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On Rectifying Slant Curves in Galilean Space

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Abstract

In this paper, we study rectifying slant curves in three-dimensional Galilean space. Further geometric properties of rectifying slant curves are also presented in Galilean space. Moreover, we obtain a admissible family of rectifying slant helices for a special case. Consequently, an example is constructed and plotted.

Keywords: *Galilean space; curves; rectifying slant curves; non-Euclidean geometry.*
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1. Introduction

At each point of the curve, the planes spanned by $\{\mathbf{t},\mathbf{n}\}$, $\{\mathbf{t},\mathbf{b}\}$, and $\{\mathbf{n},\mathbf{b}\}$ are known as the osculating plane, the rectifying plane, and the normal plane, respectively [15, 21]. The notion of rectifying curve has been introduced by Chen [22]. He showed, under which conditions, the position vector of a unit speed curve lies in its rectifying plane. After that several properties of rectifying curve have been studied by mathematicians until today. Chen and Dillen established simple relationship between rectifying curves and the notion of centrodes in mechanics [6]. In [12] the authors studied some characterization of rectifying curves in the Minkowski 3-space. Recently Oztekin give a parametrization for rectifying curves in the Galilean space [17].

In differential geometry, a general helix in Euclidean 3-space is characterized by the property that the tangent lines make a constant angle with a fixed direction [8, 19]. On the other hand, the notion of slant helix was introduced by Izuyama and Takeuchi [7]. They showed, under which conditions, a unit speed curve is a slant helix. Later Ahmed published a paper in which position vectors of some slant helices were shown [1]. In [9, 10], Kula, et al. studied slant helices. Deshmukh et al. studied rectifying curves via the dilation of unit speed curves on the unit sphere S^2 in the Euclidean space [4]. Altunkaya and Kula studied timelike rectifying slant helices in Minkowski 3-space [3]. In [2], the authors have found the general equations of the curvature and the torsion of rectifying slant helices. After that, they have obtained a family of rectifying slant helices which lie on cones. Recently timelike rectifying slant helices in Minkowski space studied in [5].

The geometry of Galilean Relativity acts like a "bridge" from Euclidean geometry to Special Relativity [18, 23]. Similar to the three dimensional Euclidean space, the norm of the cross product measures the area spanned by two vectors in the three-dimensional Galilean space [16]. Galilean geometry is described in [11, 16]. The geometry of surfaces has been largely developed by O. Röschel [14]. Öğrenmiş et all. also described helices in Galilean space [13]. The characterization of slant helix in Galilean and Pseudo Galilean space studied in [20]. The relationship between the curvatures of the parallel surfaces in Galilean space is given in [24].

The papers mentioned above led us to study on the notion of rectifying slant helices in the Galilean space. First of all, we obtained the equations of curvature and torsion of a rectifying slant helix in Galilean space $\mathbb{G}^3$. Moreover, we constructed a second-order linear differential equation to determine the positon vector of a rectifying slant helix. We solved this differential equation for a special case in order to obtain a admissible family of rectifying slant helices in Galilean space.

2. Preliminaries

Definition 2.1. *A curve is called a slant helix if its principal normal vector field makes a constant angle with a fixed line in space [8].*

Theorem 2.2. *A unit speed curve α is called a slant helix if and only if the geodesic curvature of the spherical image of the principal normal*

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indicatrix of α

$$\sigma(s)=\left(\frac{\kappa^2}{\left(\kappa^2+\tau^2\right)^{3/2}}\left(\frac{\tau}{\kappa}\right)'\right)(s) \tag{2.1}$$

is constant [8, 9].

Theorem 2.3. *A unit space curve* α *is congruent to a rectifying curve if and only if*

$$\frac{\tau(s)}{\kappa(s)}=c_1 s+c_2$$

for some constant c_1 *and* c_2 *with* $c_1\neq 0$ *[22].*

The Galilean space $\mathbb{G}_3$ is a Cayley–Klein space equipped with the projective metric of signature $(0,0,+,+)$, as in . The absolute figure of the Galilean geometry consists of an ordered triple $\{\omega,f,I\}$, where ω is the real (absolute) plane, f the real line (absolute line) in ω and I the fixed elliptic involution of points of f [23].
A plane is called Euclidean if it contains f, otherwise it is called isotropic or planes $x=constant$ are Euclidean and so is the plane w. Other planes are isotropic. A vector $u=(u_1,u_2,u_3)$ is said to non-isotropic if $u_1\neq 0$. All unit non-isotropic vectors are of the form $u=(1,u_2,u_3)$. For isotropic vectors $u_1=0$ holds [11]. We introduce homogeneous coordinates in $\mathbb{G}_3$ in such a way that the absolute plane ω is given by $x_0=0$, the absolute line f by $x_0=x_1=0$. In affine coordinates defined by $(x_0:x_1:x_2:x_3)=(1:x:y:z)$.

Definition 2.4. *A plane is called Euclidean if it contains* f*, otherwise it is called isotropic. The planes* $x=constant$ *are Euclidean plane and so is the plane* w*. Other planes are isotropic. A vector* $\mathbf{u}=(u_1,u_2,u_3)$ *is said to non-isotropic if* $u_1\neq 0$*. For isotropic vectors* $u_1=0$ *holds. All unit non-isotropic vectors are of the form* $\mathbf{u}=(1,u_2,u_3)$ *[24].*

Definition 2.5. *Let* $\boldsymbol{a}=(x,y,z)$ *and* $\boldsymbol{b}=(x_1,y_1,z_1)$ *be vectors in the Galilean space [24]. The scalar product is defined by*

$$<\boldsymbol{a},\boldsymbol{b}>=x_1x \tag{2.2}$$

The norm of $\boldsymbol{a}$ *defined by* $\|\boldsymbol{a}\|=|x|$*, and* $\boldsymbol{a}$ *is called a unit vector if* $\|a\|=1$.
The scaler product of two isotropic vectors $\boldsymbol{p}=(0,y,z)$ *and* $\boldsymbol{q}=(0,y_1,z_1)$ *in Galilean space is defined by*

$$<\boldsymbol{p},\boldsymbol{q}>_1=yy_1+zz_1 \tag{2.3}$$

The orthogonality of these vectors, $\boldsymbol{p}\perp_1\boldsymbol{q}$*, means that* $<\boldsymbol{p},\boldsymbol{q}>_1=0$*. The norm of* p *defined by* $\|\boldsymbol{p}\|_1=\sqrt{y^2+z^2}$*, and* $\boldsymbol{p}$ *is called a unit isotropic vector if* $\|\boldsymbol{p}\|_1=1$.

Definition 2.6. *The isotropic angle [11] measure* ϑ *between two vectors* $\mathbf{u}=(1,u_2,u_3)$ *and* $\mathbf{v}=(1,v_2,v_3)$ *is defined as*

$$\vartheta=\|\mathbf{u}-\mathbf{v}\|_1=\sqrt{(u_2-v_2)^2+(u_3-v_3)^2} \tag{2.4}$$

On the otherhand, the angle measure between two isotropic vectors $\boldsymbol{p}=(0,y,z)$ *and* $\boldsymbol{q}=(0,y_1,z_1)$ *is defined as*

$$\cos\theta=\frac{<\boldsymbol{p},\boldsymbol{q}>_1}{\|\boldsymbol{p}\|_1\|\boldsymbol{q}\|_1} \tag{2.5}$$

Definition 2.7. *If* $\mathbf{u}=(u_1,u_2,u_3)$ *and* $\mathbf{v}=(v_1,v_2,v_3)$ *are vectors in the Galilean space [11], we define the vector product of* $\mathbf{u}$ *and* $\mathbf{v}$ *as the following:*

$$\mathbf{u}\wedge\mathbf{v}=\begin{vmatrix}0 & e_2 & e_3\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{vmatrix}=(0,u_3v_1-u_1v_3,u_1v_2-u_2v_1) \tag{2.6}$$

.

Definition 2.8. *let* ε *be plane and* $f(\varepsilon)$ *the intersection of the absolute line* f *and* ε*. The point* $f(\varepsilon)$ *is called the absolute point of* ε*. Now, let* $f(\varepsilon)^{\perp}=I(f(\varepsilon))$ *be the point on* f *orthogonal to* $f(\varepsilon)$ *according to the elliptic involution* I.
This is elliptic involution because there is no line perpendicular to itself. In Galilean space, the elliptic involution is given by

$$(0:0:x_2:x_3)\rightarrow(0:0:x_3:-x_2) \tag{2.7}$$

.

Definition 2.9. *If an admissible curve* c *of the class* $C^r(r\geq 3)$ *is given by the parametrization*

$$r(s)=(s,y(s),z(s)) \tag{2.8}$$

then s *is a Galilean invariant of the arc length on* C *[17]. The associated invariant moving trihedron is given by*

$$\begin{aligned}\mathbf{t}&=(1,y'(s),z'(s))\\ \mathbf{n}&=\frac{1}{\kappa}(0,y''(s),z''(s))\\ \mathbf{b}&=\frac{1}{\kappa}(0,-z''(s),y''(s))\end{aligned} \tag{2.9}$$

where $\kappa=\sqrt{y''(s)^2+z''(s)^2}$ *is the curvature. The torsion of the curve is given by* $\tau=\frac{1}{\kappa^2}\det[r'(s),r''(s),r'''(s)]$ *is the torsion. Frenet formulas may be written as*

$$\frac{d}{ds}\begin{bmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{bmatrix}=\begin{bmatrix}0 & \kappa & 0\\ 0 & 0 & \tau\\ 0 & -\tau & 0\end{bmatrix}\begin{bmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{bmatrix} \tag{2.10}$$

It is easy to see that

$$r''=\kappa'\mathbf{n}+\kappa\tau\mathbf{b} \tag{2.11}$$

Theorem 2.10. *Let* $r(s)$ *be an admissible curve in Galilean space. The curve is a rectifying if and only if the following relation holds:*

$$\frac{\tau}{\kappa}=c_1s+c_2 \tag{2.12}$$

where $c_1\in\mathbb{R}-\{0\},c_2\in\mathbb{R}$ *[17].*

Theorem 2.11. *Let* $r(s)$ *be an admissible curve in Galilean space. Then the curve is a slant helix if and only if the function*

$$\frac{\kappa^2}{\tau^3}\left(\frac{\tau}{\kappa}\right)' \tag{2.13}$$

is constant [20].

3. On Rectifying Slant Helices in Galilean Space

In this section, our goal is to study the rectifying slant helices in Galilean space.

Lemma 3.1. *Let* $r(s)$ *be an admissible curve in Galilean space. If the curve* $r(s)$ *is a rectifying slant helix then the curvature and torsion of the curve satisfies following equations*

$$\kappa=\frac{c}{(c_1s+c_2)^3}$$

and

$$\tau=\frac{c}{(c_1s+c_2)^2}$$

where c,c_1 *and* c_2 *are constants.*

Proof. It is easy to see that combining (2.12) and (2.13) gives

$$\frac{c_1}{(c_1s+c_2)^2\tau}=m$$

where m is a constant. It follows that

$$\tau=\frac{c}{(c_1s+c_2)^2} \tag{3.1}$$

where $c=\frac{c_1}{m}$.

On the other hand, from (2.12) and (3.1) we have

$$\kappa=\frac{c}{(c_1s+c_2)^3}$$

□

Theorem 3.2. *Let* $r(s)$ *be an admissible curve in Galilean space. If* $r(s)$ *is a rectifying slant helix. Then, the following linear vector differential equation of second orders hold:*

$$v''+f^2\kappa^2v=0 \tag{3.2}$$

where $v=\mathbf{n}'\kappa$*.*

Proof. Assume that $r(s)$ is a admissible rectifying slant helix so one can rearrange the Frenet frame of the curve as follows

$$\frac{d}{ds}\begin{bmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{bmatrix}=\begin{bmatrix}0&\kappa&0\\ 0&0&f\kappa\\ 0&-f\kappa&0\end{bmatrix}\begin{bmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{bmatrix} \tag{3.3}$$

where $f(s)=c_1s+c_2=\dfrac{\tau}{\kappa}$.

An easy computation shows that

$$\frac{\mathbf{n}'}{\kappa}=f\mathbf{b} \tag{3.4}$$

Differentiating (3.4), then substituting (3.3) into the result gives

$$(\frac{\mathbf{n}'}{\kappa})'+f^2\kappa\mathbf{n}=f'\mathbf{b}$$

Again by differentiating above equation we have

$$(\frac{\mathbf{n}'}{\kappa})''+f^2\kappa\mathbf{n}'+((f^2\kappa)'+f'f\kappa)\mathbf{n}=0 \tag{3.5}$$

On the other hand a simple calculation shows that

$$(f^2\kappa)'+f'f\kappa=0 \tag{3.6}$$

Thus the second group of terms of the equation (3.4) is also vanishes as a consequence of equation (3.6), finally we have

$$(\frac{\mathbf{n}'}{\kappa})''+f^2\kappa\mathbf{n}'=0 \tag{3.7}$$

Substituting $\dfrac{\mathbf{n}'}{\kappa}=v$ into (3.3) gives

$$v''+f^2\kappa^2v=0 \tag{3.8}$$

Observe that the above equation is a differential equation of second order.
On the other hand, substituting $f=c_1s+c_2$ into (3.8) gives

$$v''+\frac{c^2}{(c_1s+c_2)^4}v=0 \tag{3.9}$$

□

Theorem 3.3. *Let $r(s)=(r_1,r_2,r_3)$ be a unit speed rectifying slant helix. Then, it can be parametrized as follows*

$$\begin{aligned} r_1 &= s\\ r_2 &= -\frac{c}{c_1^2}\left(\cos\frac{1}{c_2+sc_1}\right)(c_2+sc_1)\\ r_3 &= \frac{c}{c_1^2}\left(\sin\frac{1}{c_2+sc_1}\right)(c_2+sc_1)\end{aligned} \tag{3.10}$$

Proof. Assume that $v=(v_1,v_2,v_3)$. in equation (3.9). In Galilean space, the normal vector is an isotropic vector therefore we have $v_1=0$. Moreover, since the every components of the vector v must satisfy (3.8), then we can solve the equation (3.9) as follows

$$v_2=f(s)\sin(\frac{1}{mf(s)})$$

and

$$v_3=f(s)\cos(\frac{1}{mf(s)})$$

where $m=\dfrac{c_1}{c}$.
On the other hand by using $\mathbf{n}'=\kappa v$ in the above equations we have

$$\mathbf{n}_1=\int v_1\kappa ds=0 \tag{3.11}$$

$$\mathbf{n}_2=\int\sin(\frac{1}{mf(s)})\frac{c}{f(s)^2}ds=c_1\cos(\frac{1}{mf(s)})$$

and

$$\mathbf{n}_3 = \int \cos\frac{1}{mf(s)}\frac{c}{f(s)^2}ds = -c_1\sin(\frac{1}{mf(s)})$$

From (3.11) and $r'' = (r_1, r_2, r_3) = \mathbf{t}' = \kappa\mathbf{n}$, we obtain the first component of the curve as

$$r_1 = \int\int \kappa\mathbf{n}_1 ds = s$$

Similarly one gets

$$r_2 = -\frac{c}{c_1^2}\left(\cos\frac{1}{c_2+sc_1}\right)(c_2+sc_1)$$

and

$$r_3 = \frac{c}{c_1^2}\left(\sin\frac{1}{c_2+sc_1}\right)(c_2+sc_1)$$

□

Example 3.4. *If we take $c = c_1 = 1$, $c_2 = 0$ in the equation (3.10), then the rectifying slant helix can be parametrized by*

$$\alpha = (s, -s\cos\frac{1}{s}, s\sin\frac{1}{s})$$

The Frenet frame of the curve is obtained by

$$\begin{aligned}
\mathbf{t} &= \left(1, \frac{-s\cos(\frac{1}{s}) - \sin(\frac{1}{s})}{s}, \frac{s\sin(\frac{1}{s}) - \cos(\frac{1}{s})}{s}\right) \\
\mathbf{n} &= (0, \cos(\frac{1}{s}), -\sin(\frac{1}{s})) \\
\mathbf{b} &= (0, \sin(\frac{1}{s}), \cos(\frac{1}{s}))
\end{aligned}$$

The curvature and torsion of the curve as follows

$$\kappa = \frac{1}{s^3} \qquad \tau = \frac{1}{s^2}$$

Therefore the curve is a rectifying slant helix in the Galilean space shown in Figure 3.1.

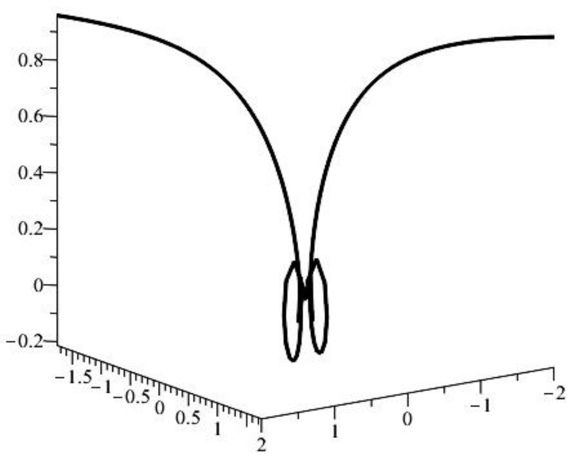

Figure 3.1: rectifying slant helix.

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