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Half-Lightlike Submanifolds of Metallic semi-Riemannian Manifolds

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ABSTRACT

The aim of the present paper is to study half-lightlike submanifolds of a semi-Riemannian manifold endowed with a metallic structure. We introduce a special half-lightlike submanifold called screen semi-invariant half lightlike submanifold in metallic semi-Riemannian manifolds and give an example. We present necessary and sufficient conditions for the distributions included in the definition of such half lightlike submanifolds to be integrable. Moreover, we analyze geometry of a screen semi-invariant half lightlike submanifold in a locally metallic semi-Riemannian manifold when it is totally geodesic and screen conformal.

Keywords: *Metallic structure, lightlike submanifolds, half lightlike submanifolds.*
AMS Subject Classification (2020): *Primary: 53C15 ; Secondary: 53C25; 53C35.*

1. Introduction

The non-degeneracy of the metric, which is induced from the ambient manifold on the submanifold, has paved the way for the concept of lightlike submanifold. This case, unlike the Riemannian case, has enabled the development of many new tools to examine the geometry of such submanifolds, allowing the examination of rich geometric properties. The theory of lightlike submanifolds was covered in detail by Duggal and Bejancu in 1996 [7] and they introduced a new concept, namely screen distribution, to overcome the difficulties encountered when studying with lightlike submanifolds. Since then lightlike submanifolds of different types of manifolds have been studied by many authors and important geometric results and classifications have been obtained. For detailed reading, we recommend article [7, 13] and the references therein.

A lightlike submanifold of codimension 2 of a semi-Riemannian manifold is called a half-lightlike submanifold if the mapping defining the radical distribution has rank 1 [8, 9]. Screen conformal half-lightlike submanifolds of semi-Riemannian manifolds are presented in [9]. Jin [11, 12] studied screen conformal real half-lighlike submanifolds and real half-lightlike submanifolds in indefinite Kaehler manifolds.

While studying the submanifold theory on manifolds equipped with various structures such as almost complex structure, almost contact structure etc., these structures allow the emergence of different geometric properties and offer very important tools in the examination of these properties. One of the manifold types that has been widely studied in recent years is (semi-)Riemannian manifolds endowed with metallic structures. On a differentiable manifold, the polynomial structure $Q\left(\tilde{J}\right) = \tilde{J}^m + a_m\tilde{J}^{m-1} + ... + a_2\tilde{J} + a_1$, where $\tilde{J}$ is a $(1,1)$-tensor field and I is an identity operator, was introduced by Goldberg et al [19, 20]. Considering the polynomial structure of degree 2, that is $\tilde{J}^2 = \tilde{p}\tilde{J} + \tilde{q}I$, with the special selection of $\tilde{p}$ and $\tilde{q}$, the well-known structures, such as almost product structure, almost para complex structure, almost complex structure, golden structure etc., are obtained.

In 2002, Spinadel [5] introduced the notion of metallic means family, which contains the silver mean, the bronze mean, the copper mean and the nickel mean etc., by calling the the positive solution of the equation $x^2 - \tilde{p}x - \tilde{q} = 0,$ for some positive integer $\tilde{p}$ and $\tilde{q}$, as a $(\tilde{p},\tilde{q})$-metallic number [4, 6]. The metallic means family

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can be considered as a generalization of the golden mean, and is expressed as follows.

$$\tilde{\sigma}_{\tilde{p},\tilde{q}} = \frac{\tilde{p} + \sqrt{\tilde{p}^2 + 4\tilde{q}}}{2}.$$

For $\tilde{p} = \tilde{q} = 1$ and $\tilde{p} = 2, \tilde{q} = 1$, it is well-known that we have the golden mean $\tilde{\sigma}_{1,1} = \tilde{\phi} = \frac{1+\sqrt{5}}{2}$ and silver mean $\tilde{\sigma}_{2,1} = 1 + \sqrt{2}$, respectively.

Inspired by polynomial structures on manifolds, Hretcanu and Crasmareanu defined golden structure [2, 3] and some generalizations of this, called metallic structure [16]. It is well-known that the most important subclasses of metallic Riemannian manifolds is the golden Riemannian manifolds.

In golden semi-Riemannian manifolds, Poyraz Önen and Yaşar [18] started lightlike geometry studies by investigating lightlike hypersurfaces, (see also, [21, 22]). Recently, half-lightlike submanifolds of golden semi-Riemannian manifolds were investigated by [23]. The study of lightlike submanifolds in metallic semi-Riemannian manifolds were initiated with two separate papers by Acet [1] and Erdoğan [14]. After then Erdoğan et al [15] presented differential geometric properties of screen transversal lightlike submanifolds in metallic semi-Riemannian manifolds.

Comparing the existing studies on the lightlike geometry of an arbitrary semi-Riemannian manifold with the studies on lightlike submanifolds of metallic semi-Riemannian manifolds, it is seen that there are still many special lightlike submanifolds of this type of manifolds that can be investigated. In the present paper, we study half-lightlike submanifolds of a metallic semi-Riemannian manifold.

2. Preliminaries

The metallic ratio family is known as the positive solution of $x^2 - \tilde{p}x - \tilde{q} = 0$, such that $\tilde{p}$ and $\tilde{q}$ are fixed positive integers [5]. These solutions are called $(\tilde{p}, \tilde{q})$-metallic numbers and denoted by

$$\tilde{\sigma}_{\tilde{p},\tilde{q}} = \frac{\tilde{p} + \sqrt{\tilde{p}^2 + 4\tilde{q}}}{2}.$$

Let $\tilde{N}$ be m-dimensional differentiable manifold. If a tensor field $\tilde{J}$ of type $(1,1)$ allows

$$\tilde{J}^2 = \tilde{p}\tilde{J} + \tilde{q}I, \tag{2.1}$$

then $\tilde{J}$ is named as a metallic structure in $\tilde{N}$.

Moreover, if a semi-Riemannian manifold $(\tilde{N}, \tilde{g})$ equipped with a metallic structure $\tilde{J}$ such that the semi-Riemannian metric $\tilde{g}$ is $\tilde{J}$ -compatible, i.e., for any $U_1, V_1 \in \Gamma(T\tilde{N})$

$$\tilde{g}(U_1, \tilde{J}V_1) = \tilde{g}(\tilde{J}U_1, V_1), \tag{2.2}$$

which is equivalent to

$$\tilde{g}(\tilde{J}U_1, \tilde{J}V_1) = \tilde{p}\tilde{g}(U_1, \tilde{J}V_1) + \tilde{q}\tilde{g}(U_1, V_1), \tag{2.3}$$

then $(\tilde{g}, \tilde{J}\,)$ (resp., $(\tilde{N}, \tilde{g}, \tilde{J}\,)$) is named as a metallic semi-Riemannian structure (resp., metallic semi-Riemannian manifold (for short, MSRM)), respectively. Note that $\tilde{J}$ is called a locally metallic structure if it is parallel with respect to the Levi-Civita connection of $\tilde{g}$ [5].

Example 2.1. [1] Consider $\tilde{N} = \mathbb{R}^5_2$ with a coordinate system $(u_1, u_2, u_3, u_4, u_5)$ and a pseudo-Euclidean metric of signature $(-,+,-,+,+)$. Taking

$$\tilde{J}(u_1, u_2, u_3, u_4, u_5) = (\tilde{\sigma}u_1, \tilde{\sigma}u_2, \tilde{\sigma}u_3, \tilde{\sigma}u_4, \tilde{\sigma}u_5),$$

then one can easily see that (2.1), which implies $\tilde{J}$ is a metallic structure on $\tilde{N}$, is satisfied .

Remark 2.1. Taking $\tilde{p} = 1 = \tilde{q}$ in (2.3) gives that $(\tilde{g}, \tilde{J})$ is a golden (semi)-Riemannian structure (see [10]).

Given an almost product structure $\tilde{F}$ on $\tilde{N}$ induces two metallic structures on the same manifold defined as

$$\tilde{J}_1 = \frac{\tilde{p}}{2}I + \left(\frac{2\tilde{\sigma}_{\tilde{p},\tilde{q}} - \tilde{p}}{2}\right)\tilde{F}, \quad \tilde{J}_2 = \frac{\tilde{p}}{2}I - \left(\frac{2\tilde{\sigma}_{\tilde{p},\tilde{q}} - \tilde{p}}{2}\right)\tilde{F}. \tag{2.4}$$

On the contrary, every metallic structure $\tilde{J}$ gives allow two almost product structures such that

$$\tilde{F} = \pm\left(\frac{2}{2\tilde{\sigma}_{\tilde{p},\tilde{q}} - \tilde{p}}\tilde{J} - \frac{\tilde{p}}{2\tilde{\sigma}_{\tilde{p},\tilde{q}} - \tilde{p}}I\right). \tag{2.5}$$

A lightlike submanifold M with codimension 2 of a semi-Riemannian manifold $\tilde{N}$ is called a half-lightlike submanifold provided that the radical distribution $RadTM$ of $\tilde{N}$ is a vector subbundle of TM and the normal bundle $TM^{\perp}$ is of rank 1. Then, it is known that there exist complementary non-degenerate distributions $S(TM)$ and $S(TM^{\perp})$ of $RadTM$ in TM and $TM^{\perp}$, respectively, which are called screen distribution and co-screen distribution.

So, we get

$$TM = RadTM \perp S(TM), \tag{2.6}$$

and

$$TM^{\perp} = RadTM \perp S(TM^{\perp}). \tag{2.7}$$

Assume that $S(TM)^{\perp}$ is the orthogonal complementary distribution to $S(TM)$ in $\tilde{N}$ and $L_1 \in \Gamma(S(TM^{\perp}))$, $g(L_1, L_1) = \varepsilon = \pm 1$. Thus we can write

$$S(TM)^{\perp} = S(TM^{\perp}) \perp S(TM^{\perp})^{\perp}, \tag{2.8}$$

where $S(TM^{\perp})^{\perp}$ is orthogonally complement to $S(TM^{\perp})$ in $S(TM)^{\perp}$. For any null section $E_1 \in \Gamma(RadTM)$ on a coordinate neighborhood $\wp \subset M$, there exists uniquely determined null vector field $N_1 \in \Gamma(ltr(TM)$ satisfying [13]

$$\tilde{g}(E_1, N_1) = 1, \quad \tilde{g}(X, N_1) = \tilde{g}(N_1, N_1) = \tilde{g}(L_1, N_1) = 0, \ \ \forall X \in \Gamma(S(TM)). \tag{2.9}$$

$ltr(TM)$ and $tr(TM) = S(TM^{\perp}) \perp ltr(TM)$ are named as lightlike vector bundle and transversal vector bundle of M with respect to $S(TM)$, respectively. Then, $T\tilde{N}$ is decomposed as follows:

$$\begin{aligned} T\tilde{N} &= TM \oplus tr(TM) = \{RadTM \oplus tr(TM)\} \perp S(TM) \\ &= \{RadTM \oplus ltr(TM)\} \perp S(TM) \perp S(TM^{\perp}). \end{aligned} \tag{2.10}$$

For the projection morphism $\omega : \Gamma(TM) \to \Gamma(S(TM))$ and $U_1, V_1 \in \Gamma(TM)$, we get

$$\dot{\tilde{\nabla}}_{U_1} V_1 = \ddot{\tilde{\nabla}}_{U_1} V_1 + B_1(U_1, V_1)N + B_2(U_1, V_1)L, \tag{2.11}$$

$$\dot{\tilde{\nabla}}_{U_1} N = -A_{N_1}U_1 + \tau(U_1)N_1 + \rho(U_1)L_1, \tag{2.12}$$

$$\dot{\tilde{\nabla}}_{U_1} L_1 = -A_L U + \mu(U_1)N_1, \tag{2.13}$$

$$\ddot{\tilde{\nabla}}_{U_1} \omega V_1 = \dot{\tilde{\nabla}}^{*}_{U_1} \omega V_1 + C(U_1, \omega V_1)E_1, \tag{2.14}$$

$$\ddot{\tilde{\nabla}}_{U_1} E_1 = -A^{*}_{E_1} U_1 - \tau(U_1)E_1. \tag{2.15}$$

Here $\dot{\tilde{\nabla}}$ denotes the Levi-Civita connection on $T\tilde{N}$, $\ddot{\tilde{\nabla}}$ and $\dot{\tilde{\nabla}}^{*}$ are induced connection on TM and $S(TM)$, respectively, B_1 and B_2 are called the local second fundamental forms of M, C is named the local second fundamental form on $S(TM)$. A_{N_1}, A_{L_1} and $A^{*}_{E_1}$ are linear operators on TM, τ, ρ and μ are $1-$forms on TM. Because of $\dot{\tilde{\nabla}}$ is torsion free, $\dot{\tilde{\nabla}}$ is also torsion free and second fundamental form B_1 and B_2 are symmetric and satisfying for all $U_1 \in \Gamma(TM)$

$$B_1(U_1, E_1) = 0, \quad B_2(U_1, E_1) = -\varepsilon\mu(U_1). \tag{2.16}$$

For the induced connection $\dot{\tilde{\nabla}}$ on M which is not a metric connection in general, we have

$$(\dot{\tilde{\nabla}}_{U_1}\tilde{g})(V_1, Z_1) = B_1(U_1, V_1)\theta(Z_1) + B_1(U_1, Z_1)\theta(V_1), \tag{2.17}$$

for any $U_1, V_1, Z_1 \in \Gamma(TM)$, where θ is a differential $1-$form such that

$$\theta(U_1) = \tilde{g}(N_1, U_1). \tag{2.18}$$

Also, for these local second fundamental forms and their related shape operators, we write

$$B_1(U_1,V_1)=\tilde{g}(A^*_{E_1}U_1,V_1),\quad \tilde{g}\,(A^*_{E_1}U_1,N_1)=0, \tag{2.19}$$

$$C(U_1,\omega V_1)=\tilde{g}(A_{N_1}U_1,\omega V_1),\quad \tilde{g}(A_{N_1}U_1,N_1)=0, \tag{2.20}$$

$$\varepsilon B_2(U_1,\omega V_1)=\tilde{g}(A_{L_1}U_1,\omega V_1),\quad \tilde{g}(A_{L_1}U_1,N_1)=\varepsilon\rho(U_1), \tag{2.21}$$

$$\varepsilon B_2(U_1,V_1)=\tilde{g}(A_{L_1}U_1,V_1)-\mu(U_1)\theta(V_1), \tag{2.22}$$

$$A^*_{E_1}E_1=0. \tag{2.23}$$

Using (2.11), (2.15) with (2.16), we find

$$\dot{\tilde{\nabla}}_{U_1}E_1=-A^*_{E_1}U_1-\tau(U_1)E_1-\varepsilon\mu(U_1)L_1. \tag{2.24}$$

Definition 2.1. A lightlike submanifold M of a semi-Riemannian manifold $\tilde{N}$ satisfying

$$h(U_1,V_1)=H\tilde{g}(U_1,V_1), \tag{2.25}$$

is called a totally umbilical lightlike submanifold [13].

Therefore, the necessary and sufficient conditions for a lightlike submanifold to be totally umbilical are the followings:

$$B_1(U_1,V_1)=\lambda\tilde{g}(U_1,V_1),\quad B_2(U_1,V_1)=\delta\tilde{g}(U_1,V_1). \tag{2.26}$$

After this section, the abbreviation MSRM will be used instead of the metallic semi-Riemannian manifold concept.

3. HALF-LIGHTLIKE SUBMANIFOLDS OF MSRMs

Assume that M is a half-lightlike submanifold of a MSRM $(\tilde{N},\tilde{g},\tilde{J})$. For any $U_1\in\Gamma(TM)$, $N_1\in\Gamma(ltr(TM))$ and $L_1\in\Gamma(S(TM^\perp))$, we get

$$\tilde{J}U_1=\varphi U_1+u(U_1)N_1+w(U_1)L_1, \tag{3.1}$$

$$\tilde{J}N_1=\xi+v(E_1)N_1+v(L_1)L_1, \tag{3.2}$$

$$\tilde{J}L_1=W_1+w(E_1)N_1+w(L_1)L_1, \tag{3.3}$$

where φU_1, $\xi, W_1\in\Gamma(TM)$, and u, v, w are 1-forms given by

$$\begin{aligned} u(U_1)&=\tilde{g}(U_1,\tilde{J}E_1),\\ v(U_1)&=\tilde{g}(U_1,\tilde{J}N_1),\\ w(U_1)&=\tilde{g}(U_1,\tilde{J}L_1). \end{aligned} \tag{3.4}$$

Lemma 3.1. *Let M be a half-lightlike submanifold of $(\tilde{N},\tilde{g},\tilde{J})$. Then we have*

$$\varphi^2U_1=\tilde{p}\varphi U_1+\tilde{q}U_1-u(U_1)\xi-w(U_1)W_1, \tag{3.5}$$

$$u(\varphi U_1)=\tilde{p}u(U_1)-u(U_1)v(E_1)-w(U_1)w(E_1), \tag{3.6}$$

$$w(\varphi U_1)=\tilde{p}w(U_1)-u(U_1)v(L_1)-w(U_1)w(L_1), \tag{3.7}$$

$$\varphi\xi=\tilde{p}\xi-v(E_1)\xi-v(L_1)W_1, \tag{3.8}$$

$$w\xi=\tilde{p}v(L_1)-v(E_1)v(L_1)-v(L_1)w(L_1), \tag{3.9}$$

$$u\xi=\tilde{p}v(E_1)+\tilde{q}-(v(E_1))^2-v(L_1)w(E_1), \tag{3.10}$$

$$\varphi W_1=\tilde{p}W_1+\tilde{q}-w(E_1)\xi-w(L_1)W_1, \tag{3.11}$$

$$uW_1=\tilde{p}w(E_1)+\tilde{q}-w(E_1)v(E_1)-w(L_1)w(E_1), \tag{3.12}$$

$$wW_1=\tilde{p}w(L_1)+\tilde{q}-w(E_1)v(L_1)-w(L_1)w(L_1), \tag{3.13}$$

$$\tilde{g}(\varphi U_1,V_1)=\tilde{g}(U_1,\varphi Y_1)+u(V_1)\theta(U_1)-u(U_1)\theta(V_1), \tag{3.14}$$

$$\begin{aligned} \tilde{g}(\varphi U_1,\varphi V_1)\;=\;&\tilde{p}\tilde{g}(U,\varphi V)+\tilde{q}\tilde{g}(U,V)+\tilde{p}u(V)\theta(U)\\ &-u(V_1)\tilde{g}(\varphi U_1,N_1)-u(U_1)\tilde{g}(\varphi V_1,N_1)\\ &-w(U_1)w(V_1). \end{aligned} \tag{3.15}$$

Lemma 3.2. *Let M be a half-lightlike submanifold of a locally MSRM $(\tilde{N}, \tilde{g}, \tilde{J})$. Then we get*

$$\begin{aligned}(\dot{\tilde{\nabla}}_{U_1}\varphi)V_1 &= u(V_1)A_{N_1}U_1 + w(V_1)A_{L_1}U_1 + B_1(U_1, V_1)\xi \\ &\quad + B_2(U_1, V_1)\xi,\end{aligned} \tag{3.16}$$

$$\begin{aligned}(\dot{\tilde{\nabla}}_{U_1}u)V_1 &= -B_1(U_1, \varphi V_1) - u(V_1)\tau(U_1) - w(V_1)\mu(U_1) \\ &\quad + B_1(U_1, V_1)v(E_1) + B_2(U_1, V_1)v(E_1),\end{aligned} \tag{3.17}$$

$$\begin{aligned}(\dot{\tilde{\nabla}}_{U_1}w)V_1 &= -B_2(U_1, \varphi V_1) - \rho(U_1)u(V_1)L_1 \\ &\quad + B_1(U_1, V_1)v(L_1) + B_2(U_1, V_1)v(L_1),\end{aligned} \tag{3.18}$$

$$\dot{\tilde{\nabla}}_{U_1}\xi = -\varphi A_{N_1}U_1 + \tau(U_1)\xi + \rho(U_1)W_1, \tag{3.19}$$

$$\begin{aligned}B_1(U_1, \xi) &= U_1(v(\xi)) - \mu(U_1)v(L_1) \\ &\quad - u(A_{N_1}U_1) + \rho(U_1)w(E_1),\end{aligned} \tag{3.20}$$

$$\begin{aligned}B_2(U_1, \xi) &= U_1(v(L_1)) - \rho(U_1)v(E_1) - w(A_{N_1}U_1) \\ &\quad - \tau(U_1)v(L_1) + \rho(U_1)w(L_1),\end{aligned} \tag{3.21}$$

$$\begin{aligned}\dot{\tilde{\nabla}}_{U_1}W_1 &= w(E_1)A_{N_1}U_1 - w(L_1)A_{N_1}U_1 - \varphi A_{L_1}U_1 \\ &\quad + \mu(U_1)\xi,\end{aligned} \tag{3.22}$$

$$\begin{aligned}B_1(U_1, W_1) &= -\tau(U_1)w(E_1) - \mu(U_1)w(L_1) - u(A_{L_1}U_1) \\ &\quad + \mu(U_1)v(E_1),\end{aligned} \tag{3.23}$$

$$\begin{aligned}B_2(U_1, W_1) &= -\rho(U_1)w(E_1) - u(w(L_1)) - w(A_{L_1}U_1) \\ &\quad - \mu(U_1)v(L_1).\end{aligned} \tag{3.24}$$

Now, we give the following definition:

Definition 3.1. Assume that M is a half-lightlike submanifold of a MSRM $(\tilde{N}, \tilde{g}, \tilde{J})$. If

$$\begin{aligned}&\tilde{J}(RadTM) \subset S(TM), \\ &\tilde{J}(ltr(TM)) \subset S(TM), \\ &\tilde{J}(S(TM^{\perp})) \subset S(TM),\end{aligned} \tag{3.25}$$

then M is called a screen semi-invariant half-lightlike submanifold of $\tilde{N}$.

Example 3.1. Let $M_1 = R_1^3$, $M_2 = R^2$ and $\tilde{N} = M_1 \times M_2$ be a semi-Riemannian product manifold with the metric tensor $g = \pi_1^* g_1 + \pi_2^* g_2$, where g_1 and g_2 are the standart metric tensors of R_1^3 and R^2 of signature $(-,+,+)$ and $(+,+)$, π_1 and π_2 are the projections of $\Gamma(T\tilde{N})$ to $\Gamma(TM_1)$ and $\Gamma(TM_2)$, respectively. Consider a submanifold M in $\tilde{N}$ given by the equations

$$\begin{aligned}x^1 &= \sqrt{2}u_1 + u_3, \quad x^2 = u_1 + u_3, \quad x^3 = u_1 + (\sqrt{2} - 1)u_3, \\ x^4 &= u_2 + (\frac{\sqrt{2} - 1}{\sqrt{2}})u_3, \qquad x^5 = u_2 - (\frac{\sqrt{2} - 1}{\sqrt{2}})u_3,\end{aligned}$$

where (u_1, u_2, u_3) are real parameters. Then TM is spanned by $\{U_1, U_2, U_3\}$, where

$$\begin{aligned}U_1 &= \sqrt{2}\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^3}, \\ U_2 &= \frac{\partial}{\partial x^4} + \frac{\partial}{\partial x^5}, \\ U_3 &= \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + (\sqrt{2} - 1)\frac{\partial}{\partial x^3} + (\frac{\sqrt{2} - 1}{\sqrt{2}})\frac{\partial}{\partial x^4} - (\frac{\sqrt{2} - 1}{\sqrt{2}})\frac{\partial}{\partial x^5}.\end{aligned}$$

Thus M is a half-lightlike submanifold with $RadTM = Span\{U_1\}$. The screen distribution $S(TM)$ and $S(TM^{\perp})$ are spanned $\{U_2, U_3\}$ and $\{H\}$, respectively, where

$$H = \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + (\sqrt{2}-1)\frac{\partial}{\partial x^3} + (\frac{\sqrt{2}-1}{\sqrt{2}})\frac{\partial}{\partial x^4} + (\frac{\sqrt{2}-1}{\sqrt{2}})\frac{\partial}{\partial x^5}.$$

Hence lightlike transversal bundle $ltrTM$ is spanned by $N_1 = -\frac{1}{2\sqrt{2}}\frac{\partial}{\partial x^1} + \frac{1}{4}\frac{\partial}{\partial x^2} + \frac{1}{4}\frac{\partial}{\partial x^3}$ [17]. If we use (2.4), then we obtain

$$\tilde{J}_1 U_1 = \frac{\tilde{p}}{2}U_1 + \left(\frac{2\tilde{\sigma}_{\tilde{p},\tilde{q}} - \tilde{p}}{2}\right)\tilde{F}U_1$$

and

$$\tilde{J}_1 U_2 = \frac{\tilde{p}}{2}U_2 + \left(\frac{2\tilde{\sigma}_{\tilde{p},\tilde{q}} - \tilde{p}}{2}\right)\tilde{F}U_2, \quad \tilde{J}_1 U_3 = \frac{\tilde{p}}{2}U_3 + \left(\frac{2\tilde{\sigma}_{\tilde{p},\tilde{q}} - \tilde{p}}{2}\right)\tilde{F}U_3.$$

Since $\tilde{F}RadTM = RadTM$, $\tilde{F}U_2 = U_2$, $\tilde{F}U_3 = S(TM^{\perp})$ and $\tilde{F}ltrTM = ltrTM$, then it can be seen that relations given by (3.25) are satisfied. Hence, M is a proper screen semi invariant half-lightlike submanifold of $M_1 \times M_2$.

Example 3.2. Consider the submanifold M in $\tilde{N} = R_1^5$ defined by

$$u_1 = u_3, \qquad u_5 = \sqrt{1-(u_2^2+u_4^2)}$$

[13] and the metallic structure on $\tilde{N}$ given by

$$\tilde{J}(u_1, u_2, u_3, u_4, u_5) = (\sigma u_1, (\tilde{p}-\tilde{\sigma})u_2, (\tilde{p}-\tilde{\sigma})u_3, (\tilde{p}-\tilde{\sigma})u_4, (\tilde{p}-\tilde{\sigma})u_5).$$

Then we have

$$TM = Span\{\xi = \partial u_1 + \partial u_3,\ V_1 = u_5\partial u_2 - u_2\partial u_5,\ V_2 = u_5\partial u_4 - u_4\partial u_5\}.$$

It follows that

$$N_1 = \frac{1}{2}(-\partial u_1 + \partial u_3),$$

and

$$v = u_2\partial u_2 + u_4\partial u_4 + \sqrt{1-(u_2^2+u_4^2)}\partial u_5.$$

Therefore we find that $\tilde{J}(RadTM) \subset S(TM)$, $\tilde{J}(ltrTM) \subset S(TM)$, $\tilde{J}(S(TM^{\perp})) \subset S(TM)$. Namely M is a screen semi-invariant half-lightlike submanifold.

Taking $\hat{\mathcal{V}}_1 = \tilde{J}\ (RadTM)$, $\hat{\mathcal{V}}_2 = \tilde{J}\ (ltr(TM))$ and $\hat{\mathcal{V}}_3 = \tilde{J}\ (S(TM^{\perp}))$, we get

$$S(TM) = \mathring{\mathcal{V}} \perp \{\hat{\mathcal{V}}_1 \oplus \hat{\mathcal{V}}_2\} \perp \hat{\mathcal{V}}_3, \tag{3.26}$$

where $\mathring{\mathcal{V}}$ is a $(n-3)$-dimensional non-degenerate distribution. Thus, we can write

$$TM = \mathring{\mathcal{V}} \perp \{\hat{\mathcal{V}}_1 \oplus \hat{\mathcal{V}}_2\} \perp \hat{\mathcal{V}}_3 \perp \{Rad(TM) \oplus ltr(TM)\}. \tag{3.27}$$

If we take $\hat{\mathcal{V}} = \mathring{\mathcal{V}} \perp Rad(TM) \perp \tilde{J}\ (Rad(TM))$ and $\hat{\mathcal{V}}^{\perp} = \hat{\mathcal{V}}_2 \perp \hat{\mathcal{V}}_3$ on M, we obtain

$$TM = \hat{\mathcal{V}} \oplus \hat{\mathcal{V}}^{\perp}, \quad T\tilde{N} = \{\hat{\mathcal{V}} \oplus \hat{\mathcal{V}}^{\perp}\} \oplus ltr(TM). \tag{3.28}$$

Also, for local lightlike vector fields $\zeta \in \Gamma(\hat{\mathcal{V}})$, $\psi \in \Gamma(\hat{\mathcal{V}}^{\perp})$, $\varrho \in \Gamma(\hat{\mathcal{V}}^{\perp})$ and for any $U_1 \in \Gamma(TM)$, we have

$$\varphi^2 U_1 = \tilde{p}\varphi U_1 + \tilde{q}U_1 - u(U_1)\zeta - w(U_1)\varrho, \tag{3.29}$$

$$u(\varphi U_1) = \tilde{p}u(U_1), \quad w(\varphi U_1) = \tilde{p}w(U_1), \quad \varphi\xi = \tilde{p}\xi, \tag{3.30}$$

$$\varphi W_1 = \tilde{p}W_1, \quad u(W_1) = 0, \quad w(W_1) = \tilde{q}, \tag{3.31}$$

$$\tilde{g}(\varphi U_1, V_1) = \tilde{g}(U_1, \varphi V_1) + u(V_1)\theta(U_1) - u(U_1)\theta(V_1), \tag{3.32}$$

$$\begin{aligned}\tilde{g}(\varphi U_1, \varphi V_1) &= \tilde{p}\tilde{g}(U_1, \varphi V_1) + q\tilde{g}(U_1, V_1) + \tilde{p}u(V_1)\theta(U_1) \\ &\quad -u(V_1)\tilde{g}(\varphi U_1, N_1) - u(U_1)\tilde{g}(\varphi V_1, N_1),\end{aligned} \tag{3.33}$$

$$\begin{aligned}(\dot{\tilde{\nabla}}_{U_1}\varphi)V_1 &= u(V_1)A_{N_1}U_1 + w(V_1)A_{L_1}U_1 + B_1(U_1, V_1)\zeta \\ &\quad +B_2(U_1, V_1)\varrho,\end{aligned} \tag{3.34}$$

$$(\dot{\tilde{\nabla}}_{U_1}u)V_1 = -B_1(U_1, \varphi V_1) - u(V_1)\tau(U_1) - \mu(U_1)w(V_1), \tag{3.35}$$

$$\begin{aligned}(\dot{\tilde{\nabla}}_{U_1}w)V_1 &= -B_2(U_1, \varphi V_1) - u(V_1)\rho(U_1) + B_1(U_1, V_1) \\ &\quad +B_2(U_1, V_1),\end{aligned} \tag{3.36}$$

$$\dot{\tilde{\nabla}}_{U_1}\zeta = -\varphi A_N U_1 + \tau(U_1)\psi + \rho(U_1)\varrho, \tag{3.37}$$

$$\dot{\tilde{\nabla}}_{U_1}\psi = -\varphi A^*_{E_1}U_1 - \tau(U_1)\psi - \mu(U_1)\varrho, \tag{3.38}$$

$$\dot{\tilde{\nabla}}_{U_1}\varrho = -\varphi A_{L_1}U_1 + \phi(U_1)\zeta, \tag{3.39}$$

$$\begin{aligned}B_1(U_1, \zeta) &= -C(U_1, \psi), \\ B_1(U_1, \varrho) &= B_2(U_1, \psi), \\ B_2(U_1, \zeta) &= -C(U_1, \varrho),\end{aligned} \tag{3.40}$$

$$B_1(U_1, \psi) = C(U_1, \zeta) = B_2(U_1, \varrho). \tag{3.41}$$

Theorem 3.1. *Assume that M is a screen semi-invariant half-lightlike submanifold of a locally MSRM $(\tilde{N}, \tilde{g}, \tilde{J})$. Then φ is a metallic structure on $\mathring{\mathcal{V}}$.*

Proof. In view of (3.29), for $U_1 \in \Gamma(\mathring{\mathcal{V}})$, we get

$$\varphi^2 U_1 = \tilde{p}\varphi U_1 + \tilde{q}U_1.$$

Also by use of (3.32), for every $U_1, V_1 \in \Gamma(\mathring{\mathcal{V}})$, we have

$$\tilde{g}(\varphi U_1, V_1) = \tilde{g}(U_1, \varphi V_1),$$

which shows that φ is a metallic structure on the distribution $\mathring{\mathcal{V}}$. □

Theorem 3.2. *Assume that M is a screen semi-invariant half-lightlike submanifold of a locally MSRM $(\tilde{N}, \tilde{g}, \tilde{J})$. Then $\mathring{\mathcal{V}}$ is integrable if and only if*

$$\begin{aligned}&B_1(\tilde{J}U_1, V_1) = B_1(U_1, \tilde{J}V_1) \qquad B_2(\tilde{J}U_1, V_1) = B_2(U_1, \tilde{J}V_1), \\ &C(\tilde{J}U_1, V_1) = C(U_1, \tilde{J}V_1) \qquad C(U_1, V_1) = C(V_1, U_1),\end{aligned} \tag{3.42}$$

for every $U_1, V_1 \in \Gamma(\mathring{\mathcal{V}})$.

Proof. From the definition of the distribution, $\mathring{\mathcal{V}}$ is integrable if and only if

$$u([U_1, V_1]) = v([U_1, V_1]) = w([U_1, V_1]) = \theta([U_1, V_1]) = 0,$$

for every $U_1, V_1 \in \Gamma(\mathring{\mathcal{V}})$. So, we get

$$\begin{aligned}0 &= u([U_1, V_1]) \\ &= \tilde{g}([U_1, V_1], \tilde{J}E) \\ &= \tilde{g}(\dot{\tilde{\nabla}}_{U_1}\tilde{J}V_1 - \dot{\tilde{\nabla}}_{V_1}\tilde{J}U_1, E_1) \\ &= B_1(U_1, \tilde{J}V_1) - B_1(\tilde{J}U_1, V_1),\end{aligned}$$

$$\begin{aligned}0 &= v([U_1, V_1]) \\ &= \tilde{g}([U_1, V_1], N) \\ &= \tilde{g}(\dot{\tilde{\nabla}}_{U_1}\tilde{J}V_1 - \dot{\tilde{\nabla}}_{V_1}\tilde{J}U_1, N_1) \\ &= C(U_1, \tilde{J}V_1) - C(V_1, \tilde{J}U_1),\end{aligned}$$

$$\begin{aligned} 0 &= w([U_1, V_1]) \\ &= \tilde{g}([U_1, V_1], \tilde{J}L_1) \\ &= \tilde{g}(\dot{\tilde{\nabla}}_{U_1} \tilde{J}V_1 - \dot{\tilde{\nabla}}_{V_1} \tilde{J}U_1, L_1) \\ &= B_2(U_1, \tilde{J}V_1) - B_2(\tilde{J}U_1, V_1), \end{aligned}$$

and

$$\begin{aligned} 0 &= \theta([U_1, V_1]) \\ &= \tilde{g}([U_1, V_1], N_1) \\ &= \tilde{g}(\dot{\tilde{\nabla}}_{U_1} V_1 - \dot{\tilde{\nabla}}_{V_1} U_1, N_1) \\ &= C(U_1, V_1) - C(V_1, U_1). \end{aligned}$$

Thus, we arrive at the equation (3.42). □

Theorem 3.3. *Assume that M is a screen semi-invariant half-lightlike submanifold of a locally MSRM $(\tilde{N}, \tilde{g}, \tilde{J})$. Then $\hat{\mathcal{V}}$ is integrable if and only if*

$$B_1(\tilde{J}U_1, V_1) = B_1(U_1, \tilde{J}V_1) \qquad B_2(\tilde{J}U_1, V_1) = B_2(U_1, \tilde{J}V_1), \tag{3.43}$$

for every $U_1, V_1 \in \Gamma(\hat{\mathcal{V}})$.

Proof. From the definition of the distribution $\hat{\mathcal{V}}$ is integrable if and only if

$$u([U_1, V_1]) = w([U_1, V_1]) = 0.$$

Therefore, we have

$$\begin{aligned} 0 &= u([U_1, V_1]) \\ &= \tilde{g}([U_1, V_1], \tilde{J}E_1) \\ &= \tilde{g}(\dot{\tilde{\nabla}}_{U_1} \tilde{J}V_1 - \dot{\tilde{\nabla}}_{V_1} \tilde{J}U_1, E_1) \\ &= B_1(U_1, \tilde{J}V_1) - B_1(\tilde{J}U_1, V_1) \end{aligned}$$

and

$$\begin{aligned} 0 &= w([U_1, V_1]) \\ &= \tilde{g}([U_1, V_1], \tilde{J}L) \\ &= \tilde{g}(\dot{\tilde{\nabla}}_{U_1} \tilde{J}V_1 - \dot{\tilde{\nabla}}_{V_1} \tilde{J}U_1, L) \\ &= B_2(U_1, \tilde{J}V_1) - B_2(\tilde{J}U_1, V_1). \end{aligned}$$

So, we complete the proof. □

Theorem 3.4. *Assume that M is a screen semi-invariant half-lightlike submanifold of a locally MSRM $(\tilde{N}, \tilde{g}, \tilde{J})$. If the distribution $\hat{\mathcal{V}}$ is parallel then $\hat{\mathcal{V}}$ is totally geodesic on M.*

Proof. From the decomposition (3.28), $\hat{\mathcal{V}}$ is parallel if and only if

$$\tilde{g}(\dot{\tilde{\nabla}}_{U_1} V_1, \psi) = \tilde{g}(\dot{\tilde{\nabla}}_{U_1} V_1, \varrho) = 0,$$

for any $U_1, V_1 \in \Gamma(\hat{\mathcal{V}})$. Thus, we have

$$0 = \tilde{g}(\dot{\tilde{\nabla}}_{U_1} V_1, \psi) = \tilde{g}(\dot{\tilde{\nabla}}_{U_1} V_1, \psi) = \tilde{g}(\dot{\tilde{\nabla}}_{U_1} V_1, \tilde{J}E_1) = B_1(U_1, \tilde{J}V_1)$$

and

$$0 = \tilde{g}(\dot{\tilde{\nabla}}_{U_1} V_1, \varrho) = \tilde{g}(\dot{\tilde{\nabla}}_{U_1} V_1, \varrho) = \tilde{g}(\dot{\tilde{\nabla}}_{U_1} V_1, \tilde{J}L_1) = B_2(U_1, \tilde{J}V_1).$$

So, we get our assertion. □

Corollary 3.1. *Assume that M is a screen semi-invariant half-lightlike submanifold of a locally MSRM $(\tilde{N}, \tilde{g}, \tilde{J})$. If the distribution $\hat{\mathcal{V}}$ is parallel then φ is parallel on M.*

Theorem 3.5. *Assume that M is a screen semi-invariant half-lightlike submanifold of a locally MSRM $(\tilde{N}, \tilde{g}, \tilde{J})$. If M is totally geodesic, then we have*

$$(\dot{\tilde{\nabla}}_{U_1}\varphi)Z_1 = 0, \quad (\dot{\tilde{\nabla}}_{U_1}\varphi)\zeta = 0, \quad (\dot{\tilde{\nabla}}_{U_1}\varphi)\varrho = 0,$$

for $U_1 \in \Gamma(TM)$ and $Z_1 \in \Gamma(\hat{\mathcal{V}})$.

Proof. Let M be totally geodesic. From (3.34), for any $Z_1 \in \Gamma(\hat{\mathcal{V}})$, we find

$$u(Z_1) = w(Z_1) = 0,$$

which yields

$$(\dot{\tilde{\nabla}}_{U_1}\varphi)Z_1 = 0.$$

Similarly taking ζ and ϱ in (3.34), we obtain

$$(\dot{\tilde{\nabla}}_{U_1}\varphi)\zeta = 0, \quad (\dot{\tilde{\nabla}}_{U_1}\varphi)\varrho = 0,$$

respectively. So, the claim holds. □

Definition 3.2. Let M be a screen semi-invariant half-lightlike submanifold of a locally MSRM $(\tilde{N}, \tilde{g}, \tilde{J})$. If

$$B_1(U_1, V_1) = B_2(U_1, V_1) = 0,$$

for any $U_1 \in \Gamma(\hat{\mathcal{V}})$ and $V_1 \in \Gamma(\hat{\mathcal{V}}^{\perp})$, then M is called a mixed totally geodesic lightlike submanifold.

Theorem 3.6. *Assume that M is a screen semi-invariant half-lightlike submanifold of a locally MSRM $(\tilde{N}, \tilde{g}, \tilde{J})$. If M is mixed totally geodesic then $A_{\tilde{J}U_1}V_1$ has no component in $\Gamma(\hat{\mathcal{V}}_2 \perp \hat{\mathcal{V}}_3)$, for any $U_1 \in \Gamma(\mathring{\mathcal{V}})$ and $V_1 \in \Gamma(\hat{\mathcal{V}}^{\perp})$.*

Proof. Suppose that M is mixed totally geodesic, i.e.,

$$B_1(U_1, V_1) = 0 = B_2(U_1, V_1).$$

Then we can write

$$\begin{aligned} B_1(U_1, V_1) &= \tilde{g}(\dot{\tilde{\nabla}}_{U_1}V_1, E_1) \\ &= \frac{1}{\tilde{q}}\tilde{g}(\dot{\tilde{\nabla}}_{U_1}\tilde{J}V_1, \tilde{J}E_1) - \frac{\tilde{p}}{\tilde{q}}\tilde{g}(\dot{\tilde{\nabla}}_{U_1}\tilde{J}V_1, E_1) \\ &= \frac{1}{\tilde{q}}\tilde{g}(A_{\tilde{J}V_1}U_1, \tilde{J}E_1) - \frac{\tilde{p}}{\tilde{q}}B_1(U_1, \tilde{J}V_1), \end{aligned} \tag{3.44}$$

and

$$\begin{aligned} B_2(U_1, V_1) &= \tilde{g}(\dot{\tilde{\nabla}}_{U_1}V_1, L_1) \\ &= \frac{1}{\tilde{q}}\tilde{g}(\dot{\tilde{\nabla}}_{U_1}\tilde{J}V_1, \tilde{J}L_1) - \frac{\tilde{p}}{\tilde{q}}\tilde{g}(\dot{\tilde{\nabla}}_{U_1}\tilde{J}V_1, L_1) \\ &= \frac{1}{\tilde{q}}\tilde{g}(A_{\tilde{J}V_1}U_1, \tilde{J}L_1) - \frac{\tilde{p}}{\tilde{q}}B_2(U_1, \tilde{J}V_1). \end{aligned} \tag{3.45}$$

From (3.45) and (3.46), we get

$$\tilde{q}B_1(U_1, V_1) + \tilde{p}B_1(U_1, \tilde{J}V_1) = \tilde{g}(A_{\tilde{J}V_1}U_1, \tilde{J}E_1),$$

and

$$\tilde{q}B_2(U_1, V_1) + \tilde{p}B_2(U_1, \tilde{J}V_1) = \tilde{g}(A_{\tilde{J}V_1}U_1, \tilde{J}L_1).$$

Thus, we proof our claim. □

Theorem 3.7. *Assume that M is a screen semi-invariant half-lightlike submanifold of a locally MSRM $(\tilde{N}, \tilde{g}, \tilde{J})$. If M is mixed totally geodesic then for any $U_1 \in \Gamma(\hat{\mathcal{V}})$ and $V_1 \in \Gamma(\hat{\mathcal{V}}^{\perp})$*

$$(\dot{\dot{\nabla}}_{U_1}\varphi)V_1 = 0. \tag{3.46}$$

Proof. For any $U_1 \in \Gamma(\hat{\mathcal{V}})$ and $V_1 \in \Gamma(\hat{\mathcal{V}}^{\perp})$, we know that

$$u(V_1) = 0 = w(V_1).$$

In viev of (3.34) with above equation, we find

$$(\dot{\dot{\nabla}}_{U_1}\varphi)V_1 = B_1(U_1, V_1)\zeta + B_2(U_1, V_1)\varrho,$$

which gives (3.46). □

Definition 3.3. Let $\tilde{N}$ be a semi-Riemannian manifold and M be a half-lightlike submanifold of $\tilde{N}$. If the shape operator A_{N_1} and $A^*_{E_1}$ of M and $S(TM)$, respectively, are related by $A_{N_1} = fA^*_{E_1}$ or equivalently

$$C(U_1, \tilde{J}V_1) = fB(U_1, V_1), \quad U_1, V_1 \in \Gamma(TM), \tag{3.47}$$

holds, then M is a screen conformal submanifold, where f is a non-vanishing smooth function on a neighborhood $\wp$ in M [13].

In view of (3.40) and (3.41) with (3.47), we have

$$\begin{aligned} h(U_1, \zeta) &= B_1(U_1, \zeta)N_1 + B_2(U_1, \zeta)L_1 \\ &= -C(U_1, \zeta) - C(U_1, \varrho) \\ &= -fB_1(U_1, \psi) - fB_1(U_1, \varrho) \\ &= fB_1(U_1, \psi) + fB_2(U_1, \psi) \\ &= fh(U_1, \psi), \end{aligned}$$

for any $U_1 \in \Gamma(TM)$, where h is global second fundamental form of M. Therefore, we arrive at

$$h(U_1, \zeta - f\psi) = 0. \tag{3.48}$$

Theorem 3.8. *Assume that M is a screen conformal half-lightlike submanifold of a locally MSRM $(\tilde{N}, \tilde{g}, \tilde{J})$. If M is totally umbilical then M and $S(TM)$ are totally geodesics.*

Proof. Let M be a screen conformal half-lightlike submanifold of a locally MSRM $\tilde{N}$. In view of (2.25) with (3.48), we get

$$Hg(U_1, \zeta - f\psi) = 0.$$

Taking U_1 for ζ in above equation, we find $H = 0$ and so $h = 0$. Hence, we obtain $B_1 = 0 = B_2$ and $C = 0$. □

Theorem 3.9. *Assume that M is a screen conformal half-lightlike submanifold of a locally MSRM $(\tilde{N}, \tilde{g}, \tilde{J})$. Then any screen distribution $S(TM)$ of M is integrable.*

Proof. In view of (2.9), $S(TM)$ is integrable if and only if, for $U_1, V_1 \in \Gamma(S(TM))$,

$$\tilde{g}([U_1, V_1], N_1) = 0.$$

Then by using (2.11) and (2.14), we get

$$\begin{aligned} 0 &= \tilde{g}(\dot{\dot{\nabla}}_{U_1}V_1, N_1) - \tilde{g}(\dot{\dot{\nabla}}_{V_1}U_1, N_1) \\ &= \tilde{g}(\dot{\dot{\nabla}}_{U_1}V_1 + B_1(U_1, V_1)N_1 + B_2(U_1, V_1)L_1, N_1) \\ &\quad -\tilde{g}(\dot{\dot{\nabla}}_{V_1}U_1 + B_1(V_1, U_1)N_1 + B_2(V_1, U_1)L_1, N_1) \\ &= \tilde{g}(\dot{\dot{\nabla}}_{U_1}V_1, N_1) - \tilde{g}(\dot{\dot{\nabla}}_{V_1}U_1, N_1) \\ &= \tilde{g}(\dot{\dot{\nabla}}^*_{U_1}V_1 + C(U_1, V_1)E_1, N_1) \\ &\quad -\tilde{g}(\dot{\dot{\nabla}}^*_{V_1}U_1 + C(V_1, U_1)E, N) \\ &= C(U_1, V_1) - C(V_1, U_1) \\ &= f(B_1(U_1, V_1) - B_1(U_1, V_1)). \end{aligned}$$

Because of B_1 is symmetric we arrive at our assertion. □

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References

[1] Acet, B. E.: *Lightlike hypersurfaces of metallic semi-Riemannian manifolds*. International Journal of Geometric Methods in Modern Physics. 2018; DOI:10.1142/S0219887818502018.
[2] Blaga, A. M., Hretcanu, C. E.: *Invariant, anti-invariant and slant submanifolds of a metallic Riemannian manifold.* Novi Sad Journal Mathematics. **48** (2), 55-80 (2018).
[3] Crasmareanu, M., Hretcanu C. E.: *Golden differential geometry.* Chaos, Solitons & Fractals **38**, 1229-1238 (2008).
[4] De Spinadel, V. W.: *On specification of the onset to chaos.* Chaos, Solitons & Fractals. **8** (10), 1631-1643 (1997).
[5] De Spinadel, V. W.: *The metallic means family and forbidden symmetries.* International Mathematics Journal. **2** (3), 279-288 (2002).
[6] De Spinadel, V. W.: *The metallic means family and renormalization group techniques.* Proceedings of the Steklov Institute of Mathematics., Control in Dynamic Systems. **1**, 194-209 (2000).
[7] Duggal, K. L., Bejancu, A.: *Lightlike submanifolds of semi-Riemannian manifolds and applications.* Mathematics and Its Applications. Kluwer Publisher, (1996)
[8] Duggal K. L., Jin, D. H.: *Half-lightlike submanifolds of codimension 2.* Math. J. Toyama Univ. **22**, 121-161 (1999).
[9] Duggal, K. L., Şahin, B.: *Screen conformal half-lightlike submanifolds.* International Journal of Matematics and Matematical Sciences. **68** , 3737-3753 (2004).
[10] Hretcanu, C. E., Crasmareanu, M.: *Applications of the golden ratio on Riemannian manifolds.* Turkish Journal of Mathematics. **33** (2), 179-191 (2009).
[11] Jin, D. H.: *Geometry of screen conformal real half-lightlike submanifolds.* Bulletin of the Korean Mathematical Society. **47** (4), 701-714 (2010).
[12] Jin, D. H.: *Real half-lightlike submanifolds of an indefinite Kaehler manifold.* Communication Korean Mathematical Society. **26** (4), 635-647 (2011).
[13] Duggal, K. L., Şahin, B.: Differential geometry of lightlike submanifolds. Frontiers in Mathematics. Birkhäuser Verlag, Basel (2010).
[14] Erdoğan, F. E.: *Transversal lightlike submanifolds of metallic semi-Riemannian manifold.* Turkish Journal of Mathematics. **42**, 3133-3148 (2018).
[15] Erdoğan, F. E., Yüksel Perktaş, S., Acet, B. E., Blaga, A. M.: *Screen transversal lightlike submanifolds of metallic semi-Riemannian manifolds.* Journal of Geometry and Physics. **142**, 111-120 (2019).
[16] Hretcanu, C. E., Crasmareanu, M.: *Metallic structures on Riemannian manifolds.* Revista de la Unión Matemática Argentina. **54**, 15-27 (2013).
[17] Kılıç, E., Şahin, B., Keleş, S.: *Screen semi invariant lightlike submanifolds of semi Riemannian product manifolds.* International Electronic Journal of Geometry. **4**, 120-135 (2011).
[18] Poyraz, NÖ., Yaşar E.: *Lightlike hypersurfaces of a golden semi-Riemannian manifold.* Mediterrenean Journal of Mathematics. **14** (2017).
[19] Goldberg, S. I., Yano, K.: *Polynomial structures on manifolds.* Kodai Mathematical Seminar Reports. **22**, 199-218 (1970).
[20] Goldberg, S. I., Petridis, N. C.: *Differentiable solutions of algebraic equations on manifolds.* Kodai Mathematical Seminar Reports. **25**, 111-128 (1973).
[21] Poyraz, N., Yaşar, E.: *Lightlike submanifolds of golden semi-Riemannian manifolds.* Journal of Geometry and Physics. **141**, 92-104 (2019).
[22] Poyraz, N.: *Screen semi invariant lightlike submanifolds of golden semi-Riemannian manifolds.* International Electronic Journal of Geometry. **14** (1), 207-216 (2021).
[23] Poyraz, N, Yaşar, E., Dönmez, D.: *Half-lightlike submanifolds of a golden semi-Riemannian manifold.* Kragujevac Journal of Mathematics. **48** (1), 99-122 (2024).

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