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On the Generalized of p-Harmonic Maps

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ABSTRACT

In this paper, we extend the definition of p-harmonic and p-biharmonic maps between Riemannian manifolds. We present some new properties for the generalized stable p-harmonic maps.

Keywords: *p-harmonic maps, p-biharmonic maps, stable p-harmonic maps.*
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1. Introduction

Consider a smooth map $\varphi : (M, g) \longrightarrow (N, h)$ between Riemannian manifolds, and let p be a smooth positive function on M such that $p(x) \geq 2$ for all $x \in M$. For any compact domain D of M the $p(\cdot)$-energy functional of φ is defined by

$$E_{p(\cdot)}(\varphi; D) = \int_D \frac{|d\varphi|^{p(x)}}{p(x)} v_g, \tag{1.1}$$

where $|d\varphi|$ is the Hilbert-Schmidt norm of the differential $d\varphi$ and v^g is the volume element on (M, g). A map is called $p(\cdot)$-harmonic if it is a critical point of the $p(\cdot)$-energy functional over any compact subset D of M. $p(\cdot)$-harmonic maps is a natural generalization of harmonic map ([1, 5]) and p-harmonic map ([2, 3, 6]). We denote by

$$\tau(\varphi) = \text{trace}_g \nabla d\varphi = \sum_{i=1}^{m} \{\nabla^{\varphi}_{e_i} d\varphi(e_i) - d\varphi(\nabla^M_{e_i} e_i)\}. \tag{1.2}$$

the tension field of φ, where $\{e_i\}_{i=1}^m$ is an orthonormal frame on (M, g), ∇^M is the Levi-Civita connection of (M, g), and ∇^φ denote the pull-back connection on $\varphi^{-1}TN$.
In this paper, we investigate some properties for $p(\cdot)$-harmonic maps between two Riemannian manifolds. In particular, we present the first and the second variation of the $p(\cdot)$-energy. We also extend the definition of p-biharmonic maps between two Riemannian manifolds ([8]).

2. $p(\cdot)$-Harmonic Maps

Theorem 2.1 (The first variation of the $p(\cdot)$-energy). *Let $\varphi : (M, g) \to (N, h)$ be a smooth map between two Riemannian manifolds, $\{\varphi_t\}_{t\in(-\epsilon,\epsilon)}$ a smooth variation of φ supported in compact domain D of M. Then*

$$\frac{d}{dt} E_{p(\cdot)}(\varphi_t; D)\Big|_{t=0} = -\int_D h(v, \tau_{p(\cdot)}(\varphi)) v_g, \tag{2.1}$$

where $\tau_{p(\cdot)}(\varphi)$ denotes the $p(\cdot)$-tension field of φ given by

$$\tau_{p(\cdot)}(\varphi) = \text{trace}_g \nabla |d\varphi|^{p(x)-2} d\varphi, \tag{2.2}$$

and $v = \frac{d\varphi_t}{dt}\big|_{t=0}$ denotes the variation vector field of $\{\varphi_t\}_{t\in(-\epsilon,\epsilon)}$.

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Proof. Let $\phi : M \times (-\epsilon, \epsilon) \longrightarrow N$ be a smooth map defined by

$$\phi(x,t) = \varphi_t(x), \quad \forall (x,t) \in M \times (-\epsilon, \epsilon).$$

We have $\phi(x,0) = \varphi(x)$ for all $x \in M$, and the variation vector field associated to the variation $\{\varphi_t\}_{t \in (-\epsilon,\epsilon)}$ is given by

$$v(x) = d_{(x,0)}\phi(\frac{\partial}{\partial t})\Big|_{t=0}, \quad \forall x \in M.$$

Let $\{e_i\}_{i=1}^m$ be an orthonormal frame on (M,g). We compute

$$\begin{aligned}
\frac{d}{dt} E_{p(\cdot)}(\varphi_t; D)\Big|_{t=0} &= \frac{d}{dt}\int_D \frac{|d\varphi_t|^{p(x)}}{p(x)} v_g\Big|_{t=0} \\
&= \int_D \frac{1}{p(x)}\frac{\partial}{\partial t}|d\varphi_t|^{p(x)}\Big|_{t=0} v_g \\
&= \int_D \frac{1}{p(x)}\frac{\partial}{\partial t}(|d\varphi_t|^2)^{\frac{p(x)}{2}}\Big|_{t=0} v_g \\
&= \sum_{i=1}^m \int_D \frac{1}{p(x)}\frac{\partial}{\partial t} h(d\varphi_t(e_i), d\varphi_t(e_i))^{\frac{p(x)}{2}}\Big|_{t=0} v_g \\
&= \sum_{i=1}^m \int_D \frac{1}{p(x)}\frac{\partial}{\partial t} h(d\phi(e_i,0), d\phi(e_i,0))^{\frac{p(x)}{2}}\Big|_{t=0} v_g \\
&= \sum_{i=1}^m \int_D h(\nabla^\phi_{\frac{\partial}{\partial t}} d\phi(e_i,0), d\phi(e_i,0))(|d\varphi_t|^2)^{\frac{p(x)}{2}-1}\Big|_{t=0} v_g. \qquad (2.3)
\end{aligned}$$

By using the property

$$\nabla^\phi_X d\phi(Y) = \nabla^\phi_Y d\phi(X) + d\phi([X,Y]),$$

with $X = \frac{\partial}{\partial t}$, $Y = (e_i, 0)$, and $[\frac{\partial}{\partial t}, (e_i,0)] = 0$, the equation (2.3) becomes

$$\begin{aligned}
\frac{d}{dt} E_{p(\cdot)}(\varphi_t; D)\Big|_{t=0} &= \sum_{i=1}^m \int_D h(\nabla^\phi_{(e_i,0)} d\phi(\frac{\partial}{\partial t}), d\phi(e_i,0))|d\varphi_t|^{p(x)-2}\Big|_{t=0} v_g \\
&= \sum_{i=1}^m \int_D h(\nabla^\varphi_{e_i} v, |d\varphi|^{p(x)-2} d\varphi(e_i)) v_g \\
&= \sum_{i=1}^m \int_D \Big[e_i h(v, |d\varphi|^{p(x)-2} d\varphi(e_i)) \\
&\quad - h(v, \nabla^\varphi_{e_i} |d\varphi|^{p(x)-2} d\varphi(e_i))\Big] v_g. \qquad (2.4)
\end{aligned}$$

Let $\omega \in \Gamma(T^*M)$ defined by

$$\omega(X) = h(v, |d\varphi|^{p(x)-2} d\varphi(X)), \quad \forall X \in \Gamma(TM)$$

The divergence of ω is given by

$$\operatorname{div}^M \omega = \sum_{i=1}^m \Big[e_i h(v, |d\varphi|^{p(x)-2} d\varphi(e_i)) - h(v, |d\varphi|^{p(x)-2} d\varphi(\nabla^M_{e_i} e_i))\Big]. \qquad (2.5)$$

By equations (2.4), (2.5), and the divergence Theorem [1], we get

$$\begin{aligned}
\frac{d}{dt} E_{p(\cdot)}(\varphi_t; D)\Big|_{t=0} &= \sum_{i=1}^m \int_D h\big(v, |d\varphi|^{p(x)-2} d\varphi(\nabla^M_{e_i} e_i) - \nabla^\varphi_{e_i} |d\varphi|^{p(x)-2} d\varphi(e_i)\big) v_g \\
&= -\sum_{i=1}^m \int_D h\left(v, \Big[\nabla_{e_i} |d\varphi|^{p(x)-2} d\varphi\Big](e_i)\right) v_g. \qquad (2.6)
\end{aligned}$$

□

Corollary 2.1. *A smooth map $\varphi : (M,g) \longrightarrow (N,h)$ between two Riemannian manifolds is $p(\cdot)$-harmonic if and only if*

$$\tau_{p(\cdot)}(\varphi) = |d\varphi|^{p(x)-2}\tau(\varphi) + d\varphi(\operatorname{grad}^M |d\varphi|^{p(x)-2}) = 0.$$

Example 2.1. The restriction of inversion

$$\varphi : \mathbb{R}^n\backslash\{0\} \longrightarrow \mathbb{R}^n\backslash\{0\}, \quad x \longmapsto \frac{x}{\|x\|^2},$$

to $M = \{x \in \mathbb{R}^n\backslash\{0\},\ \|x\|^2 > \sqrt{n}\}$ is $p(\cdot)$-harmonic map, where the function p is given by

$$p(x) = n + \frac{c}{2\ln(\|x\|^2) - \ln(n)}, \quad \forall x \in M,$$

for some constant $c \geq 0$. Here, $|d\varphi|(x) = \frac{\sqrt{n}}{\|x\|^2}$ for all $x \in \mathbb{R}^n\backslash\{0\}$.

Example 2.2. Let $F : \mathbb{R} \longrightarrow [2,\infty)$ be a smooth function. The map

$$\varphi : \mathbb{R}^n\backslash\{0\} \longrightarrow \mathbb{S}^{n-1}, \quad x \longmapsto \frac{x}{\|x\|},$$

is $p(\cdot)$-harmonic, where $p(x) = F(\|x\|^2)$ for all $x \in \mathbb{R}^n\backslash\{0\}$. The Hilbert-Schmidt norm of $d\varphi$ is given by $|d\varphi|(x) = \frac{\sqrt{n-1}}{\|x\|}$ for all $x \in \mathbb{R}^n\backslash\{0\}$.

Remark 2.1. A smooth harmonic map, i.e., $\tau(\varphi) = 0$, with constant energy density $\frac{|d\varphi|^2}{2}$ is not always $p(\cdot)$-harmonic. The previous examples prove the following results; There is no equivalence between the $p(\cdot)$-harmonicity and the harmonicity of a smooth map $\varphi : (M,g) \longrightarrow (N,h)$. There are $p(\cdot)$-harmonic maps which have non-constant Hilbert-Schmidt norm and they are not harmonic.

3. Stable $p(\cdot)$-Harmonic Maps

Theorem 3.1 (The second variation of the $p(\cdot)$-energy)**.** *Let φ be a smooth $p(\cdot)$-harmonic map between two Riemannian manifolds (M,g) and (N,h). Then we have*

$$\frac{\partial^2}{\partial t\partial s}E_{p(\cdot)}(\varphi_{t,s};D)\Big|_{t=s=0} = \int_D h(J^{\varphi}_{p(\cdot)}(v),w)v_g, \tag{3.1}$$

where $\{\varphi_{t,s}\}_{(t,s)\in(-\epsilon,\epsilon)\times(-\epsilon,\epsilon)}$ is a smooth variation supported in compact domain $D \subset M$ of φ,

$$v = \frac{\partial\varphi_{t,s}}{\partial t}\Big|_{t=s=0}, \quad w = \frac{\partial\varphi_{t,s}}{\partial s}\Big|_{t=s=0}, \tag{3.2}$$

and $J^{\varphi}_{p(\cdot)}$ the generalized Jacobi operator of φ given by

$$\begin{aligned} J^{\varphi}_{p(\cdot)}(v) = &- |d\varphi|^{p(x)-2}\operatorname{trace}_g R^N(v,d\varphi)d\varphi - \operatorname{trace}_g \nabla^{\varphi}|d\varphi|^{p(x)-2}\nabla^{\varphi}v \\ &- \operatorname{trace}_g \nabla(p(x)-2)|d\varphi|^{p(x)-4}\langle\nabla^{\varphi}v, d\varphi\rangle d\varphi. \end{aligned} \tag{3.3}$$

Here $\langle\,,\,\rangle$ denote the inner product on $T^*M \otimes \varphi^{-1}TN$.

Proof. Let $\phi : M \times (-\epsilon,\epsilon) \times (-\epsilon,\epsilon) \longrightarrow N$ be a smooth map defined by $\phi(x,t,s) = \varphi_{t,s}(x)$. We have $\phi(x,0,0) = \varphi(x)$, and the variation vectors fields associated to the variation $\{\varphi_{t,s}\}_{(t,s)\in(-\epsilon,\epsilon)\times(-\epsilon,\epsilon)}$ are given by

$$v(x) = d_{(x,0,0)}\phi(\frac{\partial}{\partial t}), \quad w(x) = d_{(x,0,0)}\phi(\frac{\partial}{\partial s}), \quad \forall x \in M. \tag{3.4}$$

Let $\{e_i\}_{i=1}^m$ be an orthonormal frame with respect to g on M such that $\nabla^M_{e_i}e_j = 0$ at $x \in M$ for all $i,j = 1,\ldots,m$. We compute

$$\begin{aligned} \frac{\partial^2}{\partial t\partial s}E_{p(\cdot)}(\varphi_{t,s};D)\Big|_{t=s=0} &= \frac{\partial^2}{\partial t\partial s}\int_D \frac{|d\varphi_{t,s}|^{p(x)}}{p(x)}v_g\Big|_{t=s=0} \\ &= \int_D \frac{1}{p(x)}\frac{\partial^2}{\partial t\partial s}|d\varphi_{t,s}|^{p(x)}\Big|_{t=s=0} v_g. \end{aligned} \tag{3.5}$$

First, note that

$$\begin{aligned}\frac{1}{p(x)}\frac{\partial^2}{\partial t\partial s}|d\varphi_{t,s}|^{p(x)} &= \frac{1}{p(x)}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial s}(|d\varphi_{t,s}|^2)^{\frac{p(x)}{2}}\right)\\ &= \frac{1}{2}\frac{\partial}{\partial t}\left((|d\varphi_{t,s}|^2)^{\frac{p(x)}{2}-1}\frac{\partial}{\partial s}|d\varphi_{t,s}|^2\right)\\ &= \sum_{i=1}^m \frac{\partial}{\partial t}\left((|d\varphi_{t,s}|^2)^{\frac{p(x)}{2}-1}h(\nabla^\phi_{\frac{\partial}{\partial s}}d\phi(e_i,0,0),d\phi(e_i,0,0))\right).\end{aligned}$$

Thus

$$\begin{aligned}\frac{1}{p(x)}\frac{\partial^2}{\partial t\partial s}|d\varphi_{t,s}|^{p(x)} &= \sum_{i=1}^m \frac{\partial}{\partial t}\left(|d\varphi_{t,s}|^2\right)^{\frac{p(x)}{2}-1} h(\nabla^\phi_{\frac{\partial}{\partial s}}d\phi(e_i,0,0),d\phi(e_i,0,0))\\ &+\sum_{i=1}^m |d\varphi_{t,s}|^{p(x)-2}h(\nabla^\phi_{\frac{\partial}{\partial t}}\nabla^\phi_{\frac{\partial}{\partial s}}d\phi(e_i,0,0),d\phi(e_i,0,0))\\ &+\sum_{i=1}^m |d\varphi_{t,s}|^{p(x)-2}h(\nabla^\phi_{\frac{\partial}{\partial s}}d\phi(e_i,0,0),\nabla^\phi_{\frac{\partial}{\partial t}}d\phi(e_i,0,0)).\end{aligned}$$

So that

$$\begin{aligned}\frac{1}{p(x)}\frac{\partial^2}{\partial t\partial s}|d\varphi_{t,s}|^{p(x)} &= \sum_{i,j=1}^m (p(x)-2)|d\varphi_{t,s}|^{p(x)-4}h(\nabla^\phi_{\frac{\partial}{\partial t}}d\phi(e_j,0,0),d\phi(e_j,0,0))\\ & h(\nabla^\phi_{\frac{\partial}{\partial s}}d\phi(e_i,0,0),d\phi(e_i,0,0))\\ &+\sum_{i=1}^m |d\varphi_{t,s}|^{p(x)-2}h(\nabla^\phi_{\frac{\partial}{\partial t}}\nabla^\phi_{\frac{\partial}{\partial s}}d\phi(e_i,0,0),d\phi(e_i,0,0))\\ &+\sum_{i=1}^m |d\varphi_{t,s}|^{p(x)-2}h(\nabla^\phi_{\frac{\partial}{\partial s}}d\phi(e_i,0,0),\nabla^\phi_{\frac{\partial}{\partial t}}d\phi(e_i,0,0)).\end{aligned}$$

By the definition of the curvature tensor of (N,h) and the properties

$$\nabla^\phi_{\frac{\partial}{\partial t}}d\phi(e_i,0,0)=\nabla^\phi_{(e_i,0,0)}d\phi(\frac{\partial}{\partial t}),\quad \nabla^\phi_{\frac{\partial}{\partial s}}d\phi(e_i,0,0)=\nabla^\phi_{(e_i,0,0)}d\phi(\frac{\partial}{\partial s}),$$

with $\left[\frac{\partial}{\partial t},(e_i,0,0)\right]=0$, we obtain the following equation

$$\begin{aligned}\frac{1}{p(x)}\frac{\partial^2}{\partial t\partial s}|d\varphi_{t,s}|^{p(x)}\Big|_{t=s=0} &= \sum_{i=1}^m h\left(\nabla^\varphi_{e_i}w,(p(x)-2)|d\varphi|^{p(x)-4}\langle\nabla^\varphi v,d\varphi\rangle d\varphi(e_i)\right)\\ &-|d\varphi|^{p(x)-2}\sum_{i=1}^m h(R^N(v,d\varphi(e_i))d\varphi(e_i),w)\\ &+\sum_{i=1}^m h\left(\nabla^\varphi_{e_i}\nabla^\phi_{\frac{\partial}{\partial t}}d\phi(\frac{\partial}{\partial s})\Big|_{t=s=0},|d\varphi|^{p(x)-2}d\varphi(e_i)\right)\\ &+\sum_{i=1}^m h\left(\nabla^\varphi_{e_i}w,|d\varphi|^{p(x)-2}\nabla^\varphi_{e_i}v\right). \end{aligned}\tag{3.6}$$

Let $\omega_1,\omega_2,\omega_3\in\Gamma(T^*M)$ defined by

$$\begin{aligned}\omega_1(X) &= h\left(w,(p(x)-2)|d\varphi|^{p(x)-4}\langle\nabla^\varphi v,d\varphi\rangle d\varphi(X)\right);\\ \omega_2(X) &= h\left(\nabla^\phi_{\frac{\partial}{\partial t}}d\phi(\frac{\partial}{\partial s})\Big|_{t=s=0},|d\varphi|^{p(x)-2}d\varphi(X)\right):\\ \omega_3(X) &= h\left(w,|d\varphi|^{p(x)-2}\nabla^\varphi_X v\right),\quad \forall X\in\Gamma(TM).\end{aligned}$$

The divergence of ω_1, ω_2, and ω_3 are given by

$$\begin{aligned}
\operatorname{div}^M \omega_1 &= \sum_{i=1}^m e_i h\left(w, (p(x)-2)|d\varphi|^{p(x)-4}\langle \nabla^\varphi v, d\varphi\rangle d\varphi(e_i)\right);\\
\operatorname{div}^M \omega_2 &= \sum_{i=1}^m e_i h\left(\nabla^\phi_{\frac{\partial}{\partial t}} d\phi(\frac{\partial}{\partial s})\Big|_{t=s=0}, |d\varphi|^{p(x)-2} d\varphi(e_i)\right):\\
\operatorname{div}^M \omega_3 &= \sum_{i=1}^m e_i h\left(w, |d\varphi|^{p(x)-2}\nabla^\varphi_{e_i} v\right), \quad \forall X \in \Gamma(TM).
\end{aligned}$$

By equations (3.5), (3.6), the $p(\cdot)$-harmonicity condition of φ, and the divergence Theorem, we obtain

$$\begin{aligned}
\frac{\partial^2}{\partial t \partial s} E_{p(\cdot)}(\varphi_{t,s}; D)\Big|_{t=s=0} &= -\int_D \sum_{i=1}^m h\left(w, \nabla^\varphi_{e_i}(p(x)-2)|d\varphi|^{p(x)-4}\langle \nabla^\varphi v, d\varphi\rangle d\varphi(e_i)\right) v_g\\
&\quad - \int_D |d\varphi|^{p(x)-2}\sum_{i=1}^m h(w, R^N(v, d\varphi(e_i))d\varphi(e_i)) v_g\\
&\quad - \int_D \sum_{i=1}^m h\left(w, \nabla^\varphi_{e_i}|d\varphi|^{p(x)-2}\nabla^\varphi_{e_i} v\right) v_g. \qquad (3.7)
\end{aligned}$$

The proof is completed. □

If (M, g) is a compact Riemannian manifold, φ be a $p(\cdot)$-harmonic map from (M, g) to Riemannian manifold (N, h), and for any vector field v along φ,

$$I^\varphi_{p(\cdot)}(v, v) \equiv \int_M h(v, J^\varphi_{p(\cdot)}(v))\, v_g \geq 0, \qquad (3.8)$$

then φ is called a stable $p(\cdot)$-harmonic map. Note that, the definition of stable $p(\cdot)$-harmonic maps is a generalization of stable harmonic maps ([10]), is also a generalization of stable p-harmonic maps ([4, 9]). By using the Green Theorem [1] it is easy to prove that

$$\begin{aligned}
I^\varphi_{p(\cdot)}(v, v) &= -\int_M |d\varphi|^{p(x)-2}\sum_{i=1}^m h(v, R^N(v, d\varphi(e_i))d\varphi(e_i)) v_g \qquad (3.9)\\
&\quad + \int_M |d\varphi|^{p(x)-2}|\nabla^\varphi v|^2 v_g + \int_M (p(x)-2)|d\varphi|^{p(x)-4}\langle \nabla^\varphi v, d\varphi\rangle^2 v_g.
\end{aligned}$$

From equation (3.9), we deduce the following result.

Proposition 3.1. *Every $p(\cdot)$-harmonic map from compact Riemannian manifold (M, g) to Riemannian manifold (N, h) has $\operatorname{Sect}^N \leq 0$ is stable.*

In the case where the codomain of the stable $p(\cdot)$-harmonic map is the standard sphere $\mathbb{S}^n$, we have the following result.

Theorem 3.2. *Let (M, g) be a compact Riemannian manifold. When $n > 2$, any stable $p(\cdot)$-harmonic map $\varphi : (M, g) \longrightarrow \mathbb{S}^n$ must be constant, where p is a smooth positive function on M such that $2 \leq p(x) < n$ for all $x \in M$.*

Proof. Choose a normal orthonormal frame $\{e_i\}_{i=1}^m$ at point x in (M, g). We set $\lambda(y) = \langle \alpha, y\rangle_{\mathbb{R}^{n+1}}$, for all $y \in \mathbb{S}^n$, where $\alpha \in \mathbb{R}^{n+1}$. Let $v = \operatorname{grad}^{\mathbb{S}^n} \lambda$. We have $\nabla^{\mathbb{S}^n}_X v = -\lambda X$ for all $X \in \Gamma(T\mathbb{S}^n)$, where $\nabla^{\mathbb{S}^n}$ is the Levi-Civita connection on $\mathbb{S}^n$ with respect to the standard metric of the sphere (see [10]). We compute

$$\begin{aligned}
\sum_{i=1}^m \nabla^\varphi_{e_i}|d\varphi|^{p(x)-2}\nabla^\varphi_{e_i}(v\circ\varphi) &= \nabla^\varphi_{\operatorname{grad}^M |d\varphi|^{p(x)-2}}(v\circ\varphi)\\
&\quad + \sum_{i=1}^m |d\varphi|^{p(x)-2}\nabla^\varphi_{e_i}\nabla^\varphi_{e_i}(v\circ\varphi). \qquad (3.10)
\end{aligned}$$

By using the property $\nabla^{\mathbb{S}^n}_X v = -\lambda X$, the first term of (3.10) is given by

$$\nabla^{\varphi}_{\mathrm{grad}^M |d\varphi|^{p(x)-2}}(v\circ\varphi) = -(\lambda\circ\varphi)d\varphi(\mathrm{grad}^M |d\varphi|^{p(x)-2}), \tag{3.11}$$

and the seconde term of (3.10) is given by

$$\begin{aligned}\sum_{i=1}^m |d\varphi|^{p(x)-2}\nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i}(v\circ\varphi) &= -\sum_{i=1}^m |d\varphi|^{p(x)-2}\nabla^{\varphi}_{e_i}(\lambda\circ\varphi)d\varphi(e_i)\\ &= -\sum_{i=1}^m |d\varphi|^{p(x)-2} < d\varphi(e_i), v\circ\varphi > d\varphi(e_i)\\ &\quad -(\lambda\circ\varphi)|d\varphi|^{p(x)-2}\tau(\varphi).\end{aligned} \tag{3.12}$$

Substituting the formulas (3.11) and (3.12) in (3.10) gives

$$\begin{aligned}\sum_{i=1}^m \nabla^{\varphi}_{e_i}|d\varphi|^{p(x)-2}\nabla^{\varphi}_{e_i}(v\circ\varphi) &= -(\lambda\circ\varphi)d\varphi(\mathrm{grad}^M |d\varphi|^{p(x)-2})\\ &\quad -\sum_{i=1}^m |d\varphi|^{p(x)-2} < d\varphi(e_i), v\circ\varphi > d\varphi(e_i)\\ &\quad -(\lambda\circ\varphi)|d\varphi|^{p(x)-2}\tau(\varphi).\end{aligned} \tag{3.13}$$

By the $p(\cdot)$-harmonicity condition of φ

$$\tau_{p(\cdot)}(\varphi) = |d\varphi|^{p(x)-2}\tau(\varphi) + d\varphi(\mathrm{grad}^M |d\varphi|^{p(x)-2}) = 0,$$

and equation (3.13), we get

$$\sum_{i=1}^m \langle\nabla^{\varphi}_{e_i}|d\varphi|^{p(x)-2}\nabla^{\varphi}_{e_i}(v\circ\varphi), v\circ\varphi\rangle = -\sum_{i=1}^m |d\varphi|^{p(x)-2}\langle d\varphi(e_i), v\circ\varphi\rangle^2. \tag{3.14}$$

Since the sphere $\mathbb{S}^n$ has constant curvature, we have

$$\begin{aligned}\sum_{i=1}^m \langle |d\varphi|^{p(x)-2}R^{\mathbb{S}^n}(v\circ\varphi, d\varphi(e_i))d\varphi(e_i), v\circ\varphi\rangle &= |d\varphi|^{p(x)}\langle v\circ\varphi, v\circ\varphi\rangle\\ &\quad -\sum_{i=1}^m |d\varphi|^{p(x)-2}\langle d\varphi(e_i), v\circ\varphi\rangle^2.\end{aligned} \tag{3.15}$$

By the definition of generalized Jacobi operator, and (3.14), (3.15), we obtain

$$\begin{aligned}\langle J^{\varphi}_f(v\circ\varphi), v\circ\varphi\rangle &= 2|d\varphi|^{p(x)-2}\sum_{i=1}^m \langle d\varphi(e_i), v\circ\varphi\rangle^2\\ &\quad -|d\varphi|^{p(x)}\langle v\circ\varphi, v\circ\varphi\rangle\\ &\quad -\sum_{i=1}^m \langle\nabla^{\varphi}_{e_i}(p(x)-2)|d\varphi|^{p(x)-4}\langle\nabla^{\varphi}v\circ\varphi, d\varphi\rangle d\varphi(e_i), v\circ\varphi\rangle,\end{aligned} \tag{3.16}$$

Using $\langle\nabla^{\varphi}v\circ\varphi, d\varphi\rangle = -(\lambda\circ\varphi)|d\varphi|^2$, and equation (3.16), we find that

$$\mathrm{trace}_{\alpha}\langle J^{\varphi}_f(v\circ\varphi), v\circ\varphi\rangle = (p(x)-n)|d\varphi|^{p(x)}. \tag{3.17}$$

Hence Theorem 3.2 follows from (3.17), and the stable $p(\cdot)$-harmonicity condition of φ, with $2 \le p(x) < n$ for all $x \in M$. □

4. $p(\cdot)$-Biharmonic Maps

Let $\varphi : (M,g) \longrightarrow (N,h)$ be a smooth map between two Riemannian manifolds, the $p(\cdot)$-bienergy of φ is defined by

$$E_{2,p(\cdot)}(\varphi; D) = \frac{1}{2}\int_D |\tau_{p(\cdot)}(\varphi)|^2 v_g, \tag{4.1}$$

where $p \geq 2$ is a smooth function on M, and D a compact subset of M. A smooth map φ is called $p(\cdot)$-biharmonic if it is a critical point of the $p(\cdot)$-bienergy functional for any compact domain D.

Theorem 4.1 (The first variation of the $p(\cdot)$-bienergy)**.** *Let φ be a smooth map between two Riemannian manifolds (M,g) and (N,h). Then we have*

$$\frac{d}{dt}E_{2,p(\cdot)}(\varphi_t; D)\Big|_{t=0} = -\int_D h(v, \tau_{2,p(\cdot)}(\varphi))v_g, \tag{4.2}$$

where $\{\varphi_t\}_{t\in(-\epsilon,\epsilon)}$ is a smooth variation of φ supported in D, $v = \frac{d\varphi_t}{dt}\big|_{t=0}$ denotes the variation vector field, and $\tau_{2,p(\cdot)}(\varphi)$ the $p(\cdot)$-bitension field of φ given by

$$\begin{aligned}\tau_{2,p(\cdot)}(\varphi) = &-|d\varphi|^{p(x)-2}\operatorname{trace}_g R^N(\tau_{p(\cdot)}(\varphi), d\varphi)d\varphi - \operatorname{trace}_g \nabla^\varphi |d\varphi|^{p(x)-2}\nabla^\varphi \tau_{p(\cdot)}(\varphi)\\ &- \operatorname{trace}_g \nabla(p(x)-2)|d\varphi|^{p(x)-4}\langle\nabla^\varphi\tau_{p(\cdot)}(\varphi), d\varphi\rangle d\varphi.\end{aligned} \tag{4.3}$$

Proof. Define $\phi : M \times (-\epsilon,\epsilon) \longrightarrow N$ by $\phi(x,t) = \varphi_t(x)$. First, note that

$$\frac{d}{dt}E_{2,p(\cdot)}(\varphi_t; D) = \int_D h(\nabla^\phi_{\frac{\partial}{\partial t}}\tau_{p(\cdot)}(\varphi_t), \tau_{p(\cdot)}(\varphi_t))\, v_g. \tag{4.4}$$

Calculating in a normal frame at $x \in M$, we have

$$\nabla^\phi_{\frac{\partial}{\partial t}}\tau_{p(\cdot)}(\varphi_t) = \sum_{i=1}^m \nabla^\phi_{\frac{\partial}{\partial t}}\nabla^\phi_{(e_i,0)}|d\varphi_t|^{p(x)-2}d\phi(e_i,0). \tag{4.5}$$

From the definition of the curvature tensor of (N,h), we obtain

$$\begin{aligned}&\sum_{i=1}^m \nabla^\phi_{\frac{\partial}{\partial t}}\nabla^\phi_{(e_i,0)}|d\varphi_t|^{p(x)-2}d\phi(e_i,0)\\ &\quad= |d\varphi_t|^{p(x)-2}\sum_{i=1}^m R^N(d\phi(\frac{\partial}{\partial t}), d\phi(e_i,0))d\phi(e_i,0)\\ &\qquad+\sum_{i=1}^m \nabla^\phi_{(e_i,0)}\nabla^\phi_{\frac{\partial}{\partial t}}|d\varphi_t|^{p(x)-2}d\phi(e_i,0).\end{aligned} \tag{4.6}$$

By using the compatibility of ∇^ϕ with h, we find that

$$\begin{aligned}&\sum_{i=1}^m h(\nabla^\phi_{(e_i,0)}\nabla^\phi_{\frac{\partial}{\partial t}}|d\varphi_t|^{p(x)-2}d\phi(e_i,0), \tau_{p(\cdot)}(\varphi_t))\\ &\quad= \sum_{i=1}^m (e_i,0)\left[h(\nabla^\phi_{\frac{\partial}{\partial t}}|d\varphi_t|^{p(x)-2}d\phi(e_i,0), \tau_{p(\cdot)}(\varphi_t))\right]\\ &\qquad-\sum_{i=1}^m h(\nabla^\phi_{\frac{\partial}{\partial t}}|d\varphi_t|^{p(x)-2}d\phi(e_i,0), \nabla^\phi_{(e_i,0)}\tau_{p(\cdot)}(\varphi_t)).\end{aligned} \tag{4.7}$$

From the property $\nabla^\phi_X d\phi(Y) = \nabla^\phi_Y d\phi(X) + d\phi([X,Y])$, with $X = \frac{\partial}{\partial t}$ and $Y = |d\varphi_t|^{p(x)-2}(e_i,0)$, we get

$$
\begin{aligned}
&\nabla^{\phi}_{\frac{\partial}{\partial t}} |d\varphi_t|^{p(x)-2} d\phi(e_i,0)\Big|_{t=0} \\
&\qquad = |d\varphi|^{p(x)-2}\nabla^{\varphi}_{e_i} v \\
&\qquad\quad +\sum_{j=1}^{m}(p(x)-2)|d\varphi|^{p(x)-4} h(\nabla^{\varphi}_{e_j} v, d\varphi(e_j))d\varphi(e_i),
\end{aligned}
\tag{4.8}
$$

for all $i = 1, ..., m$. Substituting (4.8) in (4.7), we have

$$
\begin{aligned}
&\sum_{i=1}^{m} h(\nabla^{\phi}_{(e_i,0)}\nabla^{\phi}_{\frac{\partial}{\partial t}} |d\varphi_t|^{p(x)-2} d\phi(e_i,0), \tau_{p(\cdot)}(\varphi_t))\Big|_{t=0} \\
&\qquad = \sum_{i=1}^{m} e_i h(|d\varphi|^{p(x)-2}\nabla^{\varphi}_{e_i} v, \tau_{p(\cdot)}(\varphi)) \\
&\qquad\quad +\sum_{i=1}^{m} e_i h((p(x)-2)|d\varphi|^{p(x)-4}\langle\nabla^{\varphi} v, d\varphi\rangle d\varphi(e_i), \tau_{p(\cdot)}(\varphi)) \\
&\qquad\quad -\sum_{i=1}^{m} e_i h(v, |d\varphi|^{p(x)-2}\nabla^{\varphi}_{e_i}\tau_{p(\cdot)}(\varphi)) \\
&\qquad\quad +\sum_{i=1}^{m} h(v, \nabla^{\varphi}_{e_i}|d\varphi|^{p(x)-2}\nabla^{\varphi}_{e_i}\tau_{p(\cdot)}(\varphi)) \\
&\qquad\quad -\sum_{j=1}^{m} e_j h(v, (p(x)-2)|d\varphi|^{p(x)-4}\langle\nabla^{\varphi}\tau_{p(\cdot)}(\varphi), d\varphi\rangle d\varphi(e_j)) \\
&\qquad\quad +\sum_{j=1}^{m} h(v, \nabla^{\varphi}_{e_j}(p(x)-2)|d\varphi|^{p(x)-4}\langle\nabla^{\varphi}\tau_{p(\cdot)}(\varphi), d\varphi\rangle d\varphi(e_j)).
\end{aligned}
\tag{4.9}
$$

Let $\eta_1, \eta_2, \eta_3, \eta_4 \in \Gamma(T^*M)$ defined by

$$
\begin{aligned}
\eta_1(X) &= h(|d\varphi|^{p(x)-2}\nabla^{\varphi}_{X} v, \tau_{p(\cdot)}(\varphi)); \\
\eta_2(X) &= h((p(x)-2)|d\varphi|^{p(x)-4}\langle\nabla^{\varphi} v, d\varphi\rangle d\varphi(X), \tau_{p(\cdot)}(\varphi)); \\
\eta_3(X) &= h(v, |d\varphi|^{p(x)-2}\nabla^{\varphi}_{X}\tau_{p(\cdot)}(\varphi)); \\
\eta_4(X) &= h(v, (p(x)-2)|d\varphi|^{p(x)-4}\langle\nabla^{\varphi}\tau_{p(\cdot)}(\varphi), d\varphi\rangle d\varphi(X)).
\end{aligned}
$$

Finally, we calculate the divergence of η_i $(i = 1, ..., 4)$ and substituting in (4.9). The proof of Theorem 4.1 follows by (4.4)-(4.6), (4.9), and the divergence Theorem. □

From Theorem 4.1, we deduce:

Corollary 4.1. *Let $\varphi : (M, g) \to (N, h)$ be a smooth map between Riemannian manifolds. Then, φ is $p(\cdot)$-biharmonic if and only if*

$$
\begin{aligned}
\tau_{2,p(\cdot)}(\varphi) &= -|d\varphi|^{p(x)-2}\operatorname{trace}_g R^N(\tau_{p(\cdot)}(\varphi), d\varphi)d\varphi \\
&\quad -\operatorname{trace}_g \nabla^{\varphi}|d\varphi|^{p(x)-2}\nabla^{\varphi}\tau_{p(\cdot)}(\varphi) \\
&\quad -\operatorname{trace}_g \nabla(p(x)-2)|d\varphi|^{p(x)-4}\langle\nabla^{\varphi}\tau_{p(\cdot)}(\varphi), d\varphi\rangle d\varphi = 0.
\end{aligned}
$$

Remark 4.1. For any smooth map $\varphi : (M, g) \longrightarrow (N, h)$ between two Riemannian manifolds, we have

$$
\tau_{2,p(\cdot)}(\varphi) = J^{\varphi}_{p(\cdot)}(\tau_{p(\cdot)}(\varphi)).
$$

We can extract several examples of $p(\cdot)$-biharmonic non $p(\cdot)$-harmonic maps $\varphi : (M, g) \longrightarrow \mathbb{R}^n$ where the $p(\cdot)$-tension field is parallel along φ, i.e., the components of $\tau_{p(\cdot)}(\varphi)$ are constants.

Example 4.1. Let $M = \{(x,y,z) \in \mathbb{R}^3, \ \sqrt{x^2+y^2} > 2\}$. The smooth map $\varphi : M \longrightarrow \mathbb{R}^2$ defined by

$$\varphi(x,y,z) = (\sqrt{x^2+y^2}, z), \quad \forall (x,y,z) \in M,$$

is $p(\cdot)$-biharmonic non $p(\cdot)$-harmonic, where

$$p(x,y,z) = \frac{\ln(x^2+y^2)}{\ln(2)},$$

for all $(x,y,z) \in M$. Here, $\tau_{p(\cdot)}(\varphi) = (1,0)$.

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Author's contributions

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