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Critical Oscillation Constant for Half Linear Differential Equations Which Have Different Periodic Coefficients

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ABSTRACT

In this paper, we compute explicitly the oscillation constant for certain half-linear second order differential equations having different periodic coefficients. If the periods of these functions are coincide, our result reduce to Dosly and Hasil's result, which were published in Annali di Matematica 190 (2011) 395-408. Finally some examples are also given to illustrate the result.

Keywords: *Half linear differential equation, Riemann-Weber equation, Prüfer transformation, Critical oscillation constant.*

1. INTRODUCTION

An equation of the form

$$\left(r(t)\Phi(x')\right)' + c(t)\Phi(x) = 0 \qquad (1.1)$$

$for\ \Phi(x) = |x|^{p-2}x,\ \ p > 1$. Wheere r, c are continuous functions and $r(t) > 0$ was introduced for the first time in [1] and called half-linear differential equation. The name half-linear equation was introduced in

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[2]. Since the linear Sturmian theory extends verbatim to half-linear case (for details, we refer to Section 1. 2 in [5]), we can classify Equ. (1.1) as oscillatory or non-oscillatory.

Actually, we are interested in the conditional oscillation of half-linear differential equations with different periodic coefficients. We say that the equation

$$(r(t)x')' + c(t)x = 0 \tag{1.2}$$

with positive coefficients is conditionally oscillatory if there exists a constant γ_0 such that Equ. (1.2) is oscillatory for all $\gamma > \gamma_0$ and non-oscillatory for all $\gamma < \gamma_0$. The constant γ_0 is called an oscillation constant of this equation.

Considerable effort has been made over the years to extend oscillation constant theory of half-linear differential equation Equ. (1.1), see [3, 4, 6, 7, 8, 9] and reference there in. For example it is well known that Cauchy-Euler differential equation

$$x'' + \frac{\gamma}{t^2}x = 0 \tag{1.3}$$

(which is special case $p = 2$, $r(t) = 1$ and $c(t) = \frac{1}{t^2}$ of Equ. (1.1)) is oscillatory if $\gamma > \frac{1}{4}$, non-oscillatory if $\gamma < \frac{1}{4}$. Additionally $x(t) = a\sqrt{t} + b\sqrt{t}logt$ is the general solution of Equ. (1.3) and non-oscillatory for $\gamma = \frac{1}{4}$.

In 2000 Elbert and Schneider [6] considered the half-linear Euler differential equation

$$\left(\Phi(x')\right)' + \frac{\gamma}{t^p}\,\Phi(x) = 0 \tag{1.4}$$

and showed that Equ. (1.4) is non-oscillatory if and only if $\gamma \le \gamma_p = \left(\frac{p-1}{p}\right)^p$.

In 2008 Hasil [7] considered the half-linear differential equation of the form

$$\left(r(t)\Phi(x')\right)' + \frac{\gamma c(t)}{t^p}\,\Phi(x) = 0 \tag{1.5}$$

where r, c are α-periodic positive functions and showed that Equ. (1.5) is oscillatory if $\gamma > K$ and non-oscillatory if $\gamma < K$, where K is given by

$$K = q^{-p}\left(\frac{1}{\alpha}\int_0^\alpha r^{1-q}(\tau)d\tau\right)^{1-p}\left(\frac{1}{\alpha}\int_0^\alpha c(\tau)d\tau\right)^{-1}$$

for p and q are conjugate numbers, i.e., $\frac{1}{p} + \frac{1}{q} = 1$. If the functions r, c are positive constants, then Equ. (1.5) reduce to the half-linear Euler equation Equ. (1.4), whose oscillatory properties were studied in detail [6] and references given therein.

In 2011 Dosly and Hasil [4] considered the Equ. (1.5) for r and c are α-periodic positive functions defined on $[0,\infty)$ and showed that Equ. (1.5) is non-oscillatory if and only if $\gamma \le \gamma_{rc}$, where γ_{rc} is given by

$$\gamma \le \gamma_{rc} = \frac{\alpha^p \gamma_p}{\left(\int_0^\alpha r^{1-q}(t)dt\right)^{p-1}\int_0^\alpha c(t)dt}.$$

Our goal is to find explicit oscillation constant for Equ. (1.5) with periodic coefficients which have different periods. We point out that the main motivation of our research comes from the papers [4, 7], where the oscillation constant is computed for Equ. (1.5) with the periodic coefficients which have same α-period. But in that papers the oscillation constant is not obtained for the periodic functions having different periods and consequently the number of the least common multiple of these periodic coefficients is not defined. Thus in this paper we investigate the oscillation constant for Equ. (1.5) with periodic coefficients which have different periods. For the sake of simplicity, we usually use the same notations with the papers [4,7].

2. PRELIMINARIES

First, we start this section with the recalling the concept of half-linear trigonometric functions [5]. Consider the following special half-linear equation of the form

$$\left(\Phi(x')\right)' + (p-1)\Phi(x) = 0 \tag{2.1}$$

and denote by $x = x(t)$ its solution given by the initial conditions $x(0) = 0, x'(0) = 1$. We see that the behavior of this solution is very similar to that of the classical sine function. We denote this solution by $sin_p t$ and its derivative $\left(sin_p t\right)' = cos_p t$. These functions are $2\pi_p -$ periodic, where $\pi_p := \frac{2\pi}{psin\left(\frac{\pi}{p}\right)}$ and satisfy the half-linear Pythagorean identity

$$\left|sin_p t\right|^p + \left|cos_p t\right|^p = 1, \qquad t\epsilon R. \tag{2.2}$$

Every solution of Equ. (2.1) is of the form $x(t) = Csin_p(t+\varphi)$, where C, φ are real constants, that is it is bounded together with its derivative and periodic with the period $2\pi_p$. The function $u = \Phi\left(cos_p t\right)$ is a solution of the reciprocal equation to Equ. (2.1);

$$(\Phi^{-1}(u')')' + (p-1)^{q-1}\Phi^{-1}(u) = 0, \Phi^{-1}(u) = |u|^{q-2}u, q = \frac{p}{p-1}$$

which is an equation of the form as Equ. (2.1), so the functions u and u' are also bounded.

Let x be a nontrivial solution of Equ. (1.1) and we consider the half-linear Prüfer transformation which is introduced using the half-linear trigonometric functions

$$x(t) = \rho(t) sin_p \varphi(t), x'(t) = r^{1-q}(t)\, \rho(t) cos_p \varphi(t), \tag{2.3}$$

where $\rho(t) = \sqrt[p]{|x(t)|^p + r^q(t)|x'(t)|^p}$ and Prüfer angle φ be a continuos function defined at all points where $x(t) \neq 0$.

Then φ satisfies the following differential equation

$$\varphi' = \frac{1}{t}\left[r^{1-q}(t)\left|cos_p\varphi\right|^p - \Phi\left(cos_p\varphi\right)sin_p\varphi + \frac{t^p c(t)}{p-1}\left|sin_p\varphi\right|^p\right]$$

which plays the fundamental role in this paper.

Next, we briefly mention about principal solution of non-oscilatory equation Equ. (1.1) [3], which is defined via the minimal solution of the associated Riccati equation

$$w' + c(t) + (p-1)r^{1-q}(t)|w|^q = 0, \tag{2.4}$$

where $w(t) = r(t)\frac{\Phi(x'(t))}{\Phi(x(t))}$. Non-oscillation of Equ. (1.1) implies that there exists $T \in \mathbb{R}$ and a solution $\widetilde{w}$ of Equ. (2.4) which is defined on some interval $[T_{\widetilde{w}},\infty)$. $\widetilde{w}$ called the minimal solution of among all solution of Eq. (2.4) and it satisfies the inequality $w(t) > \widetilde{w}(t)$ where w is any other solution of Equ. (2.4) which is defined on some interval $[T_w,\infty)$ and then $\tilde{x}$ is the principal solution of Equ. (1.1) via the formula $\widetilde{w}(t)= r(t)\frac{\Phi(\tilde{x}'(t))}{\Phi(\tilde{x}(t))}$.

3. MAIN RESULTS

We need the following two lemmas for proving the main theorem of this paper.

Lemma 1 *Let* $\varphi = \varphi_1 + \varphi_2 + \varphi_3 + M$, *(where* M *is a suitable constant)be a solution of the equation*

$$\varphi' = \varphi'_1 + \varphi'_2 + \varphi'_3$$

where

$$\varphi_1' = \frac{1}{t} r^{1-q}(t)\left|cos_p\varphi\right|^p,$$

$$\varphi_2' = -\tfrac{1}{t}\, \Phi\left(cos_p\varphi\right)sin_p\varphi$$

$$\varphi_3' = \frac{c(t)}{(p-1)t}\left|sin_p\varphi\right|^p$$

with r, c *are positive defined functions which have different* β_1, β_2 *−periods, respectively and*

$$\theta(t) = \left[\frac{1}{\beta_1}\int_t^{t+\beta_1}\varphi_1(s)ds + \frac{1}{\xi}\int_t^{t+\xi}\varphi_2(s)ds + \frac{1}{\beta_2}\int_t^{t+\beta_2}\varphi_3(s)ds\right],$$

where ξ *is one of the periods* β_1 *or* β_2. *Then* θ *is a solution of*

$$\theta'(t) = \frac{1}{t}\left[R\left|cos_p\theta\right|^p - \Phi\left(cos_p\theta\right)sin_p\theta + C\left|sin_p\theta\right|^p\right] + 0\left(\frac{1}{t^2}\right)$$

where

$$R = \frac{1}{\beta_1}\int_0^{\beta_1} r^{1-q}(\tau)d\tau\,, \qquad C = \frac{1}{\beta_2(P-1)}\int_0^{\beta_2} c(\tau)d\tau$$

and $\varphi(\tau) - \theta(t) = o(1)$ $as\ t \to \infty$.

Proof *Taking derivative of* $\theta(t)$, *we have*

$$\theta'(t)=\left[\frac{1}{\beta_1}\int_t^{t+\beta_1}\varphi'_1(s)ds+\frac{1}{\xi}\int_t^{t+\xi}\varphi'_2(s)ds+\frac{1}{\beta_2}\int_t^{t+\beta_2}\varphi'_3(s)ds\right]$$
$$=\frac{1}{\beta_1}\int_t^{t+\beta_1}\frac{1}{s}r^{1-q}(s)\left|cos_p\varphi(s)\right|^p ds\;-\frac{1}{\xi}\int_t^{t+\xi}\frac{1}{s}\,\Phi\left(cos_p\varphi(s)\right)sin_p\varphi(s)ds$$
$$+\frac{1}{\beta_2}\int_t^{t+\beta_2}\frac{c(s)}{(p-1)s}\left|sin_p\varphi(s)\right|^p ds\,.$$

Using integration by parts, we get

$$\theta'(t)=\frac{1}{\beta_1 t}\int_t^{t+\beta_1}r^{1-q}(\tau)\left|cos_p\varphi(\tau)\right|^p d\tau-\frac{1}{\xi t}\int_t^{t+\xi}\Phi\left(cos_p\varphi(\tau)\right)sin_p\varphi(\tau)d\tau$$
$$+\frac{1}{\beta_2 t}\int_t^{t+\beta_2}\frac{c(\tau)}{(p-1)}\left|sin_p\varphi(\tau)\right|^p d\tau$$
$$-\frac{1}{\beta_1}\int_t^{t+\beta_1}\frac{1}{s^2}\int_s^{t+\beta_1}r^{1-q}(\tau)\left|cos_p\varphi(\tau)\right|^p d\tau ds$$
$$+\frac{1}{\xi}\int_t^{t+\xi}\frac{1}{s^2}\int_s^{t+\xi}\Phi\left(cos_p\varphi(\tau)\right)sin_p\varphi(\tau)d\tau ds$$
$$-\frac{1}{\beta_2}\int_t^{t+\beta_2}\frac{1}{s^2}\int_s^{t+\beta_2}\frac{c(\tau)}{(p-1)}\left|sin_p\varphi(\tau)\right|^p d\tau ds\,.$$

By using the fact, $\int_t^{t+T}f(s)ds=\int_0^T f(s)ds$ *for any* $T-$*periodic function and half-linear Pythagorean identity, the expressions*

$$r^{1-q}(\tau)\left|cos_p\varphi(\tau)\right|^p,\Phi\left(cos_p\varphi(\tau)\right)sin_p\varphi(\tau),\frac{c(\tau)}{(p-1)}\left|sin_p\varphi(\tau)\right|^p$$

are bounded. Thus we get

$$\theta'(t)=\frac{1}{\beta_1 t}\int_t^{t+\beta_1}r^{1-q}(\tau)\left|cos_p\varphi(\tau)\right|^p d\tau\;-\frac{1}{\xi t}\int_t^{t+\xi}\Phi\left(cos_p\varphi(\tau)\right)sin_p\varphi(\tau)d\tau$$
$$+\frac{1}{\beta_2 t}\int_t^{t+\beta_2}\frac{c(\tau)}{(p-1)}\left|sin_p\varphi(\tau)\right|^p d\tau+0\left(\frac{1}{t^2}\right).$$

We can rewrite this equation as

$$\theta'(t)=\frac{1}{\beta_1 t}\int_t^{t+\beta_1}r^{1-q}(\tau)\left|cos_p\theta(t)\right|^p d\tau-\frac{1}{\xi t}\int_t^{t+\xi}\Phi\left(cos_p\theta(t)\right)sin_p\theta(t)d\tau$$
$$+\frac{1}{\beta_2 t}\int_t^{t+\beta_2}\frac{c(\tau)}{(p-1)}\left|sin_p\theta(t)\right|^p d\tau$$
$$+\frac{1}{\beta_1 t}\int_t^{t+\beta_1}r^{1-q}(\tau)\left\{\left|cos_p\varphi(\tau)\right|^p-\left|cos_p\theta(t)\right|^p\right\}d\tau$$
$$-\frac{1}{\xi t}\int_t^{t+\xi}\left\{\Phi\left(cos_p\varphi(\tau)\right)sin_p\varphi(\tau)-\;\Phi\left(cos_p\theta(t)\right)sin_p\theta(t)\right\}\,d\tau$$
$$+\frac{1}{\beta_2 t}\int_t^{t+\beta_2}\frac{c(\tau)}{(p-1)}\left\{\left|sin_p\varphi(\tau)\right|^p-\left|sin_p\theta(t)\right|^p\right\}d\tau+0\left(\frac{1}{t^2}\right).$$

By using the definition of R *and* C*, we get*

$$\theta'(t)=\frac{1}{t}\left[R\left|cos_p\theta(t)\right|^p-\Phi\left(cos_p\theta(t)\right)sin_p\theta(t)+C\left|sin_p\theta(t)\right|^p\right]$$

$$+\frac{1}{\beta_1 t}\int_t^{t+\beta_1} r^{1-q}(\tau)\left\{\left|cos_p\varphi(\tau)\right|^p - \left|cos_p\theta(t)\right|^p\right\}d\tau$$
$$-\frac{1}{\xi t}\int_t^{t+\xi}\left\{\Phi\left(cos_p\varphi(\tau)\right)sin_p\varphi(\tau) - \Phi\left(cos_p\theta(t)\right)sin_p\theta(t)\right\}d\tau$$
$$+\frac{1}{\beta_2 t}\int_t^{t+\beta_2}\frac{c(\tau)}{(p-1)}\left\{\left|sin_p\varphi(\tau)\right|^p - \left|sin_p\theta(t)\right|^p\right\}d\tau + O\left(\frac{1}{t^2}\right).$$

And using the half-linear trigonometric functions, we have

$$\left|\left|cos_p\varphi(\tau)\right|^p - \left|cos_p\theta(t)\right|^p\right| \le p\left|\int_{\theta(t)}^{\varphi(\tau)}\left|\Phi\left(cos_p s\right)\left(cos_p s\right)'\right|ds\right| \le (constant)|\varphi(\tau)-\theta(t)|,$$

$$\left|\Phi\left(cos_p\varphi(\tau)\right)sin_p\varphi(\tau) - \Phi\left(cos_p\theta(t)\right)sin_p\theta(t)\right| \le \left|\int_{\theta(t)}^{\varphi(\tau)}\left|\left[\Phi\left(cos_p s\right)\left(sin_p s\right)\right]'\right|ds\right|$$
$$\le (constant)|\varphi(\tau)-\theta(t)|$$

and

$$\left|\left|sin_p\varphi(\tau)\right|^p - \left|sin_p\theta(t)\right|^p\right| \le (constant)|\varphi(\tau)-\theta(t)|.$$

By the mean value theorem we can write

$$\theta(t) = \varphi_1(t_1) + \varphi_2(t_2) + \varphi_3(t_3)$$

for $t_1 \in [t, t+\beta_1], t_2 \in [t, t+\xi], t_3 \in [t, t+\beta_2]$. *Thus*

$$|\varphi(\tau)-\theta(t)| \le |\varphi_1(\tau)-\varphi_1(t_1)| + |\varphi_2(\tau)-\varphi_2(t_2)| + |\varphi_3(\tau)-\varphi_3(t_3)| \le o\left(\frac{1}{t}\right).$$

This implies that

$$|\varphi(\tau)-\theta(t)| \le o\left(\frac{1}{t}\right) \text{ as } t\to\infty, \varphi(\tau)-\theta(t) = o(1).$$

Hence we get

$$\theta'(t) = \frac{1}{t}\left[R\left|cos_p\theta(t)\right|^p - \Phi\left(cos_p\theta(t)\right)sin_p\theta(t) + C\left|sin_p\theta(t)\right|^p\right] + O\left(\frac{1}{t^2}\right).$$

The computation of oscillation constant in Equ. (1.5) is based on the following lemma.

Lemma 2 *Suppose that* θ *is a solution of the differential equation*

$$\theta'(t) = \frac{1}{t}\left[R\left|cos_p\theta(t)\right|^p - \Phi\left(cos_p\theta(t)\right)sin_p\theta(t) + C\left|sin_p\theta(t)\right|^p\right] + o\left(\frac{1}{t}\right),$$

where R, C *are as in Lemma 1.*

If $\left(\int_0^{\beta_1} r^{1-q}(t)dt\right)^{p-1}\int_0^{\beta_2} c(t)dt > \beta_1^{p-1}\beta_2\gamma_p$, *then* $\theta(t)\to\infty$ *as* $t\to\infty$.

If $\left(\int_0^{\beta_1} r^{1-q}(t)dt\right)^{p-1}\int_0^{\beta_2} c(t)dt < \beta_1^{p-1}\beta_2\gamma_p$, *then* $\theta(t)$ *is bounded for large t.*

Proof *Consider the extremal problem*

$$R|x|^p - \Phi(x)y + C|y|^p \to \min(\max), \text{ subject to } |x|^p + |y|^p = 1. \tag{3.1}$$

Using method of the Lagrange multipliers, we obtain Lagrange function

$$L(x,y,\lambda) = R|x|^p - \Phi(x)y + C|y|^p - \lambda(|x|^p + |y|^p - 1)$$

which leads to

$$L_x = |x|^{p-2}[p(R-\lambda)x-(p-1)y] = 0 \tag{3.2}$$

and

$$L_y = -\Phi(x) + p\Phi(y)(C-\lambda) = 0 \tag{3.3}$$

together with the restriction $|x|^p + |y|^p = 1$.

Applying the function Φ^{-1} *to Equ. (3.2) and Equ. (3.3), we get*

$$p(R-\lambda)x-(p-1)y = 0$$

$$-x+\Phi^{-1}(p)\Phi^{-1}(C-\lambda)y = 0$$

Hence λ *must be a root of the equation*

$$\Phi(\lambda-R)(\lambda-C)-\frac{1}{p}\left(\frac{p-1}{p}\right)^{p-1}. \tag{3.4}$$

Denote

$$F(\lambda) = \Phi(\lambda-R)(\lambda-C),\ \ \tilde{\lambda} = \frac{1}{p}\left(\frac{p-1}{p}\right)^{p-1}$$

we have

$$\frac{d}{d\lambda}F(\lambda) = |\lambda-R|^{p-2}[(p-1)(\lambda-C)+(\lambda-R)].$$

Hence $\lambda_* = \frac{1}{p}R+\frac{p-1}{p}C$ *is minimum point and* $F(\lambda_*) = -\tilde{\lambda}|R-C|^p < 0$ *is minimum value. This means that Equ. (3.4) has two real roots* $\lambda_{min} < \lambda_* < \lambda_{max}$. *Moreover,*

$$\Phi(R)C = \tilde{\lambda} \Leftrightarrow \lambda_{min} = 0$$

$$\Phi(R)C > \tilde{\lambda} \Leftrightarrow \lambda_{max} > 0$$

$$\Phi(R)C < \tilde{\lambda} \Leftrightarrow \lambda_{min} < 0 < \lambda_{max}.$$

Multiplying Equ. (3.2) by x, Equ. (3.3) by y and adding the obtained equations, we get

$$R|x|^p - \Phi(x)y + C|y|^p = \lambda.$$

Hence $\big((x_{min}, y_{min}), \lambda_{min}\big)$ *and* $\big((x_{max}, y_{max}), \lambda_{max}\big)$ *are extremum for the function in Equ. (3.1). Let* $\theta_{min}, \theta_{maks}$ *be determined by*

$$\cos\theta_{min} = x_{min},\ \sin\theta_{min} = y_{min}$$

$$\cos\theta_{max} = x_{max},\ \sin\theta_{max} = y_{max}.$$

If $\lambda_{min} > 0$, *then for large t (when the term* $o\left(\frac{1}{t}\right)$ *is less than* $\frac{\lambda_{min}}{2}$*) we have* $\theta'(t) \geq \frac{\lambda_{min}}{2}$. *Hence if* $(p-1)\Phi(R)C > \gamma_P$ *then* $\theta(t) \to \infty$ *as* $t \to \infty$.

If $\lambda_{min} < 0 < \lambda_{max}$, $\theta'(t) > 0$ *when* $\theta(t)$ *is in a right neighborhood of* θ_{max} *and* $\theta'(t) < 0$ *when* $\theta(t)$ *is in a left neighborhood of* θ_{min} *for* $\theta_{max} < \theta_{min}$. *Hence if* $(p-1)\Phi(R)C > \gamma_P$, *then* $\theta(t)$ *is bounded for large* t.

The main result of this paper as follows.

Theorem 1 *Let* r *and* c *be positive defined functions which have different* β_1, β_2 *periods respectively in Equ. (1.5). Then Equ. (1.5) is non-oscillatory if and only*

$$\gamma < \gamma_* = \frac{\beta_1^{p-1}\beta_2\gamma_p}{\left(\int_0^{\beta_1} r^{1-q}(t)dt\right)^{p-1}\int_0^{\beta_2} c(t)dt}$$

where $\gamma_p = \left(\frac{p-1}{p}\right)^p$.

Proof *Let x is the nontrivial solution of Equ. (1.5) and φ is the Prüfer angle of Equ. (1.5) given by Equ. (2.3). Then φ is the solution of*

$$\varphi' = \frac{1}{t}\left[r^{1-q}(t)\left|cos_p\varphi\right|^p - \Phi(cos_p\varphi)sin_p\varphi + \frac{\gamma c(t)}{p-1}\left|sin_p\varphi\right|^p\right].$$

By the help of Lemma 1the function θ satisfies the equation

$$\theta'(t) = \frac{1}{t}\left[R\left|cos_p\theta(t)\right|^p - \Phi\left(cos_p\theta(t)\right)sin_p\theta(t) + \gamma C\left|sin_p\theta(t)\right|^p\right] + o\left(\frac{1}{t}\right),$$

where R, C is as given in Lemma 1.

Again by Lemma 1 φ and θ are at the same time bounded or unbounded. By the help of Lemma 2 if

$$\gamma < \gamma_*,$$

then $\theta(t)$ is bounded for large t and φ is bounded, then Equ. (1.5) is non-oscillatory and if

$$\gamma > \gamma_*,$$

then $\theta(t) \to \infty$ as $t \to \infty$ and φ is unbounded, then Equ. (1.5) is oscillatory.

Remark 1 *If the periods of the functions r, c in Equ. (1.5) are coincide with α −period, which is given in [4] then our oscillation constants γ_* reduce to γ_{rc} is given in [4] and the main result complies with the result given by [4].*

Example 1 *Consider the equation Equ (1.5) for $p = 2,\ r(t) = \frac{1}{2+cos6t}$ and $c(t) = 2 + cos8t$*

$$\left(\left(\frac{1}{2+cos6t}\right)x'\right)' + \gamma\frac{2+cos8t}{t^2}x = 0. \qquad (3.5)$$

In this case $r(t)$ is positive defined for all $t \in \mathbb{R}$ and $\frac{\pi}{3}$ periodic function and $c(t)$ is positive defined for all $t \in \mathbb{R}$ and $\frac{\pi}{4}$ periodic function. Thus we can apply Theorem 1 and we obtain an oscillation constant for Equ. (3.5)

$$\gamma_* = \frac{\frac{\pi}{4}\frac{\pi}{3}\frac{1}{4}}{\left(\int_0^{\frac{\pi}{3}}(2+cos6t)dt\right)\int_0^{\frac{\pi}{4}}(2+cos8t\,)dt} = \frac{1}{16}$$

and Equ. (3.5) is non-oscillatory if and only if $\gamma < \frac{1}{16}$.

Remark 2 *It is well known that if f is any periodic function with period P, then kP $(k \in \mathbb{N})$ is also period of the same function. If we use this fact for the functions $r(t)$ and $c(t)$, we can choose these functions having π-period. In this case we can apply Theorem 3. 1 in [4] to the above example and we get oscillation constant as*

$$\gamma_{rc} = \frac{\alpha^p\gamma_p}{\left(\int_0^\alpha r^{1-q}(t)dt\right)^{p-1}\int_0^\alpha c(t)dt} = \frac{1}{16}$$

and Equ. (1.5)is non-oscillatoryif and only if $\gamma < \frac{1}{16}$.

Example 2 *Consider the linear equation*

$$\left((1-\varepsilon)x'\right)' + \frac{1+\varepsilon}{8t^2}x = 0 \qquad (3.6)$$

for $p = 2,\ r(t) = 1 - \varepsilon,\ c(t) = 1 + \varepsilon$ and $\gamma = \frac{1}{8}$ in Equ (1.5). This equation is non-oscillatory when $\varepsilon > 0$ sufficiently small (oscillation constant in the Equ. (1.3) is $\gamma = \frac{1}{4}$). Now, consider the equation

$$\left((1+\varepsilon sint)x'\right)' + \frac{1+f(t)}{8t^2}x = 0 \qquad (3.7)$$

with small $\varepsilon > 0$ *and* f *periodic with period* $\sqrt{71}$*, satisfying* $|f(t)| < \varepsilon$*. Under assumptions on* f*, Equ. (3.7) is a Sturmian minorant of Equ. (3.6), since*

$$1 + f(t) < 1 + \varepsilon, \quad 1 + \varepsilon sint \geq 1 - \varepsilon.$$

i.e., it is also non-oscillation. Equ. (3.7) is a particular case of Equ. (1.5) with

$$p = 2, r(t) = 1 + \varepsilon sint, c(t) = 1 + f(t), \gamma = \frac{1}{8}, \beta_1 = 2\pi, \beta_2 = \sqrt{71}$$

and satisfying all conditions of Theorem 1. Thus we can apply Theorem 1 and we obtain an oscillation constant for Equ. (3.7)

$$\gamma_* \to \frac{1}{4} \quad as\ \varepsilon \to 0_+.$$

Consequently for $\varepsilon > 0$ *sufficiently small we have* $\gamma = \frac{1}{8} < \gamma_* = \frac{1}{4}$ *which means Equ. (3.7) is non-oscillatory by Theorem 1. But it is well known that* $lcm\,(2\pi, \sqrt{71})$ *is not defined. Thus, we can not apply the Theorem 3. 1 in* [4] *for Equ. (3.7).*

The important point to note here is that the recent results obtained by P. Hasil in [7] and O. Dosly and P. Hasil in [4] and the others do not apply to Equ. (1.5) with periodic coefficients having different periods, when the least common multiple of these periods not defined.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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