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Uniqueness of the Solution to the Inverse Problem of Scattering Theory for the Sturm-Liouville Operator System with a Spectral Parameter in the Boundary Condition

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ABSTRACT

In this paper, we consider the boundary value problem (bvp)

$$y_j'' + \lambda^2 y_j = \sum_{k=1}^{n} V_{jk}(x) y_k \, , x \in \mathbb{R}_+ := (0, \infty) \qquad y_j'(0) + (\alpha_0 + i\alpha_1\lambda + \alpha_2\lambda^2) y_j(0) = 0, j = 1,2, \dots, n$$

where λ is the spectral parameter and $V(x) = \left|\left|V_{jk}(x)\right|\right|_1^n$ is a Hermitian matrix such that

$$\sigma_1(x) = \int_x^\infty t|V(t)|dx < \infty,\ x \in \mathbb{R}_+$$

and α_0, α_1 and α_2 are real numbers, $\alpha_1 \geq 0, \alpha_2 > 0, \alpha_0 + i\alpha_1\lambda + \alpha_2\lambda^2 \neq 0$ for $\lambda = i\mu, \mu > 0$.

We have obtained the uniqueness of the solution to the inverse problem of scattering theory on the semiaxis for the boundary value problem with a spectral parameter.

Keywords: *scattering theory, inverse problem, spectral parameter*

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1. INTRODUCTION

The inverse problem of scattering theory for

$$-y'' + q(x)y = \lambda^2 y \quad (1)$$

with boundary condition not containing a spectral parameter were studied in [1-3]. The direct and inverse scattering problems for a selfadjoint infinite system second-order difference equations with operator-valued coefficients are considered in [4]. The uniqueness of the solution to the inverse problem of scattering theory for the following equation

$$-y'' + q(x)y = \lambda^2 y \quad (2)$$

with a spectral parameter in the boundary condition

$$y'(0) + (\alpha_0 + i\alpha_1\lambda + \alpha_2\lambda^2)y(0) = 0 \quad (3)$$

was studied by Kh.R.Mamedov [5].

We consider the bvp

$$y_j'' + \lambda^2 y_j = \sum_{k=1}^{n} V_{jk}(x)y_k, \;\; x \in \mathbb{R}_+ := (0,\infty) \quad (4)$$

$$y_j'(0) + (\alpha_0 + i\alpha_1\lambda + \alpha_2\lambda^2)y_j(0) = 0, \; j = 1,2,\dots,n \quad (5)$$

where λ is the spectral parameter and

$V(x) = \left\|V_{jk}(x)\right\|_1^n$ is a Hermitian matrix such that

$$\sigma_1(x) = \int_x^\infty t|V(t)|dx < \infty, \; x \in \mathbb{R}_+ \quad (6)$$

and α_0, α_1 and α_2 are real numbers, $\alpha_1 \geq 0, \alpha_2 > 0, \alpha_0 + i\alpha_1\lambda + \alpha_2\lambda^2 \neq 0$ for $\lambda = i\mu, \mu > 0$.

Clearly we can study the matrix differential equation

$$Y'' + \lambda^2 Y = V(x)Y, x \in \mathbb{R}_+ \quad (7)$$

instead of the system (4). It is well known that (see [2]) under the condition (6) Equation (7) has a solution $E(x,\lambda)$ given by

$$E(x,\lambda) = e^{i\lambda x}I + \int_x^\infty K(x,t)e^{i\lambda t}dt, \quad (8)$$

where the matrix $K(x,t)$ satisfies the inequality

$$|K(x,t)| \leq \frac{1}{2}e^{\sigma_1(x)}\sigma\left(\frac{x+t}{2}\right)$$

$$\sigma(x) = \int_x^\infty |V(t)|dt. \quad (9)$$

for $x \in \mathbb{R}_+$ (and for $x = 0$ as well, in the event that $\sigma_1(0) < \infty$), $E(x,\lambda)$ is regular in $\mathbb{C}_+$ and continuous on $\overline{\mathbb{C}}_+$. Moreover, the matrix $K(x,t)$ and potential are related to

$$K(x,x) = \frac{1}{2}\int_x^\infty V(t)dt. \quad (10)$$

In this paper we shall use the following notations:

$\mathbb{C}_+ = \{\lambda : \lambda \in \mathbb{C}, Im\lambda > 0\}$

$\overline{\mathbb{C}}_+ = \{\lambda : \lambda \in \mathbb{C}, Im\lambda \geq 0\}$

I is the identity matrix,

A^* is the complex conjugate transpose of the matrix A.

$$|A| = \max_j \sum_k |a_{jk}|$$

denotes absolute value of a matrix $A = \|a_{jk}\|$.

A matrix is said to be continuous if all its elements are continuous functions. In the same sense, we shall refer to a matrix as being summable, differentiable, regular, etc.

$L^2_{(n)}(\alpha,\beta)$ denotes the Hilbert space of vector functions $f(x) = \{f_1(x),\dots,f_n(x)\}$ with components square summable over (α,β) and with scaler product defined by

$$(f,g)_{\langle\alpha,\beta\rangle} = \int_\alpha^\beta \sum_{k=1}^n f_k(x)\,\overline{g_k(x)}dx$$

Moreover we have

$$W\{E^*(x,\lambda), E(x,\lambda)\} = \begin{cases} 2i\lambda I & for\; Im\lambda = 0 \\ 0 & for\; Re\lambda = 0, Im\lambda > 0 \end{cases} \quad (11)$$

Here W denotes the Wronskian of E^* and E. As λ is real and nonzero $E(x,\lambda)$ and $E(x,-\lambda)$ form the fundamental system of solutions of the equation (4) and the Wronskian of this system is

$W\{E(x,\lambda), E(x,-\lambda)\} = -2i\lambda I$ for $Im\lambda = 0$

Consider the solution $w(x,\lambda)$ of equation (4) satisfying the initial conditions

$$w(0,\lambda) = I, w(0,\lambda) = -(\alpha_0 + i\alpha_1\lambda + \alpha_2\lambda^2)I.$$

Now we have the following:

Lemma 1:

For $\lambda \epsilon R \setminus \{0\}$ the following expression for $w(x,\lambda)$ is valid

$$w(x,\lambda) = \frac{1}{2}[E(x,-\lambda) - E(x,\lambda)S(\lambda)][E^{*'}(-\lambda) + (\alpha_0 + i\alpha_1\lambda + \alpha_2\lambda^2)E^*(-\lambda)] \quad (12)$$

where

$$S(\lambda) = [E^{*'}(\lambda) + (\alpha_0 + i\alpha_1\lambda + \alpha_2\lambda^2)E^*(\lambda)][E^{*'}(-\lambda) + (\alpha_0 + i\alpha_1\lambda + \alpha_2\lambda^2)E^*(-\lambda)]^{-1} \quad (13)$$

with $S^*(\lambda) = S(-\lambda)$

Proof:

Since $E(x,\lambda)$ and $E(x,-\lambda)$ form the fundamental solution system of equation (4) for $\lambda\epsilon R \setminus \{0\}$ then (12) can be obtained easily. From the initial conditions we deduce the following result

$$E'(-\lambda)+(\alpha_0+i\alpha_1\lambda+\alpha_2\lambda^2)E(-\lambda)][E^{*'}(-\lambda)+(\alpha_0+i\alpha_1\lambda+\alpha_2\lambda^2)E^*(-\lambda)] \\ =E'(\lambda)+(\alpha_0+i\alpha_1\lambda+\alpha_2\lambda^2)E(\lambda)]\,[E^{*'}(\lambda)+(\alpha_0+i\alpha_1\lambda+\alpha_2\lambda^2)E^*(\lambda)] \tag{14}$$

By (14) we obtain that $S^*(\lambda)=S(-\lambda)$.

Lemma 2:

The function $detF(\lambda)$,

in which

$F(\lambda):=E'(0,\lambda)+(\alpha_0+i\alpha_1\lambda+\alpha_2\lambda^2)E(0,\lambda)$

can have only a finite number of zeros in C_+ . Moreover, all these zeros are simple and lie on the imaginary axis.

Proof:

From (8), (9) we obtain that

$F(\lambda)=\alpha_2\lambda^2 I+O(\lambda)$

when $|\lambda|\to\infty$, $\lambda\epsilon\overline{\mathbb{C}}$, so that the zeros of $detF(\lambda)$ form a bounded set with at most one possible limiting point $\lambda=0$. (Since $F(x,\lambda)$ is nonsingular $detF(\lambda)\neq 0$ for $\lambda\epsilon\mathbb{R}$) . We can show the method given in [2] that the number of zeros of $detF(\lambda)$ is finite even if $detF(\lambda)=0$. So we get that the matrix function $F^{-1}(\lambda)$ is regular in $\mathbb{C}_+$ with the possible exception of a finite number of points where $detF(\lambda)=0$. (i.e. $F^{-1}(\lambda)$ has poles). Let us now show that all the singularities of the matrix function $F^{-1}(\lambda)$ lie on the imaginary axis: Let λ_1 and λ_2 be some poles of $F^{-1}(\lambda)$.

Consider the following differential equation for λ_1

$E''(x,\lambda_1)+\lambda_1^2E(x,\lambda_1)=V(x)\,E(x,\lambda_1)$

and its complex conjugate transpose for λ_2

$(E^*)''(x,\lambda_2)+\left(\bar{\lambda}_2\right)^2E^*(x,\lambda_2)=V(x)\,E^*(x,\lambda_2)$

Multiplying the first equation by $E^*(x,\lambda_2)$ and the second equation by $E(x,\lambda_1)$, subtracting the second resulting relation from the first, and integrating the result from zero to infinity, we get

$$(\lambda_1^2{}_-\left(\bar{\lambda}_2\right)^2)\int_0^\infty E^*(x,\lambda_2)E(x,\lambda_1)dx-W\{E^*(x,\lambda_1)E(x,\lambda_2)\}_{x=0}=0 \tag{15}$$

where λ_1,λ_2 are the points for which the inverse of $F(\lambda)$ does not exist. Hence obviously there exists a nonzero vector a such that

$F(\lambda_i)a=0$, i=1,2

Thus the solution $E(x,\lambda_i)a$ of the system (4) satisfies (5).

Therefore Wronskian in (15) takes the form

$$W\{E^*(x,\lambda_1)E(x,\lambda_2)\}_{x=0}=E^*(0,\lambda_1)E'(0,\lambda_2)-(E^*)'(0,\lambda_1)E(0,\lambda_2).$$

Multiplying the last equation from the right by a vector a and from the left by a^* and using the fact that $F(\lambda_i)a=0$ for i=1,2 . Therefore follows that

$$W\{E^*(x,\lambda_1)E(x,\lambda_2)\}_{x=0}=\left[-i\alpha_1(\overline{\lambda_1}+\lambda_1)+\alpha_2\left(\left(\bar{\lambda}_1\right)^2-\lambda_1^2\right)\right]a^*E^*(0,\lambda_1)\,E(x,\lambda_1)a$$

for $\lambda_1=\lambda_2$. Hence substituting the last equation into equation (15) we obtain that

$$(\lambda_1+\overline{\lambda_1})\left[(\lambda_1-\overline{\lambda_1})\int_0^\infty E^*(x,\lambda_1)E(x,\lambda_1)dx+i\alpha_1a^*E^*(0,\lambda_1)E(0,\lambda_1)a+\alpha_2\,(\lambda_1-\overline{\lambda_1})\,a^*E^*(0,\lambda_1)E(0,\lambda_1)a\right]=0 \tag{16}$$

It follows from (16) that the zeros of $detF(\lambda)$ are of the form $\lambda=i\mu,\ \mu>0$.

Now let us show that all the singularities of the matrix function $F^{-1}(\lambda)$ in $\mathbb{C}_+$ are

simple poles.

Differentiating the equation

$E''(x,\lambda)+\lambda^2 E(x,\lambda)=V(x)E(x,\lambda)$

with respect to λ and then taking the complex conjugate transpose of both sides of the resulting equation, we deduce

$(\dot{E}^*(x,\lambda))''+2\bar{\lambda}E^*(x,\lambda)+\overline{(\lambda)}^2\dot{E}^*(x,\lambda)=\dot{E}^*(x,\lambda)V(x)$

for $\lambda\epsilon\mathbb{C}_+$. Postmultiplying the last equation by $E(x,\lambda)$ and subtracting the first resulting equation from the second, after first premultiplying it by $E^*(x,\lambda)$ we obtain that

$\dot{E}^*(x,\lambda)E''(x,\lambda)-(\dot{E}^*(x,\lambda))''E(x,\lambda)=2\lambda E^*(x,\lambda)E(x,\lambda)$

Integrating both sides of the last equation from 0 to ∞ we obtain that

$$\dot{E}^*(x,\lambda)E'(x,\lambda)-(\dot{E}^*(x,\lambda))'E(x,\lambda)=2\lambda\int_x^\infty E^*(x,\lambda)E(x,\lambda)\,dx \quad (17)$$

for $\lambda\epsilon\mathbb{C}_+$.

Let $\lambda_{0=}i\mu_0$, $\mu_0\epsilon\mathbb{C}_+$ be a pole of $F^{-1}(\lambda)$. Then we have $detF(\lambda_0)=0$ and hence there exists a nonzero vector a such that

$$F(\lambda_0)a=0 \quad (18)$$

Thus the solution $E(x,\lambda_0)a$ of the system (1) satisfies

$$E'(0,\lambda_0)a+(\alpha_0+i\alpha_1\lambda_0+\alpha_2\lambda_0^2)E(0,\lambda_o)a=0$$

for $\lambda=\lambda_0$.

Since

$$W\{w(x,\lambda_0)E(x,\lambda_0)\}_{x=0}=F(\lambda_0)$$

then

$$E(x,\lambda_0)a=w(x,\lambda_0)a_1$$

here a_1 is some vector and then

$$\lim_{x\to 0}E(x,\lambda_0)a=E(\lambda_0)a=w(\lambda_0)a_1=a_1$$

exist and

$$E(x,\lambda_0)a=w(x,\lambda_0)a_1$$

Substituting $\lambda=\lambda_0$ into (17), multiplying this equation from the left by a^* and from the right by a and then letting x tend to zero, we get that

$$-ia^*\dot{F}^*(i\mu_0)E(i\mu_0)a=2\mu_0\int_0^\infty[E(x,i\mu_0)a]^*\left[E(x,i\mu_{0)}a\right]dx+(\alpha_1+2\alpha_2\mu_0)a^*E^*(i\mu_0)E(i\mu_0)a\neq 0 \quad (19)$$

In addition the condition (16) we now suppose that the vector a satisfies the equation

$$F(\lambda_0)b+\dot{F}(\lambda_0)\mathrm{a}=0 \quad (20)$$

where b is some other vector.

Taking the complex conjugate transpose of both sides of the matrix equation (20) and postmultiplying the resulting equation by $E'(\lambda_0)a$ we deduce that

$$b^*F^*(\lambda_0)E'(\lambda_0)a+a^*\dot{F}^*(\lambda_0)E'(\lambda_0)a=0$$

By the definition of $F(\lambda)$ and (18), the first term of the left hand side of the last equation is

$$b^*F^*(\lambda_0)E'(\lambda_0)a=b^*(E^*)'(\lambda_0)E(\lambda_0)a=0$$

and hence the second term is

$$a^*\dot{F}^*(\lambda_0)E'(\lambda_0)a=0$$

which gives a contradiction to (19) by the definition of $F(\lambda_0)$. This shows that the vector satisfying (18) and (20) simultaneously must be zero. Therefore by Lemma 2.2.1 in [2] we get that $F^{-1}(\lambda)$ has a simple pole at λ_0 as claimed. (i.e. all zeros of $detF(\lambda)$ are simple).

By the definition of $F(\lambda)$, we can obtain the following asymptotic equality

$$F(\lambda) = \alpha_2\lambda^2\left[I + O\left(\frac{1}{\lambda}\right)\right]$$

as $|\lambda| \to \infty$. Therefore $S(\lambda)$ is called the scattering matrix and satisfies the asymptotic equality $S(\lambda) = I + O(\frac{1}{\lambda})$ as $|\lambda| \to \infty$. Hence $I - S(\lambda) \epsilon L^2_{(n)}(-\infty, \infty)$ and therefore the function

$$F_s(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\big(I - S(\lambda)\big)e^{i\lambda t}d\lambda$$

belongs to $L^2_{(n)}(-\infty, \infty)$.

To derive the main equation we rewrite (12) in the following form:

$$2iw(x,\lambda)\left[E^{*\prime}(-\bar{\lambda}) + (\alpha_0 + i\alpha_1\lambda + \alpha_2\lambda^2)E^*(-\bar{\lambda})\right]^{-1} = E(x,-\lambda) - E(x,\lambda)S(\lambda)$$

and substitute $E(x,\lambda)$ in this by its expression (8). Then we get that

$$\begin{aligned} &2iw(x,\lambda)\left[(E^*)'(-\bar{\lambda}) + (\alpha_0 + i\alpha_1\lambda + \alpha_2\lambda^2)E^*(-\bar{\lambda})\right]^{-1} + 2isin\lambda xI = \\ &e^{i\lambda x}[I - S(\lambda)] + \int_x^{\infty}K(x,t)e^{-i\lambda t}dt + \int_x^{\infty}K(x,t)e^{i\lambda t}dt[I - S(\lambda)] - \int_x^{\infty}K(x,t)e^{i\lambda t}dt \end{aligned} \quad (21)$$

We now multiply the left hand side of (21) by $\frac{e^{i\lambda y}}{2\pi}$ and integrate the result from $-\infty\ to\ \infty$ over λ. On the left hand side we get

$$-\sum_{k=1}^{p} 2i\mu_k w(x,\mu_k)R_k^*\, e^{-\mu_k y} \quad (22)$$

where R_k is the residue of the matrix $F^{-1}(\lambda)$ at the pole $\lambda_k = i\mu_k$, $\mu_k > 0$. Since the second term $2isin\lambda xI$ of the left hand side of the equation (21) is an entire function of λ. Therefore the integral will be zero. On the right hand side, since $K(x,t) = 0$ when $x > t$, we obtain that

$$F_s(x+y) + K(x,y) + \int_x^{\infty}K(x,t)F_s(t+y)dt \quad (23)$$

for $0 \le x < y$. Taking (22) and (23) into account then (21) takes the form

$$F_s(x+y) + K(x,y) + \int_x^{\infty}K(x,t)F_s(t+y)dt = -2i\sum_{k=1}^{p}\mu_k w(x,i\mu_k)R_k^* e^{-\mu_k y}, 0 \le x < y \quad (24)$$

Taking into account (17) for $\lambda = \lambda_k$ we convert the right hand side of (24) into another form. For this purpose, we have

$$\dot{E}^*(x,\lambda_k)E'(x,\lambda_k) - (\dot{E}^*(x,\lambda_k))'E(x,\lambda_k) = 2\lambda_k\int_x^{\infty}E^*(t,\lambda_k)E(t,\lambda_k)dt \quad (25)$$

using (17) and the relations

$$F(\lambda_k)R_k = R_kF(\lambda_k) = 0.$$

$$F(\lambda_k)R_k^{(0)} + \dot{F}(\lambda_k)R_k - R_k^{(0)}F(\lambda_k) + R_k\dot{F}(\lambda_k) = I. \quad (26)$$

Let P_k denote a Hermitian matrix which is a projection onto the null space of the matrix $F(\lambda_k)$ so that $F(\lambda_k)P_k = 0$. The set of vectors with form $R_k a$, a is an arbitrary vector, coincides with the null space of $F(\lambda_k)$. Hence

$rankR_k = rankP_k$

And

$$P_kR_k = R_k \quad [2] \quad (27)$$

multiplying (25) from the left by R_k^* and from the right by P_k and letting $x \to 0$ in the result we reach the following equation :

$$R_k^*\dot{F}^*(\lambda_k)E(0,\lambda_k)P_k - (i\alpha_1 + 2\alpha_2\lambda_k)R_k^*E^*(0,\lambda_k)E(0,\lambda_k)P_k = 2\lambda_kR_k^*A_kP_k \quad (28)$$

where

$$A_k := \int_0^{\infty}E^*(t,\lambda_k)E(t,\lambda_k)dt$$

is a positive definite Hermitian matrix,

and

$$E(0,\lambda_k)P_k = \lim_{x\to 0} E(x,\lambda_k)\, P_k\,.$$

Using the second equation in (26) and the fact that

$$W\{E^*(x,\lambda), E(x,\lambda)\}_{x=0} = 0 \text{ for } \lambda\epsilon C_+, Re\lambda = 0$$

we arrive at

$$R_k^* \dot{F}^*(\lambda_k) E(0,\lambda_k)P_k = E(0,\lambda_k)P_k\ .$$

Hence taking the last equation and (26) into account we obtain for $\lambda_k = i\mu_k$ that

$$E(0,i\mu_k)P_k - i(\alpha_1 + 2\alpha_2\mu_k)R_k^*)E^*(0,i\mu_k)E(0,i\mu_k)P_k - 2i\mu_k R_k^* A_k P_k \tag{29}$$

where A_k is the matrix given in (26).

Now we have

$$E(x,i\mu_k)P_k = w(x,i\mu_k)E(0,i\mu_k)P_k$$

Since each side of the last equation is a solution of matrix Equation (4) when $\lambda = i\mu_k$ and satisfies the same initial conditions at $x = 0$. The last equation takes the form

$$E(x,i\mu_k)P_k = 2i\mu_k w(x,i\mu_k)R_k^* B_k \tag{30}$$

where

$$B_k = P_k A_k P_k + \left(\frac{\alpha_1 + 2\alpha_2\mu_k}{2\mu_k}\right) P_k E^*(0,i\mu_k)E(0,i\mu_k)P_k + I - P_k$$

by considering (27) in (29). Postmultiplying (28) by B_k^{-1} we arrive at

$$E(x,i\mu_k)M_k^2 = w(x,i\mu_k)R_k^* \tag{31}$$

here

$M_k^2 = P_k B_k^{-1}$

$M_1, \dots, M_p$ will be referred to as the normalization matrices. Taking (31) into account , then (22) takes the form

$$-\sum_{k=1}^{p} 2i\mu_k w(x,i\mu_k)R_k^*\, e^{-\mu_k y} = -\sum_{k=1}^{p} E(x,i\mu_k)M_k^2 e^{-\mu_k y}$$

where

$M_k^2 = P_k B_k^{-1}$.

Using the expression for $E(x,i\mu_k)$ given by (8) in the last form of (22) obtained above, we finally deduce that the kernel $K(x,y)$ satisfies the linear integral equation

$$F(x+y) + K(x,y) + \int_x^{\infty} K(x,t)F(t+y)dt = 0 \tag{32}$$

for $0 \le x < y$, where

$$F(t) = \sum_{k=1}^{p} M_k^2 e^{-\mu_k t} + F_s(t) = \sum_{k=1}^{p} M_k^2 e^{-\mu_k t} + \frac{1}{2\pi}\int_{-\infty}^{\infty} [I - S(\lambda)]e^{i\lambda t} d\lambda \tag{33}$$

Equation (22) is called the fundamental equation of the inverse problem of scattering theory.

Therefore we have proved the following.

Theorem 1:

The kernel $K(x,y)$ of the transformation operator (6) satisfies the fundamental equation (32) for ≥ 0 .

We know that in order to construct the fundamental equation (32), it is sufficient to state the matrix $F(t)$ (kernel of the fundamental equation). In turn, in order to construct $F(t)$, it is sufficient to know the quantities

$$S(\lambda), \lambda_k^2, \mu_k^2\ , k = 1, \dots, p$$

which are called the scattering data of the problem (4)-(5). From (33) we can deduce that $F(t)$ is a Hermitian matrix.

Given the scattering data, using (33) we can obtain the matrix $F(t)$ and hence the fundamental equation (32) for the unknown matrix $K(x,y)$. Solving this equation we find the kernel K of the transformation operator. From (10) we reach to the potential such that

$$q(x) = -\frac{1}{2}\frac{d}{dx}K(x,y)$$

Theorem 2:

The equation (32) has a unique solution $K(x,y)\epsilon L_1[x,\infty)$.

Proof:

We need to show that the homogeneous equation

$$x(t) + \int_x^\infty x(\xi)F(t+\xi)d\xi = 0 \tag{34}$$

has only the zero solution in $L^2_{(n)}(0,\infty)$. We assume that (34) has a different zero solution. By forming the scalar product of both sides of (34) with $x(t)$ and integrating;

$$\int_x^\infty \big(x(t),x(t)\big)dt + \int_x^\infty (\int_x^\infty x(\xi)F(t+\xi)d\xi, x(t))dt = 0$$

By using last equation and (33)

$$\int_x^\infty \big(x(t),x(t)\big)dt + \int_x^\infty (\int_x^\infty x(\xi)F(t+\xi)d\xi, x(t))dt + \int_x^\infty (\sum_{k=1}^p \int_x^\infty x(\xi)M_k^2 e^{-\mu_k(t+\xi)}d\xi, x(t))dt = 0 \tag{35}$$

In (35) interchanging integrals and using $\sum_{k=1}^p e^{-\mu_k(t+\xi)}\varphi(t)$ series uniform converges (35) can be integrated by terms. So we obtain following

$$\int_x^\infty x^2(t)dt + \sum_{k=1}^p (\int_x^\infty x(t)\, e^{-\mu_k t}dt)^2 + \int_x^\infty (\int_x^\infty x(t)e^{i\mu t}dt)^2 d\xi \tag{36}$$

By using Parseval equation of Fourier transformation in (36)

$$\frac{1}{2\pi}\int_{-\infty}^\infty |\phi^+(\xi)|^2 d\xi + \sum_{k=1}^p (\int_x^\infty x(t)e^{-\mu_k t}dt)^2 + \frac{1}{2\pi}\int_{-\infty}^\infty [\phi^+(\xi)]^2 d\xi = 0 \tag{37}$$

where Parseval equation

$$\int_x^\infty x^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^\infty |\phi^+(\xi)|^2 d\xi \ .$$

From

$\arg\big(\phi(\xi)\big) = \theta(\xi)$

(37) rewrite as polar formata, we obtaine

$$\sum_{k=1}^p \left(\int_x^\infty x(t)e^{-\mu t}dt\right)^2 + \frac{1}{2\pi}\int_{-\infty}^\infty |\phi^+(\xi)|^2\big(1 - e^{i(\eta(k)+2\theta(k))}\big)d\xi = 0. \tag{38}$$

For $Re\big(e^{i(\eta(k)+2\theta(k))}\big) = \cos(\eta(k) + 2\theta(k))$

Real part of (38) is

$$\sum_{k=1}^p \left(\int_x^\infty x(t)e^{-\mu t}dt)\right)^2 + \frac{1}{2\pi}\int_{-\infty}^\infty |\phi^+(\xi)|^2\big(1 - \cos(\eta(k) + 2\theta(k))\big)dk = 0$$

For this equation is equal to zero only situation is

$\phi(\xi) = 0, x(t) = 0.$

This is a contradiction. So equation (32) has a unique solution for finite x.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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