(s)\sqrt{1 + f^2(s)} = f'(s) / \left(\kappa(s)\left(1 + f^2(s)\right)\right)$. Therefore, torsion of the tangent indicatrix of the curve remains invariant under $\mathcal{SPT}$ if and only if $\kappa(s) = \overline{\kappa}(s)$ or $|\boldsymbol{w}(s)| = 1$.

It is well known that the curvatures of slant helices can be given by $\kappa(s) = (1/m)\,\varphi'(s)\cos\varphi(s)$ and $\tau(s) = (1/m)\,\varphi'(s)\sin\varphi(s)$, where φ is a differentiable function of s and $m = \cot\theta \neq 0$ is a constant, see [13]. Depends on intrinsic equations of slant helices, we give the following theorem:

Theorem 3. *The proper slant helices remain invariant under the $\mathcal{SPT}$ if and only if φ is a linear function of arc-length parameter of the curve.*

Proof. Let the $\mathcal{SPT}$ leaves the slant helices invariant, that is, both base curve and image curve be a *proper* slant helix. The curvatures of the base curve are given by $\kappa(s) = (1/m)\,\varphi'(s)\cos\varphi(s)$, and $\tau(s) = (1/m)\,\varphi'(s)\sin\varphi(s)$, where m is nonzero constant. By definition of $\mathcal{SPT}$ we obtain $(\overline{\tau}/\overline{\kappa})(s) = (\tau/\kappa)(s) = \tan\varphi(s)$ and $\overline{\kappa}(s) = m\cos\varphi(s)/\varphi'(s)$. The geodesic curvature of the normal indicatrix of $\overline{\alpha}$ as follows:

$$\overline{\sigma} = \left(\frac{1}{\overline{\kappa}\left(1 + \left(\frac{\overline{\tau}}{\overline{\kappa}}\right)^2\right)^{3/2}}\left(\frac{\overline{\tau}}{\overline{\kappa}}\right)'\right)(s) = \frac{\varphi'(s)}{m}.$$

Since $\overline{\sigma}$ is constant, φ is a linear function of s. Conversely, if φ is a linear function of s, it is enough to prove that $\overline{\alpha}$ is a slant helix when α is a slant helix and $\mathfrak{s} : \alpha \to \overline{\alpha}$ is an $\mathcal{SPT}$. From the assumption, we find $\varphi' = c$, and $\kappa(s) = a\cos\varphi(s)$, $\tau(s) = a\sin\varphi(s)$, where both c and $a = c/m$ are constant. It follows from $\overline{\kappa}(s) = \kappa(s) / \left(\kappa^2(s) + \tau^2(s)\right) = \cos\varphi(s)/a$ and $(\overline{\tau}/\overline{\kappa})(s) = (\tau/\kappa)(s) = \tan\varphi(s)$ that

$$\overline{\sigma}(s) = \left(\frac{1}{\overline{\kappa}\left(1 + \left(\frac{\overline{\tau}}{\overline{\kappa}}\right)^2\right)^{3/2}}\left(\frac{\overline{\tau}}{\overline{\kappa}}\right)'\right)(s) = a,$$

which is constant. Hence, under the $\mathcal{SPT}$, the image curve is also slant helix and this completes the proof. □

Corollary 2. *The proper slant helix α remains invariant under the $\mathcal{SPT}$ if and only if α is a constant precession curve.*

3.3. **Bertrand and Mannheim curves.** Let $\alpha : I \to \mathbb{E}^3$ be a parameterized regular curve (not necessarily by arc length) with $\kappa(t) \neq 0$, $\tau(t) \neq 0$, $t \in I$. The curve α is called a *Bertrand curve* if there exits a curve $\alpha^* : I \to \mathbb{E}^3$ such that the normal lines of α and α^* at $t \in I$ are equal. In this case, α^* called a *Bertrand mate* of α, and we can write

$$\alpha^*(t) = \alpha(t) + \lambda \boldsymbol{n}(t). \tag{6}$$

Notice that λ is constant in Eq. (6). Moreover, α is a Bertrand curve if and only if there exits a linear relation $A\kappa(t) + B\tau(t) = 1$, $t \in I$,where A, B are nonzero constants and κ and τ are the curvature and torsion of α, respectively [5]. On the other hand, if $\alpha : I \to \mathbb{E}^3$ is a non-degenerate curve with the arc-length parameter, then $B\kappa(s) - A\tau(s) \neq 0$ for all $s \in I$. Furthermore, the curvature κ^* and the torsion τ^* of α^* are given by

$$\kappa^*(s) = \frac{|B\kappa(s) - A\tau(s)|}{(A^2 + B^2)\,|\tau(s)|}; \quad \tau^*(s) = \frac{1}{(A^2 + B^2)\,\tau(s)}. \tag{7}$$

see [8].

Let $\alpha : I \to \mathbb{E}^3$ be a unit speed Bertrand curve, α^* be a Bertrand mate of α, and $\mathfrak{s} : \alpha \to \alpha^*$ be an $\mathcal{SPT}$ *in Euclidean space* $\mathbb{E}^3$. It follows from $\tau^*(s) = \tau(s) / \left(\kappa^2(s) + \tau^2(s)\right)$, $A\kappa(s) + B\tau(s) = 1$ and Eq. (7) that,

$$\frac{\kappa}{\tau} = \pm\sqrt{A^2 + B^2 - 1}. \tag{8}$$

Here we assume that $A^2 + B^2 > 1$. Otherwise, we can't find the curvatures of the curve. From Eq. (8) and $A\kappa(s) + B\tau(s) = 1$, we obtain both the curvature κ and the torsion τ are constant. Therefore, both α and α^* are circular helix. Consequently, if α and α^* are Bertrand curves under $\mathcal{SPT}$*,* then *both* α and α^* are circular helix. In fact, there is no non-degenerate Bertrand curves with respect to $\mathcal{SPT}$. Assume that $(\alpha, \widehat{\alpha})$ is a Mannheim pair under the $\mathcal{SPT}$ i.e., α is a Mannheim curve and $\widehat{\alpha}$ is Mannheim mate of α when $\mathfrak{s} : \alpha \to \widehat{\alpha}$ is an $\mathcal{SPT}$, in Euclidean space $\mathbb{E}^3$. It is well-known that if Mannheim curve is a generalized helix, then Mannheim mate is a straight line, see [12]. From the definition of Mannheim curve and $\mathcal{SPT}$, the curvature of the image curve is obtained as $\widehat{\kappa}(s) = \kappa(s) / \left(\kappa^2(s) + \tau^2(s)\right) = \lambda$ which is constant. It follows from $\widehat{\tau}(s) = \tau(s) / \left(\kappa^2(s) + \tau^2(s)\right) = \left(\kappa^2(s) + \tau^2(s)\right) / \tau(s)$ that, the torsion of the image curve $\widehat{\tau} = \pm 1$ is also constant, i.e., Mannheim mate is circular helix. In this case, curvatures of Mannheim curve satisfy $\tau(s) = \pm(\kappa^2(s) + \tau^2(s))$. Furthermore, $\lambda = \kappa(s) / \left(\kappa^2(s) + \tau^2(s)\right)$ is constant. Hence, curvatures of the base curve satisfy $\tau/\kappa = \pm 1/\lambda$ which is constant. The last equality requires that the base curve (Mannheim curve) is generalized helix which is contradiction. Consequently, there is no suitable Mannheim pair with respect to $\mathcal{SPT}$ in Euclidean space $\mathbb{E}^3$. Therefore, we will consider only Mannheim curve (with its curvature and torsion), not the Mannheim pairs, under the $\mathcal{SPT}$ in the next.

4. Associated Curves of $\mathcal{SPT}$

4.1. Slant helices.

Proposition 4. *Let $\mathfrak{s}:\alpha\to\overline{\alpha}$ be an $\mathcal{SPT}$ in Euclidean space $\mathbb{E}^3$, and the normal vector of $\overline{\alpha}$ makes constant angle with a fixed line. Then α is a Mannheim curve with non-constant slope if and only if $\overline{\alpha}$ is a Salkowski curve.*

Proof. Let α be a Mannheim curve with non-constant slope. Since α is a Mannheim curve, $\overline{\kappa}(s)=\kappa(s)/\left(\kappa^2(s)+\tau^2(s)\right)=\lambda$ is constant. Because of $(\tau/\kappa)'(s)\neq 0$,

$$\overline{\tau}(s)=\frac{\tau(s)}{\kappa^2(s)+\tau^2(s)}=\frac{\lambda\tau(s)}{\kappa(s)}$$

is non-constant. From the assumption, and up to rigid movements or up to the antipodal map, $\overline{\alpha}$ is a Salkowski curve. Conversely, if $\overline{\alpha}$ is a Salkowski curve, then $\overline{\kappa}(s)=\kappa(s)/\left(\kappa^2(s)+\tau^2(s)\right)$ is constant and $\overline{\tau}(s)$ is non-constant. This completes the proof. □

Corollary 3. *Let $\mathfrak{s}:\alpha\to\overline{\alpha}$ be an $\mathcal{SPT}$ in Euclidean space $\mathbb{E}^3$, and the normal vector of $\overline{\alpha}$ makes constant angle with a fixed line. Then α is a constant pitch curve with non-constant slope if and only if $\overline{\alpha}$ is an anti-Salkowski curve.*

Theorem 4. *Let $\mathfrak{s}:\alpha\to\overline{\alpha}$ be an $\mathcal{SPT}$ in Euclidean space $\mathbb{E}^3$. If α is a Salkowski curve, then the followings are hold:*

(1) $\overline{\alpha}$ is a Mannheim curve.

(2) The curvatures of Mannheim mate of $\overline{\alpha}$ satisfy

$$\widetilde{\kappa}(s)\,\widetilde{\tau}(s)\mp c\left(1+\widetilde{\tau}^2(s)\right)=0,$$

where $\widetilde{\kappa}$ and $\widetilde{\tau}$ are the curvature and torsion of the Mannheim mate $\widetilde{\alpha}$ respectively, and c is a nonzero constant.

Proof. Let α be a Salkowski curve and $\overline{\alpha}$ be an image curve with respect to $\mathcal{SPT}$. Since the base curve is Salkowski curve, the curvature of α is constant but its torsion is non-constant. Without loss of generality, we can assume $\kappa\equiv 1$. From Corollary 1, the torsion of the base curve is given by $\tau(s)=cs/\sqrt{1-c^2s^2}$, where c is a nonzero constant. Since $\overline{\alpha}$ is an image curve with respect to $\mathcal{SPT}$, the curvatures of this curve satisfy

$$\frac{\overline{\kappa}(s)}{\overline{\kappa}^2(s)+\overline{\tau}^2(s)}=\frac{\frac{\kappa(s)}{\kappa^2(s)+\tau^2(s)}}{\left(\frac{\kappa(s)}{\kappa^2(s)+\tau^2(s)}\right)^2+\left(\frac{\tau(s)}{\kappa^2(s)+\tau^2(s)}\right)^2}=\kappa=1.$$

Hence, $\overline{\alpha}$ is a Mannheim curve. Respectively, the curvature and torsion of $\overline{\alpha}$ are given by

$$\overline{\kappa}(s)=\frac{\kappa(s)}{\kappa^2(s)+\tau^2(s)}=\frac{1}{1+\frac{c^2s^2}{1-c^2s^2}}=1-c^2s^2,$$

and

$$\overline{\tau}(s)=\frac{\tau(s)}{\kappa^2(s)+\tau^2(s)}=\frac{\frac{cs}{\sqrt{1-c^2s^2}}}{1+\frac{c^2s^2}{1-c^2s^2}}=cs\sqrt{1-c^2s^2}.$$

Suppose that $\widetilde{\alpha}$ is a Mannheim mate of $\overline{\alpha}$. By straightforward calculations, we find the curvature and the torsion of $\widetilde{\alpha}$ as follow:

$$\widetilde{\kappa}(s)=\frac{\overline{\kappa}(s)\left(\overline{\kappa}(s)\overline{\tau}'(s)-\overline{\kappa}'(s)\overline{\tau}(s)\right)}{|\lambda\overline{\tau}(s)|\left(\overline{\kappa}^2(s)+\overline{\tau}^2(s)\right)^{3/2}}=\mp\frac{1}{s\sqrt{1-c^2s^2}},\tag{9}$$

and

$$\widetilde{\tau}(s)=\frac{\overline{\kappa}^2(s)+\overline{\tau}^2(s)}{\overline{\tau}(s)}=\frac{\sqrt{1-c^2s^2}}{cs}.\tag{10}$$

From Eqs. (9) and (10) we obtain,

$$\frac{\widetilde{\kappa}(s)\,\widetilde{\tau}(s)}{1+\widetilde{\tau}^2(s)}=\mp c,$$

which completes the proof. □

Proposition 5. *Let $\gamma(s):I\to\mathbb{E}^3$ be a unit-speed Frenet curve with constant torsion $\tau\equiv1$ and non-constant curvature $\kappa(s)$. If normal vectors of γ make a constant angle with a fixed line, then γ is an anti-Salkowski curve with curvature*

$$\kappa(s)=\frac{|\varphi(s)|}{\sqrt{1-\varphi^2(s)}},$$

where φ is a linear function of arc-lenght parameter of γ.

Proof. Let γ be a curve with constant torsion $\tau\equiv1$ and non-constant curvature κ. If normal vectors of γ make a constant angle with a fixed line, then γ is an anti-Salkowski curve, see [14]. By Theorem 1 we have

$$\kappa(s)=\frac{\sqrt{1-f^2(s)}}{f(s)},\tag{11}$$

where $f(s)=c\int\kappa ds$ and c is a nonzero constant. Eq. (11) leads to the differential equation

$$f'(s)\,f(s)-c\sqrt{1-f^2(s)}=0,$$

which has solution

$$f(s)=\mp\sqrt{1-(cs+k)^2},\tag{12}$$

where $k\in\mathbb{R}$. It follows from Eqs. (11) and (12) that, $\kappa(s)=|\varphi(s)|/\sqrt{1-\varphi^2(s)}$, where $\varphi(s)=cs+k$. □

Corollary 4. *Let $\mathfrak{s}:\gamma\to\overline{\gamma}$ be an $\mathcal{SPT}$ in Euclidean space $\mathbb{E}^3$. If γ is an anti-Salkowski curve, then the followings are hold:*

(1) $\overline{\gamma}$ is a constant pitch curve.

(2) The curvatures of $\overline{\gamma}$ are given by

$$\overline{\kappa}(s) = |\varphi(s)| \sqrt{1-\varphi^2(s)}; \qquad \overline{\tau}(s) = 1-\varphi^2(s),$$

where φ is a linear function of arc-lenght parameter of γ.

Now we give the curvatures of a curve that is both a Mannheim curve and a slant helix (shortly Mannheim slant helix) in the following:

Proposition 6. *Let $(\alpha, \widehat{\alpha})$ be a Mannheim pair. If α is a slant helix, then the curvatures of the Mannheim pair as follows:*

$$\kappa(s) = \frac{1}{\lambda} \sec h^2 \varphi(s); \qquad \tau(s) = \frac{1}{\lambda} \sec h \varphi(s) \tanh \varphi(s),$$

and

$$\widehat{\kappa}(s) = \csc h \varphi(s); \qquad \widehat{\tau}(s) = \frac{1}{\lambda} \csc h \varphi(s),$$

where φ is a linear function of arc-lenght parameter of α and λ is nonzero constant.

Proof. Let α be a Mannheim slant helix. The equations $\kappa(s) = \lambda\left(\kappa^2(s) + \tau^2(s)\right)$ and $\sigma(s) = \left(\frac{\kappa^2}{(\kappa^2+\tau^2)^{3/2}} \left(\frac{\tau}{\kappa}\right)'\right)(s)$ leads to the differential equation

$$\sigma(s) = \pm \frac{\lambda}{2} \frac{\kappa'(s)}{\kappa(s)\sqrt{1-\lambda\kappa(s)}}.$$

Since $\sigma = c$ is constant, it follows:

$$\pm \frac{\lambda}{2} \kappa'(s) - c\kappa(s)\sqrt{1-\lambda\kappa(s)} = 0,$$

which has solution $\kappa(s) = \sec h^2 \varphi(s)/\lambda$, where $\varphi(s) = \frac{c}{\lambda} s \pm \frac{c_1}{2}$ with $c_1 \in \mathbb{R}$. On the other hand, by Theorem 1 we obtain

$$f(s) = c \int \kappa ds = \tanh \varphi(s) + c_2,$$

where $c_2 \in \mathbb{R}$. Without loss of generality, we can assume that $c_2 = 0$. Hence, we find the torsion of α as

$$\tau(s) = \frac{\kappa(s) f(s)}{\sqrt{1-f^2(s)}} = \frac{1}{\lambda} \sec h \varphi(s) \tanh \varphi(s).$$

Finally, the curvatures of $\widehat{\alpha}$ as follow:

$$\widehat{\kappa}(s) = \frac{\kappa(s)\left(\kappa(s)\tau'(s) - \kappa'(s)\tau(s)\right)}{|\lambda\tau(s)|\left(\kappa^2(s)+\tau^2(s)\right)^{3/2}} = \frac{1}{\sinh \varphi(s)},$$

and

$$\widehat{\tau}(s) = \frac{\kappa^2(s)+\tau^2(s)}{\tau(s)} = \frac{1}{\lambda \sinh \varphi(s)}.$$

Furthermore, $\widehat{\tau}/\widehat{\kappa} = 1/\lambda$ i.e., $\widehat{\alpha}$ is a generalized helix. The proof is complete. □

Corollary 5. *Let $\mathfrak{s}:\alpha\rightarrow\overline{\alpha}$ be an $\mathcal{SPT}$ in Euclidean space $\mathbb{E}^3$. If the base curve is a Mannheim slant helix, then curvature of the normal indicatrix of the image curve is equal to the curvature of the base curve, i.e. $\overline{\sigma}=\kappa$.*

Proof. Let α be a Mannheim slant helix. Since $\kappa(s)=\sec h^2\varphi(s)/\lambda$, $\tau(s)=\sec h\varphi(s)\tanh\varphi(s)/\lambda$, and $\tau/\kappa=\overline{\tau}/\overline{\kappa}$, we obtain

$$\overline{\sigma}(s)=\left(\frac{1}{\overline{\kappa}\left(1+\left(\frac{\tau}{\kappa}\right)^2\right)^{3/2}}\left(\frac{\tau}{\kappa}\right)'\right)(s)=\frac{1}{\overline{\kappa}(s)}\sec h^2\varphi(s).$$

Besides, $\overline{\kappa}(s)=\kappa(s)/\left(\kappa^2(s)+\tau^2(s)\right)=\lambda$. Thus, $\overline{\sigma}(s)=\sec h^2\varphi(s)/\lambda=\kappa(s)$ which is intended. □

Opposite of the Corollary 5 is not true in general, but we have the following result:

Corollary 6. *The ratio of curvatures of the Mannheim slant helix α satisfy $(\tau/\kappa)(s)=\sinh(s+c)$, $c\in\mathbb{R}$ when $\mathfrak{s}:\alpha\rightarrow\overline{\alpha}$ is an $\mathcal{SPT}$ and $\overline{\sigma}=\kappa$.*

Proof. Assume that $\mathfrak{s}:\alpha\rightarrow\overline{\alpha}$ is an $\mathcal{SPT}$ and $\overline{\sigma}=\kappa$. Then we obtain

$$\frac{\kappa^2(s)}{\kappa^2(s)+\tau^2(s)}=\left(\frac{1}{\left(1+\left(\frac{\tau}{\kappa}\right)^2\right)^{3/2}}\left(\frac{\tau}{\kappa}\right)'\right)(s). \tag{13}$$

By substituting $f(s)=(\tau/\kappa)(s)$ in Eq. (13), it follows:

$$f'(s)-\sqrt{1+f^2(s)}=0,$$

which has solution $f(s)=\sinh(s+c)$, where $c\in\mathbb{R}$. This completes the proof. □

Remark 5. *If α is both a constant pitch curve and a slant helix (shortly constant slant pitch curve), then α has reversed curvatures with respect to Proposition 6, i.e. the curvature and the torsion of α as follows,*

$$\kappa(s)=\frac{1}{\lambda}\sec h\varphi(s)\tanh\varphi(s);\qquad\tau(s)=\frac{1}{\lambda}\sec h^2\varphi(s),$$

where φ is a linear function of arc-length parameter of α. Furthermore, $\overline{\kappa}(s)=\lambda\sinh\varphi(s)$ and $\overline{\tau}$ is constant when the base curve is a constant slant pitch curve.

In the light of Proposition 6, one can consider both Mannheim curve and rectifying curve or namely Mannheim rectifying curve. The curvatures of this curve satisfy both $\kappa(s)/\left(\kappa^2(s)+\tau^2(s)\right)=\lambda$ and $(\tau/\kappa)(s)=\varphi(s)$, where λ is nonzero constant and φ is a linear function of arc-length parameter of the curve. Therefore, we obtain the curvatures of Mannheim rectifying curve as

$$\kappa(s)=\frac{1}{\lambda\left(1+\varphi^2(s)\right)};\qquad\tau(s)=\frac{\varphi(s)}{\lambda\left(1+\varphi^2(s)\right)}.$$

If α is a Mannheim rectifying curve and $\mathfrak{s}:\alpha\to\overline{\alpha}$ is an $\mathcal{SPT}$, then we find the curvatures of $\overline{\alpha}$ as $\overline{\kappa}=\lambda$ and $\overline{\tau}(s)=\lambda\varphi(s)$. Thus, $\mathcal{SPT}$ maps Mannheim rectifying curve to the rectifying curve, but not to the Mannheim curve.

Remark 6. *We can suggest that* $\kappa(s)=\varepsilon\sec h^2\phi(s)$ *and* $\tau(s)=\varepsilon\sec h\phi(s)\tanh\phi(s)$ *for curvatures of Mannheim rectifying curve, where* $\varepsilon\in\mathbb{R}$, $\phi(s)=\arcsin h\varphi(s)$ *and* φ *is linear function of arc-length parameter of the curve. These curvatures are similar to the curvatures of the Mannheim slant helix apart from* ϕ *being non-linear.*

4.2. **Evolute of the image curve and quasi-slant helix.** Let α and $\overline{\alpha}: I\to\mathbb{E}^3$ be non-degenerate curves. If the tangent vectors of the curves satisfy $\boldsymbol{t}(s)\cdot\overline{\boldsymbol{t}}(s)=0$ for all $s\in I$, then $\overline{\alpha}$ is called as involute of α, and its parametric equation given by

$$\overline{\alpha}(s)=\alpha(s)+(-s+c)\,\boldsymbol{t}(s),$$

where $c\in\mathbb{R}$. In this case, α is called as evolute of $\overline{\alpha}$. The curvatures of $\overline{\alpha}$ can be given by curvatures of α as follows:

$$\overline{\kappa}(s)=\frac{\sqrt{\kappa^2(s)+\tau^2(s)}}{|(-s+c)|\,\kappa(s)};\quad \overline{\tau}(s)=\frac{\kappa(s)\,\tau'(s)-\kappa'(s)\,\tau(s)}{(-s+c)\,\kappa(s)\,(\kappa^2(s)+\tau^2(s))}$$

see [7].

Proposition 7. *Let* α *and* $\overline{\alpha}: I\to\mathbb{E}^3$ *be different non-degenerate curves and* $\mathfrak{s}:\alpha\to\overline{\alpha}$ *be an* $\mathcal{SPT}$. *There is no slant helix as a base curve of* $\mathcal{SPT}$ *when the image curve is involute of* α.

Proof. Let α be a slant helix and $\overline{\alpha}$ be an involute curve of α. Respectively the curvature and the torsion of the image curve are as follow:

$$\overline{\kappa}(s)=\frac{\sqrt{\kappa^2(s)+\tau^2(s)}}{|(-s+c)|\,\kappa(s)}=\frac{\kappa(s)}{\kappa^2(s)+\tau^2(s)}, \tag{14}$$

and

$$\overline{\tau}(s)=\frac{\kappa(s)\,\tau'(s)-\kappa'(s)\,\tau(s)}{(-s+c)\,\kappa(s)\,(\kappa^2(s)+\tau^2(s))}=\frac{\tau(s)}{\kappa^2(s)+\tau^2(s)}. \tag{15}$$

From Eqs. (14) and (15), it follows:

$$\left(\frac{\overline{\tau}}{\overline{\kappa}}\right)(s)=\left(\frac{\tau}{\kappa}\right)(s)=\pm\sigma(s).$$

Since α is a slant helix, $\sigma=\pm\tau/\kappa$ is constant. On the other hand, from Eq. (15) we obtain

$$-s+c=\left(\frac{\tau}{\kappa}\right)'(s)\left(\frac{\kappa}{\tau}\right)(s)=0,$$

which is a contradiction. Therefore, there is no slant helix as a base curve when the image curve is involute curve. □

In accordance with Proposition 7, the following question occurs:

"*Which base curves are the evolute of the image curve with respect to* $\mathcal{SPT}$?"

To answer this question, we define a *new curve* (family) as follows:

Definition 3. *Let $\alpha : I \to \mathbb{E}^3$ be a C^3 space curve with non-constant slope. If the curvatures of α satisfy*

$$\left(\frac{\tau}{\kappa}\right)(s) = \frac{\kappa^2(s)}{\left(\kappa^2(s) + \tau^2(s)\right)^{3/2}} \left(\frac{\tau}{\kappa}\right)'(s), \tag{16}$$

then α is called as quasi-slant helix.

In fact, the definition of quasi-slant helix determines family of curve in Euclidean space. Therefore, one can find different quasi slant helices whose have different intrinsic equations. The following example clarify this case:

Example 1. *Let $\mathcal{Q}$ be the family of quasi-slant helices, and $\alpha : I \to \mathbb{E}^3$ be a C^3 space curve with non-constant slope. The curvatures of $\kappa_\alpha(s) = \cos^2 s/\sin s$ and $\tau_\alpha(s) = \cos s$ satisfy Eq. (16). Therefore $\alpha \in \mathcal{Q}$. Furthermore, the curvatures of α satisfy the following algebraic equation:*

$$\mathcal{P}_\alpha(\kappa_\alpha, \tau_\alpha) = \tau_\alpha^4 + \kappa_\alpha^2\tau_\alpha^2 - \kappa_\alpha^2 = 0.$$

Here we can call $\mathcal{P}_\alpha$ as a curvature polynomial of α. Moreover, the curve $\gamma : I \to \mathbb{E}^3$ with curvatures $\kappa_\gamma(s) = \sec h^2 s \csc hs$ and $\tau_\gamma(s) = \sec h^2 s$ is also quasi-slant helix i.e., $\gamma \in \mathcal{Q}$. This curve has the same slope with Mannheim slant helix (Proposition 6) when $\lambda = 1$ and $\varphi(s) = s$. Furthermore, the curvature polynomial of γ can be given by

$$\mathcal{P}_\gamma(\kappa_\gamma, \tau_\gamma) = \tau_\gamma^3 + \kappa_\gamma^2\tau_\gamma - \kappa_\gamma^2 = 0.$$

One of the rational parameterization of the curvatures of quasi-slant helices can be given by

$$\kappa_\beta(s) = \frac{1}{s^2(s^2-1)} \ ; \ \tau_\beta(s) = \frac{1}{s^2\sqrt{s^2-1}},$$

where β is a quasi-slant helix. Hence, the curvature polynomial of β is as follows:

$$\mathcal{P}_\beta(\kappa_\beta, \tau_\beta) = \tau_\beta^4 + \kappa_\beta^2\tau_\beta^2 - \kappa_\beta^3 = 0.$$

The example above obviously shows that we can find different quasi-slant helices whose have different intrinsic equations.

Proposition 8. *Let $\mathfrak{s} : \alpha \to \overline{\alpha}$ be an $\mathcal{SPT}$ in Euclidean space $\mathbb{E}^3$. $\overline{\alpha}$ is an involute of the base curve if and only if α is a quasi-slant helix with curvatures*

$$\kappa(s) = \frac{-s+c}{(1+\xi^2(s))^{3/2}}; \quad \tau(s) = \frac{(-s+c)\xi(s)}{(1+\xi^2(s))^{3/2}},$$

where $\xi(s) = c_1 e^{cs - s^2/2}$ and c, c_1 are constant.

Proof. Let $\overline{\alpha}$ be an involute of α. The curvatures of $\overline{\alpha}$ are given by

$$\overline{\kappa}(s) = \frac{\sqrt{\kappa^2(s) + \tau^2(s)}}{|-s+c|\,\kappa(s)} \ ; \ \overline{\tau}(s) = \frac{\kappa(s)\tau'(s) - \kappa'(s)\tau(s)}{(-s+c)\,\kappa(s)\left(\kappa^2(s) + \tau^2(s)\right)}, \tag{17}$$

where c is a constant. By the parameter change, we can assume that $-s+c>0$. From Eq. (17) we obtain

$$\left(\frac{\overline{\tau}}{\overline{\kappa}}\right)(s)=\left(\frac{\tau}{\kappa}\right)(s)=\frac{\kappa(s)\tau'(s)-\kappa'(s)\tau(s)}{\left(\kappa^2(s)+\tau^2(s)\right)^{3/2}}, \tag{18}$$

which proves that α is a quasi-slant helix. Since $\overline{\kappa}(s)=\kappa(s)/(\kappa^2(s)+\tau^2(s))$, we find

$$(-s+c)\kappa^2(s)=\left(\kappa^2(s)+\tau^2(s)\right)^{3/2}. \tag{19}$$

Eq. (18) and $\xi(s)=(\tau/\kappa)(s)$ leads to the differential equation

$$\xi'(s)-(-s+c)\xi(s)=0,$$

which has solution

$$\xi(s)=(\tau/\kappa)(s)=c_1e^{cs-s^2/2}, \tag{20}$$

where $c_1\in\mathbb{R}$. From Eqs. (19) and (20) we obtain the curvatures of α. Conversely, by straightforward calculations we obtain

$$\overline{\kappa}(s)=\kappa(s)/\left(\kappa^2(s)+\tau^2(s)\right)=\frac{\sqrt{1+\xi^2(s)}}{(-s+c)},$$

where $-s+c>0$ and $\xi(s)=(\tau/\kappa)(s)$. Besides,

$$\overline{\tau}(s)=\tau(s)/\left(\kappa^2(s)+\tau^2(s)\right)=\frac{\sqrt{\kappa^2(s)+\tau^2(s)}}{(-s+c)\kappa(s)}\frac{\tau(s)}{\kappa(s)}. \tag{21}$$

By substituting $(\tau/\kappa)(s)=\sigma(s)$ in (21) we find

$$\overline{\tau}(s)=\frac{\tau'(s)\kappa(s)-\tau(s)\kappa'(s)}{(-s+c)\kappa(s)\left(\kappa^2(s)+\tau^2(s)\right)},$$

which completes the proof. □

Corollary 7. *Let $\mathfrak{s}:\alpha\to\overline{\alpha}$ be an $\mathcal{SPT}$ in Euclidean space $\mathbb{E}^3$. $\mathfrak{s}$ preserves quasi-slant helices if and only if $\overline{\sigma}=\sigma$.*

Remark 7. *The curvatures of slant helices satisfy $f'(s)/\left(1+f^2(s)\right)^{3/2}=\lambda\kappa(s)$, where $f(s)=(\tau/\kappa)(s)$ with $\lambda\in\mathbb{R}$. Slightly different, curvatures of quasi-slant helices satisfy $f'(s)/\left(1+f^2(s)\right)^{3/2}=f(s)\kappa(s)$.*

4.3. **Curves in kinematics.** It is a well-known fact that the Frenet-Serret motion based on a curve is persistent if and only if $\tau(s)/\left(\kappa^2(s)+\tau^2(s)\right)$ is a constant, where $\kappa(s)$ and $\tau(s)$ are the curvature and torsion functions of the curve, see [4,20]. Moreover, if $\tau(s)/\left(\kappa^2(s)+\tau^2(s)\right)=p$, $p\neq 0$ then we say that the curve generates a $p-$persistent Frenet-Serret motion.

Remark 8. *Let $\mathfrak{s}:\alpha\to\overline{\alpha}$ be an $\mathcal{SPT}$ in Euclidean space $\mathbb{E}^3$. It is easy to see that the Frenet-Serret motion is persistent on the base curve (resp. image curve) of $\mathcal{SPT}$ if and only if torsion of the image curve (resp. base curve) is constant.*

In the following, we give the requirement that the Frenet-Serret motion is persistent on the quasi-slant helices.

Lemma 1. *Assume that* α *is a quasi-slant helix in Euclidean space* $\mathbb{E}^3$. *The Frenet-Serret motion is persistent on* α *if and only if* $\sigma(s)\,\rho(s)\,/\left(1+\sigma^2(s)\right)$ *is constant, where* $\rho(s)=1/\kappa(s)$ *is the radius of the curvature of* α.

Proof. Let α be a quasi-slant helix with curvature $\kappa(s)$ and torsion $\tau(s)$. The pitch function $\tau(s)\,/\left(\kappa^2(s)+\tau^2(s)\right)$ is constant when the Frenet-Serret motion is persistent on α. Since α is a quasi-slant helix, $\tau(s)=\sigma(s)\,\kappa(s)$. Therefore,

$$\frac{\tau(s)}{\kappa^2(s)+\tau^2(s)}=\frac{\sigma(s)\,\kappa(s)}{\kappa^2(s)+\sigma^2(s)\,\kappa^2(s)}=\frac{\sigma(s)\,\rho(s)}{1+\sigma^2(s)},$$

which is constant. Conversely, we obtain

$$\sigma(s)\,\rho(s)\,/\left(1+\sigma^2(s)\right)=\tau(s)\,/\left(\kappa^2(s)+\tau^2(s)\right)=\lambda$$

where λ is a nonzero constant. The proof is complete. □

Corollary 8. *Assume that* α *is a quasi-slant helix and* $\mathfrak{s}:\alpha\rightarrow\overline{\alpha}$ *is an* $\mathcal{SPT}$ *in Euclidean space* $\mathbb{E}^3$. *The Frenet-Serret motion is persistent on* α *if and only if*

$$\frac{\overline{\sigma}(s)}{\overline{\kappa}(s)\left(\overline{\sigma}^2(s)+\frac{1}{|\overline{w}(s)|^2}\right)}$$

is constant.

Proof. Since $\overline{\sigma}(s)=\frac{1}{\overline{\kappa}(s)\left(1+\left(\frac{\overline{\tau}(s)}{\overline{\kappa}(s)}\right)^2\right)^{3/2}}\left(\frac{\overline{\tau}}{\overline{\kappa}}\right)'(s)$ and $\overline{\tau}/\overline{\kappa}=\tau/\kappa$, it is written $\overline{\sigma}(s)=\frac{1}{\overline{\kappa}(s)\left(1+\left(\frac{\tau(s)}{\kappa(s)}\right)^2\right)^{3/2}}\left(\frac{\tau}{\kappa}\right)'(s)=\frac{\sigma(s)\kappa(s)}{\overline{\kappa}(s)}$ or $\sigma(s)=\frac{\overline{\sigma}(s)\overline{\kappa}(s)}{\kappa(s)}$. From Lemma 1 we get the intended. □

Theorem 5. *Let* $\mathfrak{s}:\alpha\rightarrow\overline{\alpha}$ *be an* $\mathcal{SPT}$ *and* $\overline{\alpha}$ *be a slant helix in Euclidean space* $\mathbb{E}^3$. *If* α *generates* $p-$*persistent Frenet-Serret motion* $(p\neq 0)$, *then the geodesic curvature of the normal indicatrix of* $\overline{\alpha}$ *satisfies the following:*

$$\frac{1}{2}\,|p|\leq|\overline{\sigma}(s)|\,.$$

Proof. Assume that α generates the $p-$persistent Frenet-Serret motion in $\mathbb{E}^3$. The curvatures of this curve are given by

$$\kappa(\theta)=\frac{1}{2p}\cos\theta;\qquad\tau(\theta)=\frac{1}{2p}\left(1+\sin\theta\right),$$

where θ is a parameter. Thereby, the curvatures of $\overline{\alpha}$ as follow:

$$\overline{\kappa}(s)=\frac{\kappa(s)}{\kappa^2(s)+\tau^2(s)}=\frac{p\cos\theta}{1+\sin\theta};\qquad\overline{\tau}(s)=\frac{\tau(s)}{\kappa^2(s)+\tau^2(s)}=p.$$

Since $\overline{\alpha}$ is slant helix, it follows:

$$\overline{\sigma}=\frac{1}{\overline{\kappa}\left(s\right)\left(1+\left(\frac{\overline{\tau}(s)}{\overline{\kappa}(s)}\right)^{2}\right)}\left(\frac{\overline{\tau}\left(s\right)}{\overline{\kappa}\left(s\right)}\right)'=\frac{p}{\sqrt{2}}\frac{\sqrt{1-\sin\theta}}{\cos\theta}.$$

Hence $\sqrt{1-\sin\theta}/\cos\theta=\lambda$, where λ is nonzero constant. This leads to $\lambda^{2}=1/\left(1+\sin\theta\right)\Rightarrow\sin\theta=1/\lambda^{2}-1$. By definition of *sinus function* we obtain $1/\sqrt{2}\leq\left|\lambda\right|$. From $\lambda=\sqrt{2}\overline{\sigma}/p$, we have the intended. □

Let $\mathfrak{s}:\alpha\rightarrow\overline{\alpha}$ be an $\mathcal{SPT}$ in Euclidean space $\mathbb{E}^{3}$. Let us recall the fixed axode along any space curve α, $\boldsymbol{a}\left(s,\lambda\right)=\alpha(s)+\frac{\kappa(s)}{\kappa^{2}(s)+\tau^{2}(s)}\boldsymbol{n}(s)+\lambda\boldsymbol{w}(s)$. Since $\overline{\kappa}\left(s\right)=\kappa\left(s\right)/\left(\kappa^{2}\left(s\right)+\tau^{2}\left(s\right)\right)$ and $\overline{\tau}\left(s\right)=\tau\left(s\right)/\left(\kappa^{2}\left(s\right)+\tau^{2}\left(s\right)\right)$, the fixed axode along $\overline{\alpha}$ is given by

$$\overline{\boldsymbol{a}}\left(s,\lambda\right)=\overline{\alpha}(s)+\kappa\left(s\right)\overline{\boldsymbol{n}}(s)+\overline{\lambda}\overline{\boldsymbol{w}}(s).$$

where $\overline{\boldsymbol{n}}$ is the principal normal, $\overline{\boldsymbol{w}}$ is the Darboux vector of $\overline{\alpha}$. Thus, we can conclude the following:

Theorem 6. *Let $\mathfrak{s}:\alpha\rightarrow\overline{\alpha}$ be an $\mathcal{SPT}$ in Euclidean space $\mathbb{E}^{3}$, and both base curve and image curve be a unit speed curve with the same arc-length parameter s. In this case, $\mathfrak{s}$ preserves the distribution parameter of fixed axode of α if and only if $\overline{\kappa}\left(s\right)=\kappa\left(s\right)+c$, where c is a constant.*

Proof. Assume that $\mathfrak{s}$ preserves the distribution parameter of fixed axode of α. Let P and $\overline{P}$ be the distribution parameters of fixed axodes of α and $\overline{\alpha}$, respectively. The distribution parameters of fixed axodes are given by

$$P=\frac{\det\left(\Delta'\left(s\right),\boldsymbol{w}\left(\boldsymbol{s}\right),\boldsymbol{w}'\left(s\right)\right)}{\boldsymbol{w}'\left(s\right)\cdot\boldsymbol{w}'\left(s\right)}\ ;\ \overline{P}=\frac{\det\left(\overline{\Delta}'\left(s\right),\overline{\boldsymbol{w}}\left(s\right),\overline{\boldsymbol{w}}'\left(s\right)\right)}{\overline{\boldsymbol{w}}'\left(s\right)\cdot\overline{\boldsymbol{w}}'\left(s\right)},$$

where $\Delta\left(s\right)=\alpha(s)+\frac{\kappa(s)}{\kappa^{2}(s)+\tau^{2}(s)}\boldsymbol{n}(s)$ and $\overline{\Delta}\left(s\right)=\overline{\alpha}(s)+\kappa\overline{\boldsymbol{n}}(s)$. By straightforward calculations we obtain $\Delta'\left(s\right)=\left(1-\kappa\left(s\right)\overline{\kappa}\left(s\right)\right)\boldsymbol{t}\left(\boldsymbol{s}\right)+\overline{\kappa}'\left(s\right)\boldsymbol{n}\left(\boldsymbol{s}\right)+\overline{\kappa}\left(s\right)\tau\left(s\right)\boldsymbol{b}\left(s\right)$ and $\boldsymbol{w}'\left(s\right)=\tau'\left(s\right)\boldsymbol{t}\left(\boldsymbol{s}\right)+\kappa'\left(s\right)\boldsymbol{b}\left(\boldsymbol{s}\right)$. It can be seen that

$$\Delta'\left(s\right)\times\boldsymbol{w}\left(\boldsymbol{s}\right)=\overline{\kappa}'\left(s\right)\left(\kappa\left(s\right)\boldsymbol{t}\left(s\right)-\tau\left(s\right)\boldsymbol{b}\left(s\right)\right)$$

and $\left(\Delta'\left(s\right)\times\boldsymbol{w}\left(s\right)\right)\cdot\boldsymbol{w}'\left(s\right)=\overline{\kappa}'\left(s\right)\left(\kappa\left(s\right)\tau'\left(s\right)-\tau\left(s\right)\kappa'\left(s\right)\right)$. Hence, the distribution parameter of fixed axode of α is as follows:

$$P=\frac{\overline{\kappa}'\left(s\right)\left(\kappa\left(s\right)\tau'\left(s\right)-\tau\left(s\right)\kappa'\left(s\right)\right)}{\left(\kappa'\left(s\right)\right)^{2}+\left(\tau'\left(s\right)\right)^{2}}.\tag{22}$$

Similarly, we find the distribution parameter of fixed axode of $\overline{\alpha}$ as

$$\overline{P}=\frac{\kappa'\left(s\right)\left(\kappa\left(s\right)\tau'\left(s\right)-\tau\left(s\right)\kappa'\left(s\right)\right)}{\left(\kappa'\left(s\right)\right)^{2}+\left(\tau'\left(s\right)\right)^{2}}.\tag{23}$$

From Eqs. (22) and (23), $\overline{\kappa}'(s) = \kappa'(s)$ or $\overline{\kappa}(s) = \kappa(s) + c$, where c is constant. Conversely, $\overline{\kappa}(s) = \kappa(s) + c$ requires $\overline{\kappa}'(s) = \kappa'(s)$. It is easily obtain that $P = \overline{P}$ which completes the proof. □

5. Examples

In this section, some curves and their images under $\mathcal{SPT}$ are illustrated by *Mathematica* software. First, we recall the following informations and give an example of Mannheim slant helix. The intrinsic equations of slant helices are presented by Menninger in [13] as follows:

$$\kappa(s) = c\beta'(s)\cos\beta(s); \quad \tau(s) = c\beta'(s)\sin\beta(s),$$

where c is constant and β is differentiable function of arc-length parameter of the curve. Furthermore, the tangent vectors of slant helices are characterized by

$$T(s) = \frac{1}{2}\begin{pmatrix} \xi_1\cos\xi_2\Pi(s) + \xi_2\cos\xi_1\Pi(s) \\ \xi_1\sin\xi_2\Pi(s) + \xi_2\sin\xi_1\Pi(s) \\ 2\frac{n}{m}\sin n\Pi(s) \end{pmatrix}, \tag{24}$$

where $\Pi(s) = \beta(s)/n$, $\xi_1 = 1 - n$, $\xi_2 = 1 + n$, with $n = \cos\theta$ and $m = \cot\theta$.

In the following example, parametric equation of Mannheim slant helix, its picture, and its image curve under $\mathcal{SPT}$ are obtained in accordance with [13].

Example 2. *Substituting $\lambda = 1$ and $\varphi(s) = s$ in Proposition 6 gives $\kappa(s) = 1/\cosh^2 s$ and $\tau(s) = \sec hs \tanh s$. This leads to the differential equation*

$$c\beta'(s)\cos\beta(s) = \frac{1}{\cosh^2 s},$$

which has solution

$$\beta(s) = \arcsin\left(\frac{ck + \tanh s}{c}\right), \tag{25}$$

where $k \in \mathbb{R}$. To simplify Eq. (25), we can take $k = 0$ and $c = 1$. It follows from Eq. (24) that

$$T(s) = \begin{pmatrix} \sec h^3 s, \\ \frac{3}{2}\tanh s - \tanh^3 s, \\ \frac{\sqrt{3}}{2}\tanh s \end{pmatrix},$$

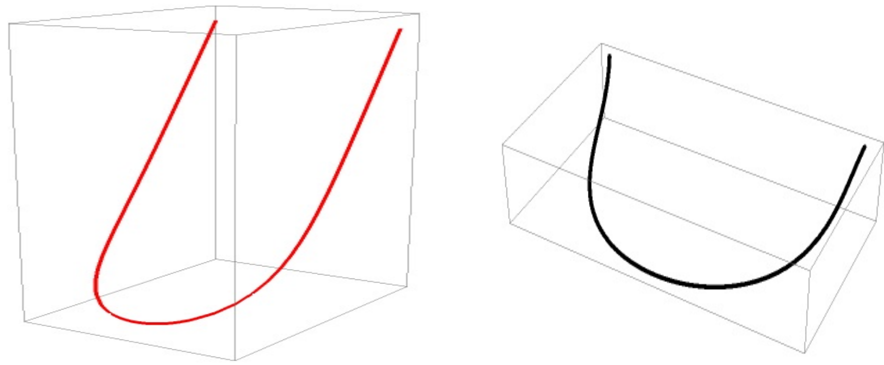
where $\theta = \pi/3$, $n = 1/2$, $m = 1/\sqrt{3}$, $\xi_1 = 1/2$, $\xi_2 = 3/2$, $\beta(s) = \arcsin(\tanh s)$, $\Pi(s) = \beta(s)/n = 2\arcsin(\tanh s)$. Thus, we obtain the parametric equation and picture of the Mannheim slant helix (Figure 1 (a)) as follows:

$$\alpha(s) = \begin{pmatrix} \arctan\left(\tanh\frac{s}{2}\right) + \frac{1}{2}\tanh s\sec hs, \\ \frac{1}{2}\log(\cosh s) - \frac{1}{2}\sec h^2 s, \\ \frac{\sqrt{3}}{2}\log(\cosh s) \end{pmatrix}.$$

Curvatures of the image curve $\overline{\alpha}$ with respect to $\mathfrak{s} : \alpha \to \overline{\alpha}$ can be given by

$$\overline{\kappa}(s) = 1; \quad \overline{\tau}(s) = \sinh s.$$

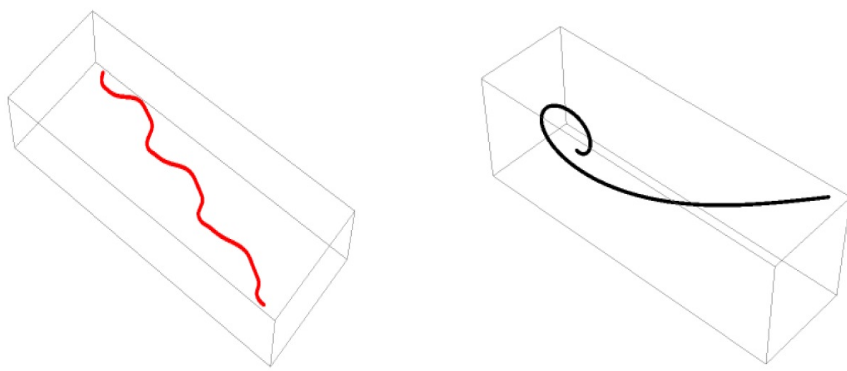
Picture of the image curve $\overline{\alpha}$ illustrated by Figure 1 (b).



(a) Mannheim slant helix (b) Image of Mannheim slant helix

FIGURE 1. Mannheim slant helix and image curve

Example 3. *Let us recall the trigonometric parameterization of curvatures of the constant pitch curves, $\kappa(\theta) = \cos\theta/2p$ and $\tau(\theta) = (1 + \sin\theta)/2p$. By substituting $p = 1$ we obtain $\kappa(\theta) = \cos\theta/2$ and $\tau(\theta) = (1 + \sin\theta)/2$. We give the picture of the curve with curvature $\kappa(\theta)$ and torsion $\tau(\theta)$ by Figure 2 (a).*



(a) Constant pitch curve (b) Image of constant pitch curve

FIGURE 2. Constant pitch curve and image curve

Moreover, curvatures of the image curve of constant pitch curve under $\mathcal{SPT}$ can be given by

$$\overline{\kappa}(\theta) = \frac{\cos\theta}{1 + \sin\theta}; \quad \overline{\tau}(\theta) = 1.$$

The picture of this curve illustrated by Figure 2 (b).

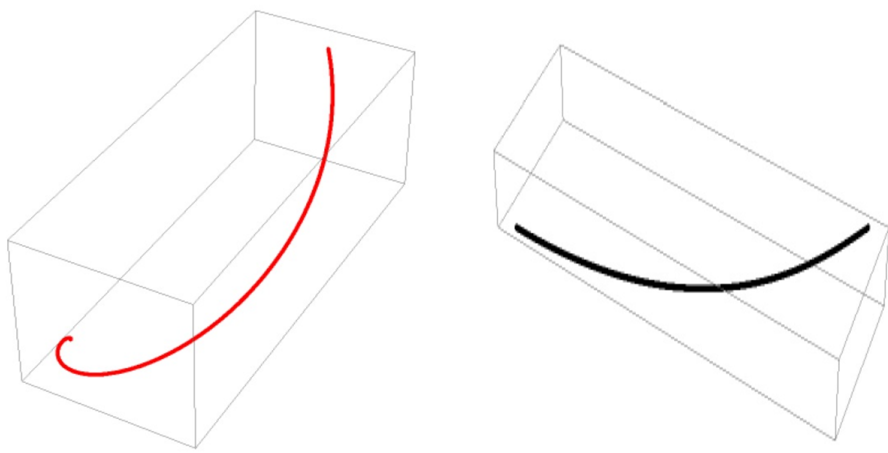
Example 4. *One of trigonometric reparameterization of curvatures of the quasi-slant helix can be given by*

$$\kappa(s) = \frac{\cos^2\left(\frac{2s-1}{2}\right)}{\sin\left(\frac{2s-1}{2}\right)}; \quad \tau(s) = \cos\left(\frac{2s-1}{2}\right).$$

The picture of this curve illustrated by Figure 3 (a), where $1/3 \leq s < 1/2$.

$$\overline{\kappa}\,(s) = \sin\left(\frac{2s-1}{2}\right); \quad \overline{\tau}\,(s) = \frac{\sin^2\left(\frac{2s-1}{2}\right)}{\cos\left(\frac{2s-1}{2}\right)}.$$

The picture of this curve illustrated by Figure 3 (b), where $1 \leq s \leq 2$.



(a) Quasi-slant helix (b) Image of quasi-slant helix

FIGURE 3. Quasi-slant helix and image curve

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