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Common Fixed Points for $(\psi,\mathcal{F},\alpha,\beta)$-Weakly Contractive Mappings in Generalized Metric Spaces via New Functions

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ABSTRACT

In this article, we establish some common fixed point theorems for new type generalized contractive mappings involving C-class functions in generalized metric spaces. We provide an example in order to support the useability of our results. The proofs of all our results are without using Hausdorff assumption. These results generalize some well-known results in the literature.

Key words: C-class functions, generalized metric space, weakly contractive condition, contraction of integral type, common fixed point.

1. INTRODUCTION AND PRELIMINARIES

In 2000, Branciari [1] introduced a concept of generalized metric space where the triangle inequality of a metric space has been replaced by an inequality involving three terms instead of two. As such, any metric space is a generalized metric space but the converse is not true [1]. He proved the Banach's fixed point theorem in such a space. After that, many fixed point results were established for this interesting space. For more, the reader can refer to [2-13].

It is also known that common fixed point theorems are generalizations of fixed point theorems. Recently, there have been many researchers who have interested in generalizing fixed point theorems to coincidence point theorems and common fixed point theorems. In a recent paper, Choudhury and Kundu [14] established the (ψ,α,β)-weak contraction principle to coincidence point and common fixed point results in partially ordered metric spaces.

We start by recalling some definitions and notions.

In the sequel, the letters $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}$ will denote the set of all real numbers, the set of all non negative real numbers and the set of all natural numbers, respectively.

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Definition 1 ([1]). Let X be a non-empty set and $d: X \times X \to \mathbb{R}^+$ be a mapping such that for all $x, y \in X$ and for all distinct points $u, v \in X$, each of them different from x and y, one has

(i) $d(x,y) = 0$ if and only if $x = y$,

(ii) $d(x,y) = d(y,x)$,

(iii) $d(x,y) \leq d(x,u) + d(u,v) + d(v,y)$ (the rectangular inequality).

Then (X,d) is called a generalized metric space (or for short g.m.s.).

Definition 2 ([1]). Let (X,d) be a g.m.s., $\{x_n\}$ be a sequence in X and $x \in X$.

(i) We say that $\{x_n\}$ is g.m.s. convergent to x if and only if $d(x_n, x) \to 0$ as $n \to +\infty$. We denote this by $x_n \to x$.

(ii) We say that $\{x_n\}$ is a g.m.s. Cauchy sequence if and only if for each $\varepsilon > 0$ there exists a natural number $n(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$ for all $n > m > n(\varepsilon)$.

(iii) (X,d) is called a complete g.m.s. if every g.m.s. Cauchy sequence is g.m.s. convergent in X.

It is well known that generalized metric spaces in the sense of Branciari might not be Hausdorff and, hence, there may exist sequences in them having more than one limit. Thus, in most of the fixed point results obtained recently in such spaces, Hausdorffness was additionally assumed. Recently, Kadelburg and Radenović [15] showed that, nevertheless, most of these results remain valid without this additional assumption.

Lemma 1 ([15]). Let (X,d) be a g.m.s. and let $\{x_n\}$ be a Cauchy sequence in X such that $x_m \neq x_n$ whenever $m \neq n$. Then $\{x_n\}$ can converge to at most one point.

Definition 3 ([15]). A pair (f,T) of self-mappings on a set X is said to be weakly compatible if f and T commute at their coincidence point (i.e. $fTx = Tfx$, $x \in X$ whenever $fx = Tx$). A point $y \in X$ is called a point of coincidence of two self-mappings f and T on X if there exists a point $x \in X$ such that $y = Tx = fx$. Here, $x \in X$ is called coincidence point of f and T and if $y = x$, we say that x is a common fixed point of f and T.

Very recently, Ansari [16] defined the concept of C-class functions and presented new fixed point results which improve and extend some results in the literature. For more details, also see [17, 18].

Definition 4 ([16]). A mapping $\mathcal{F}: \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is called C-class function if it is continuous and satisfies following axioms:

$(\mathcal{F}_1)$ $\mathcal{F}(s,t) \leq s$;

$(\mathcal{F}_2)$ $\mathcal{F}(s,t) = s$ implies that either $s = 0$ or $t = 0$ for all $s, t \in \mathbb{R}^+$.

Note that $\mathcal{F}(0,0)=0$.

We denote C-class functions as $\mathcal{C}$.

Example 1 ([16]). The following functions $\mathcal{F}: \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ are elements of $\mathcal{C}$, for all $s, t \in \mathbb{R}^+$:

(1) $\mathcal{F}(s,t) = s - t$, if $\mathcal{F}(s,t) = s \Rightarrow t = 0$;

(2) $F(s,t) = ks$ for $0 < k < 1$, if $\mathcal{F}(s,t) = s \Rightarrow s = 0$;

(3) $\mathcal{F}(s,t) = \frac{s}{(1+t)^r}$ for $r \in (0, +\infty)$, if $\mathcal{F}(s,t) = s \Rightarrow s = 0 \; or \; t = 0$;

(4) $\mathcal{F}(s,t) = \log_a \frac{t+a^s}{1+t}$ for $a > 1$, if $\mathcal{F}(s,t) = s \Rightarrow s = 0 \; or \; t = 0$;

(5) $\mathcal{F}(s,1) = \ln\left(\frac{1+a^s}{2}\right)$ for $a > e$, if $\mathcal{F}(s,1) = s \Rightarrow s = 0$;

(6) $\mathcal{F}(s,t) = s\log_{t+a} a$ for $a > 1$, if $\mathcal{F}(s,t) = s \Rightarrow s = 0 \; or \; t = 0$;

(7) $\mathcal{F}(s,t) = (s+l)^{\frac{1}{(1+t)^r}} - l$ for $l > 1$, $r \in (0, \infty)$, if $\mathcal{F}(s,t) = s \Rightarrow t = 0$.

We denote by Ψ the set of functions $\psi: \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following hypotheses:

(ψ_1) ψ is monotone nondecreasing,

(ψ_2) $\psi(t) = 0$ if and only if $t = 0$.

We denote by Φ the set of functions $\varphi: \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following hypotheses:

(φ_1) φ is continuous,

(φ_2) $\varphi(t) = 0$ if and only if $t = 0$.

In this paper, we present new type generalized contractions involving C-class functions and establish several common fixed point theorems for this class of mappings defined on generalized metric spaces. The obtained results extend many recent results in the literature. Also, we give an example to illustrate effectiveness of the obtained results.

2. MAIN RESULTS

Before proceeding to our results, let us give following lemma which will be used efficiently in the proof of main result.

Lemma 2. Let $\{a_n\}$ be a sequence of non-negative real numbers. If

$$\psi(a_{n+1}) \leq \mathcal{F}(\alpha(a_n), \beta(a_n)), \qquad (2.1)$$

for all $n \in \mathbb{N}$, where $\psi \in \Psi$, $\alpha, \beta \in \Phi$ and $\mathcal{F} \in \mathcal{C}$ and

$$\psi(t) - \mathcal{F}(\alpha(t), \beta(t)) > 0 \text{ for all } t > 0. \qquad (2.2)$$

Then the following conditions hold:

(i) $a_{n+1} < a_n$ if $a_n > 0$,

(ii) $a_n \to 0$ as $n \to +\infty$.

Proof. (i) Let, if possible, $a_n \leq a_{n+1}$ for some $n \in \mathbb{N}$. Then, using monotone property of ψ and (2.1), we have

$$\psi(a_n) \leq \psi(a_{n+1}) \leq \mathcal{F}(\alpha(a_n), \beta(a_n)),$$

which implies that $a_n = 0$ by (2.2), a contradiction with $a_n > 0$. Therefore, for all $n \in \mathbb{N}$

$$a_{n+1} < a_n.$$

(ii) By (i) the sequence $\{a_n\}$ is decreasing, hence there is $a \geq 0$ such that $a_n \to a$ as $n \to +\infty$. Letting $n \to +\infty$ in (2.1), we obtain

$$\psi(a) \leq \lim_{n\to+\infty} \psi(a_{n+1}) \leq \mathcal{F}(\alpha(a), \beta(a)),$$

which implies, by (2.2), $a = 0$. ■

Our main result is as follows:

Theorem 1. Let (X, d) be a complete g.m.s. and let $T, f: X \to X$ be self-mappings such that $TX \subseteq fX$, and fX is a closed subspace of X, and following condition holds:

$$\psi(d(Tx, Ty)) \leq \mathcal{F}(\alpha(d(fx, fy)), \beta(d(fx, fy))), \quad (2.3)$$

for all $x, y \in X$, where $\psi \in \Psi$, $\alpha, \beta \in \Phi$ and $\mathcal{F} \in \mathcal{C}$, and satisfying condition (2.2). Then T and f have a unique point of coincidence in X. Moreover, if T and f are weakly compatible, then T and f have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in X. Since $TX \subseteq fX$, we can define the sequences $\{x_n\}$ and $\{y_n\}$ in X by

$$y_n = Tx_n = fx_{n+1} \quad \text{for all } n \in \mathbb{N}_0, \quad (2.4)$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Substituting $x = x_n$ and $y = x_{n+j}$ for every $j \in \mathbb{N}$ in (2.3), and using (2.4), we have

$$\begin{aligned}&\psi\left(d(y_n, y_{n+j})\right) = \psi\left(d(Tx_n, Tx_{n+j})\right)\\ &\leq \mathcal{F}\left(\alpha\left(d(fx_n, fx_{n+j})\right), \beta\left(d(fx_n, fx_{n+j})\right)\right)\\ &\leq \mathcal{F}\left(\alpha\left(d(y_{n-1}, y_{n+j-1})\right), \beta\left(d(y_{n-1}, y_{n+j-1})\right)\right).\end{aligned}$$

By (ii) of Lemma 1, we obtain that

$$\lim_{n\to+\infty} d(y_n, y_{n+j}) = 0. \quad (2.5)$$

Suppose that $y_n \neq y_m$ for all $n \neq m$ and prove $\{y_n\}$ is a g.m.s. Cauchy sequence. Suppose that $\{y_n\}$ is not a g.m.s. Cauchy sequence. Then, there exists $\varepsilon > 0$ for which we can find subsequences $\{y_{n_k}\}$ and $\{y_{m_k}\}$ of $\{y_n\}$ with $n_k > m_k > k$ such that

$$d(y_{n_k}, y_{m_k}) \geq \varepsilon. \quad (2.6)$$

Further, corresponding to m_k, we can choose n_k in such a way that it is the smallest integer with $n_k > m_k$ and satisfying (2.6). Then

$$d(y_{n_{k-1}}, y_{m_k}) < \varepsilon. \quad (2.7)$$

Now, using (2.6), (2.7) and the rectangular inequality, we have

$$\begin{aligned}\varepsilon &\leq d(y_{n_k}, y_{m_k})\\ &\leq d(y_{n_k}, y_{n_{k-2}}) + d(y_{n_{k-2}}, y_{n_{k-1}}) + d(y_{n_{k-1}}, y_{m_k})\\ &< d(y_{n_k}, y_{n_{k-2}}) + d(y_{n_{k-2}}, y_{n_{k-1}}) + \varepsilon.\end{aligned}$$

Letting $k \to +\infty$ in the above inequality, using (2.5) with $j = 1,2$, we obtain

$$\lim_{k\to+\infty} d(y_{n_k}, y_{m_k}) = \varepsilon. \quad (2.8)$$

Again, the rectangular inequality gives us

$$\begin{aligned}d(y_{n_k}, y_{m_k}) &\leq d(y_{n_k}, y_{n_{k-1}}) + d(y_{n_{k-1}}, y_{m_{k-1}})\\ &\quad + d(y_{m_{k-1}}, y_{m_k}),\end{aligned}$$

and

$$\begin{aligned}d(y_{n_{k-1}}, y_{m_{k-1}}) &\leq d(y_{n_{k-1}}, y_{n_k}) + d(y_{n_k}, y_{m_k})\\ &\quad + d(y_{m_k}, y_{m_{k-1}}).\end{aligned}$$

Taking $k \to +\infty$ in the above inequalities and using (2.5) and (2.8), we get

$$\lim_{k\to+\infty} d(y_{n_{k-1}}, y_{m_{k-1}}) = \varepsilon. \quad (2.9)$$

Substituting $x = x_{n_k}$ and $y = x_{m_k}$ in (2.3), we have

$$\begin{aligned}&\psi\left(d(y_{n_k}, y_{m_k})\right)\\ &\leq \mathcal{F}\left(\alpha\left(d(y_{n_{k-1}}, y_{m_{k-1}})\right), \beta\left(d(y_{n_{k-1}}, y_{m_{k-1}})\right)\right)\end{aligned}$$

Letting $k \to +\infty$ in the above inequality and using (2.8) and (2.9), we deduce

$$\psi(\varepsilon) \leq \lim_{k\to+\infty} \psi\left(d(y_{n_k}, y_{m_k})\right) \leq \mathcal{F}(\alpha(\varepsilon), \beta(\varepsilon)),$$

which implies that $\varepsilon = 0$, by (2.2), a contradiction with $\varepsilon > 0$. It follows that $\{y_n\}$ is a g.m.s. Cauchy sequence and hence $\{y_n\}$ is convergent in the complete g.m.s. (X, d). Since fX is closed, there exists a $w \in fX$ such that

$$\lim_{n\to+\infty} y_n = \lim_{n\to+\infty} fx_{n+1} = \lim_{n\to+\infty} Tx_n = w. \quad (2.10)$$

We can find an element y in X such that $fy = w$. From (2.3), we get

$$\begin{aligned}&\psi(d(Tx_n, Ty))\\ &\leq \mathcal{F}\left(\alpha(d(fx_n, fy)), \beta(d(fx_n, fy))\right)\end{aligned}$$

Since $y_{n-1} = fx_n \to fy$ and $\mathcal{F}(0,0) = 0$, letting $n \to \infty$ in the above inequality, we obtain that $\lim_{n\to+\infty} \psi(d(Tx_n, Ty)) = 0$ implies that $y_n = Tx_n \to Ty$. Then, by Lemma 1, we obtain

$$w = fy = Ty. \quad (2.11)$$

Therefore, w is a point of coincidence of T and f. The uniqueness of the point of coincidence is a consequence of the condition (2.3) and so we omit the details.

Since T and f are weakly compatible, by (2.11), we have

$$Tw = Tfy = fTy = fw, \quad (2.12)$$

and so $Tw = fw$. Uniqueness of the point of coincidence implies $w = fw = Tw$. Consequently, w is a unique common fixed point of T and f.■

If we take $f = I_x$ in Theorem 1, we have the following corollary.

Corollary 1. Let (X, d) be a complete g.m.s. and let $T: X \to X$ be self-mapping such that

$$\psi(d(Tx, Ty)) \leq \mathcal{F}(\alpha(d(x, y)), \beta(d(x, y))), \quad (2.13)$$

for all $x, y \in X$, where $\psi \in \Psi$, $\alpha, \beta \in \Phi$ and $\mathcal{F} \in \mathcal{C}$, and satisfying condition (2.2). Then T have a unique fixed point in X.

Example 2. Let $X = \{1,2,3,4\}$ and define $d: X \times X \to \mathbb{R}^+$ as follows:

$$d(1,2) = d(2,1) = 1.3, \quad d(2,3) = d(3,2) = 0.7,$$
$$d(1,3) = d(3,1) = 0.2, \quad d(2,4) = d(4,2) = 1.1,$$
$$d(1,4) = d(4,1) = 0.4, \quad d(3,4) = d(4,3) = 0.8,$$
$$d(1,1) = d(2,2) = d(3,3) = d(4,4) = 0.$$

Then (X,d) is a complete generalized metric space, but it is not a metric space. Indeed,

$$1.3 = d(1,2) \nleq \quad d(1,3) + d(3,2) = 0.9.$$

Let $T: X \to X$ be defined by

$$Tx = \begin{cases} 2, & x \in \{1,2,3\}, \\ 1, & x = 4. \end{cases}$$

Also, we define the mappings $\mathcal{F}: \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by $\mathcal{F}(s,t) = \frac{s}{1+t}$ and $\psi, \alpha, \beta: \mathbb{R}^+ \to \mathbb{R}^+$ by $\psi(t) = t$, $\alpha(t) = \frac{t}{2}$ and $\beta(t) = \frac{t}{4}$. We next verify that the mapping T satisfies the inequality (2.13). For that, given $x, y \in X$, we have the following cases:

Case 1. If $x, y \in \{1,2,3\}$, then $d(Tx,Ty) = d(2,2) = 0$ and hence (2.13) trivially holds.

Case 2. If $x \in \{1,2,3\}$, $y = 4$, then $d(Tx,Ty) = d(2,1) = 1.3$.

If $x = 1$, then

$$\psi\big(d(Tx,Ty)\big) - \mathcal{F}\left(\alpha\big(d(x,y)\big), \beta\big(d(x,y)\big)\right)$$
$$= 1.3 - \frac{\frac{1}{5}}{1 + \frac{1}{10}} > 0.$$

If $x = 2$, then

$$\psi\big(d(Tx,Ty)\big) - \mathcal{F}\left(\alpha\big(d(x,y)\big), \beta\big(d(x,y)\big)\right)$$
$$= 1.3 - \frac{\frac{11}{20}}{1 + \frac{11}{40}} > 0.$$

If $x = 3$, then

$$\psi\big(d(Tx,Ty)\big) - \mathcal{F}\left(\alpha\big(d(x,y)\big), \beta\big(d(x,y)\big)\right)$$
$$= 1.3 - \frac{\frac{2}{5}}{1 + \frac{1}{5}} > 0.$$

Hence, the inequality (2.13) is satisfied in this case.

Case 3. If $x = 4$, $y \in \{1,2,3\}$, then, since d is symmetric, so the inequality (2.13) is satisfied from Case 2.

Therefore, since all the hypotheses of Corollary 1 is satisfied, then T has a unique fixed point. Here 2 is the unique fixed point of T.■

If we take $\mathcal{F}(s,t) = ks$ in Theorem 1, we have the following corollary.

Corollary 2. Let (X,d) be a complete g.m.s. and let $T, f: X \to X$ be self-mappings such that $TX \subseteq fX$, and fX is a closed subspace of X, and following condition holds:

$$\psi\big(d(Tx,Ty)\big) \le k\alpha\big(d(fx,fy)\big), \tag{2.14}$$

for all $x, y \in X$, where $k \in (0,1)$, $\psi \in \Psi$ and $\alpha \in \Phi$. Then T and f have a unique point of coincidence in X. Moreover, if T and f are weakly compatible, then T and f have a unique common fixed point.

If we take $\mathcal{F}(s,t) = s - t$ in Theorem 1, we have the following corollary.

Corollary 3 ([**2**]). Let (X,d) be a complete g.m.s. and let $T, f: X \to X$ be self-mappings such that $TX \subseteq fX$, and fX is a closed subspace of X, and following condition holds:

$$\psi\big(d(Tx,Ty)\big) \le \alpha\big(d(fx,fy)\big) - \beta(d(fx,fy))), \tag{2.15}$$

for all $x, y \in X$, where $\psi \in \Psi$ and $\alpha, \beta \in \Phi$, and satisfying condition $\psi(t) - \alpha(t) + \beta(t) > 0$ for all $t > 0$. Then T and f have a unique point of coincidence in X. Moreover, if T and f are weakly compatible, then T and f have a unique common fixed point.

If we take $\mathcal{F}(s,t) = \frac{s}{(1+t)^r}$ in Theorem 1, we have the following corollary.

Corollary 4. Let (X,d) be a complete g.m.s. and let $T, f: X \to X$ be self-mappings such that $TX \subseteq fX$, and fX is a closed subspace of X, and following condition holds:

$$\psi\big(d(Tx,Ty)\big) \le \frac{\alpha\big(d(fx,fy)\big)}{(1+\beta(d(fx,fy))))^r}, \tag{2.16}$$

for all $x, y \in X$, where r>0, $\psi \in \Psi$ and $\alpha, \beta \in \Phi$. Then T and f have a unique point of coincidence in X. Moreover, if T and f are weakly compatible, then T and f have a unique common fixed point.

If we take $\mathcal{F}(s,t) = s\log_{t+a} a$ in Theorem 1, we have the following corollary.

Corollary 5. Let (X,d) be a complete g.m.s. and let $T, f: X \to X$ be self-mappings such that $TX \subseteq fX$, and fX is a closed subspace of X, and following condition holds:

$$\psi\big(d(Tx,Ty)\big) \le \alpha\big(d(fx,fy)\big)\log_{\beta(d(fx,fy))+a} a, \tag{2.17}$$

for all $x, y \in X$, where $a > 1$, $\psi \in \Psi$ and $\alpha, \beta \in \Phi$. Then T and f have a unique point of coincidence in X. Moreover, if T and f are weakly compatible, then T and f have a unique common fixed point.

If we take $\mathcal{F}(s,t) = \log_a\left(\frac{t+a^s}{1+t}\right)$ in Theorem 1, we have the following corollary.

Corollary 6. Let (X,d) be a complete g.m.s. and let $T, f: X \to X$ be self-mappings such that $TX \subseteq fX$, and fX is a closed subspace of X, and following condition holds:

$$\psi\big(d(Tx,Ty)\big) \le \log_a \frac{\beta(d(fx,fy)) + a^{\alpha(d(fx,fy))}}{1 + \beta(d(fx,fy))}, \tag{2.18}$$

for all $x, y \in X$, where $a > 1$, $\psi \in \Psi$ and $\alpha, \beta \in \Phi$. Then T and f have a unique point of coincidence in X. Moreover, if T and f are weakly compatible, then T and f have a unique common fixed point.

If we take $\mathcal{F}(s,t) = (s+l)^{\frac{1}{(1+t)^r}} - l$ in Theorem 1, we have the following corollary.

Corollary 7. Let (X,d) be a complete g.m.s. and let $T,f: X \to X$ be self-mappings such that $TX \subseteq fX$, and fX is a closed subspace of X, and following condition holds:

$$\psi\big(d(Tx,Ty)\big) \le \big(\alpha\big(d(fx,fy)\big) + l\big)^{\frac{1}{(1+\beta(d(fx,fy)))^r}} - l, \quad (2.19)$$

for all $x,y \in X$, where $l > 1$, $r > 0$, $\psi \in \Psi$ and $\alpha,\beta \in \Phi$. Then T and f have a unique point of coincidence in X. Moreover, if T and f are weakly compatible, then T and f have a unique common fixed point.

3. APPLICATIONS

Denote by Λ the set of functions $\gamma: \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following hypotheses:

(h_1) γ is a Lebesgue-integrable mapping on each compact subset of $\mathbb{R}^+$.

(h_2) For every $\varepsilon > 0$, we have

$$\int_0^{\varepsilon} \gamma(s)\,ds > 0.$$

We have the following result.

Theorem 2. Let (X,d) be a complete g.m.s. and let $T,f: X \to X$ be self-mappings such that $TX \subseteq fX$, and fX is a closed subspace of X, and following condition holds:

$$\int_0^{d(Tx,Ty)} \gamma_1(s)\,ds \le \mathcal{F}\left(\int_0^{d(fx,fy)} \gamma_2(s)\,ds, \int_0^{d(fx,fy)} \gamma_3(s)\,ds\right), \quad (2.20)$$

for all $x,y \in X$, where $\gamma_1,\gamma_2,\gamma_3 \in \Lambda$ and $\mathcal{F} \in \mathcal{C}$, and satisfying condition (2.2). If T and f are weakly compatible, then T and f have a unique fixed point.

Proof. Follows from Theorem 1, by taking $\psi(t) = \int_0^t \gamma_1(s)\,ds$, $\alpha(t) = \int_0^t \gamma_2(s)\,ds$ and $\beta(t) = \int_0^t \gamma_3(s)\,ds$. ∎

Taking $\mathcal{F}(s,t) = ks$ in Theorem 2, we obtain the following result.

Corollary 8. Let (X,d) be a complete g.m.s. and let $T,f: X \to X$ be self-mappings such that $TX \subseteq fX$, and fX is a closed subspace of X, and following condition holds:

$$\int_0^{d(Tx,Ty)} \gamma_1(s)\,ds \le k \int_0^{d(fx,fy)} \gamma_2(s)\,ds, \quad (2.21)$$

for all $x,y \in X$, where $\gamma_1,\gamma_2,\gamma_3 \in \Lambda$ and $k \in (0,1)$. If T and f are weakly compatible, then T and f have a unique fixed point.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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