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ANALYTICAL SOLUTIONS OF (1+1)- DIMENSIONAL DISTRIBUTED LONG WAVE (DLW) EQUATION WITH AUXILIARY EQUATION METHOD

AHMET MUCAHID GOKTAS, KORAY YILMAZ, AND ORKUN TASBOZAN

0000-0001-7237-8027, 0000-0002-8641-0603 and 0000-0001-5003-6341

Abstract. In this article, the exact solutions of the (1+1)-dimensional distributed long wave(DLW) equation, a fractional partial differential equation in conformable sense, which is a nonlinear partial differential equation, are given. The analytical solutions revealed with the auxiliary equation method are also seen to satisfy the equation with the aid of the Mathematica program.

1. Introduction

Partial derivative differential equation models may not correspond exactly to events occurring in nature when they contain derivatives with integer order. Partial differential equations with fractional derivatives are more likely to correspond exactly to events in nature than equations with integer derivatives. The reason for this is that fractional order computation provides an easier expression with nonlinear partial differential equations for the parameters that occur during the physical event compared to integer computation.

There are several methods on analytical solutions of fractional partial differential equations for Conformable[10], Riemann-Liouville and Caputo fractional derivative approaches. Some of these methods are tangent hyperbolic method[11], explicit equation method[7-8], homotopy analysis method[1], elliptic function method[6], F-expansion method[3,5,13], auxiliary equation method[4], new extended direct algebraic method[12]. In this article, analytical solutions of the (1+1)-dimensional distributed long wave (DLW) equation are examined which is a fractional partial differential equation. Analytical solutions of the equation were obtained by using the auxiliary equation method.

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2. Preliminaries

The auxiliary equation method was introduced by Jiong and Sirendaoreji[9] to find analytical solutions of partial differential equations. The work to find analytical solutions of partial differential equations by Jiong and Sirendaoreji[9] can be given as follows. They used the ordinary differential equation of the form

$$\frac{dz}{d\xi} = \sqrt{az^2(\xi) + bz^4(\xi) + cz^6} \tag{2.1}$$

to find partial differential equations. In another study conducted in 2008,

$$(\frac{dz}{d\xi})^2 = az^2 + bz^3 + cz^4 \tag{2.2}$$

With the help of the ordinary differential equation, analytical solutions have been obtained for the generalized Whitham-Broer-Kaup, Zakharov, Schrodinger equations, which are nonlinear partial differential equations [2]. The ordinary differential equation given by (2.2) contains more analytical solutions than the ordinary differential equation given by (2.1).

We give the application of the auxiliary equation method to any partial differential equation below from [2]. The general form of a nonlinear partial differential equation can be written as

$$Q(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, ...) = 0 \tag{2.3}$$

Using the equation given by (2.3), applying the wave transform

$$\xi = x + mt \tag{2.4}$$

we get a differential equation

$$G(U, U_\xi, U_{\xi\xi}, U_{\xi\xi\xi}, ...) = 0 \tag{2.5}$$

where $u(x,t) = U(\xi)$and m is the velocity of the wave. The integer value of n is found with the help of the derivative of the highest order and the nonlinear term from the highest order in the obtained equation. Using n in

$$U(\xi) = \sum_{i=0}^{n} a_i z^i(\xi) \tag{2.6}$$

the analytical solution of equation (2.5) is investigated in this form. Here, the coefficients $a_i (i = 0, 1, 2, ..., n)$ will be determined in the following sections. From the solutions of the differential equation given by (2.2), $z(\xi)$ can be obtained. The equation obtained contains the powers of $z(\xi)$. By arranging this equation according to the powers of the term $z(\xi)$, an equation system is obtained by equalizing the coefficients of the terms to zero. By solving this system of equations, the coefficients a, b, c, m, a_i are determined. After determining these coefficients, the analytical solutions of the partial differential equation are obtained with the help of Table 1.

3. Analytical Solutions of The Fractional Order (1+1) Dimensional DLW Equation

In this section, analytical solutions of the fractional order (1+1) dimensional DLW equation, which is a no linear fractional partial differential equation, are obtained using the auxiliary equation method. Conformable fractional order (1+1) dimensional DLW equation is given in the form

$$\begin{aligned} \frac{\partial^p u}{\partial t^p} + u\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} &= 0 \\ \frac{\partial^p v}{\partial t^p} + \frac{\partial u}{\partial x}.\frac{\partial v}{\partial x} + \frac{1}{3}\frac{\partial^3 u}{\partial x^3} &= 0 \end{aligned} \tag{3.1}$$

Let

$$u(x,t) = u(\xi),\ v(x,t) = v(\xi)$$

where m is the speed of wave. Applying wave transform

$$\xi = x + m\frac{t^p}{p}$$

to equation (2.7), the equation system

$$\begin{aligned} mu_\xi + uu_\xi + v_\xi &= 0 \\ v_\xi + (uv)_\xi + \tfrac{1}{3}u_{\xi\xi\xi} &= 0 \end{aligned} \tag{3.2}$$

is obtained. By making the necessary adjustments in equation (2.8), we get

$$\begin{aligned} 2mu + u^2 + 2v &= 0 \\ 3mv + uv + u_{\xi\xi} &= 0 \end{aligned} \tag{3.3}$$

From the first equation in the system (2.9) the value of v is

$$v = -mu - \frac{1}{2}u^2 \tag{3.4}$$

By substituting this value in the second equation in (2.9) we get

$$-6m^2u - 9mu^2 - 3u^3 + 2u_{\xi\xi} = 0 \tag{3.5}$$

From the relation between the highest order derivative term $u_{\xi\xi}$ and the highest order nonlinear term u^3 in the ordinary derivative differential equation in (2.11), $n = 1$ is obtained. In this case, the analytical solution of equation (2.11) is investigated as

$$u(\xi) = a_0 + a_1 z(\xi) \tag{3.6}$$

The equation system

$$\begin{aligned} z^0(\xi): &\quad -6m^2a_0 - 9ma_0^2 - 3a_0^3 \\ z^1(\xi): &\quad -6m^2a_1 - 18ma_0a_1 - 9a_0^2a_1 + 2aa_1 \\ z^2(\xi): &\quad -9ma_1^2 - 9a_0a_1^2 + 3a_1b \\ z^3(\xi): &\quad -3a_1^3 + 4a_1c \end{aligned}$$

is obtained by the coefficients of the powers of z(ξ) from the algebraic equation obtained by substituting the equation given by (2.12) in the differential equation (2.11). The solution sets of this system are

$$a_0 = 0,\ a = 3m^2,\ b = 3a_1m,\ c = \frac{3a_1^2}{4} \tag{3.7}$$

$$a_0 = -2m,\ a = 3m^2,\ b = -3a_1m,\ c = \frac{3a_1^2}{4} \tag{3.8}$$

$$a_0 = -m,\ a = \frac{-3m^2}{2},\ b = 0,\ c = \frac{3a_1^2}{4} \tag{3.9}$$

By substituting the solution set given in (2.13) and the $z(\xi)$ values given in Table 1 in the equations (2.12) and (2.10), the analytical solutions of the DLW equation given by (2.7) are as follows:

$$\begin{aligned}
u_{1,2}(x,t) &= -\tfrac{4m\sec h^2(\omega)}{4-(1\pm\tanh(\omega))^2} \\
v_{1,2}(x,t) &= \tfrac{-8m\sec h^4(\omega)}{(4-(1\pm\tanh(\omega))^2)^2} + \tfrac{4m\sec h^2(\omega)}{4-(1\pm\tanh(\omega))^2} \\
u_{3,4}(x,t) &= \tfrac{4m\csc h^2(\omega)}{4-(1\pm\coth(\omega))^2} \\
v_{3,4}(x,t) &= \tfrac{-8m\csc h^4(\omega)}{(4-(1\pm\coth(\omega))^2)^2} - \tfrac{4m\csc h^2(\omega)}{4-(1\pm\coth(\omega))^2} \\
u_{5,6}(x,t) &= -\tfrac{m\sec h^2(\omega)}{1\pm\tanh(\omega)} \\
v_{5,6}(x,t) &= \tfrac{m^2\sec h^2(\omega)}{1\pm\tanh(\omega)} - \tfrac{m^2\sec h^4(\omega)}{2(1\pm\tanh(\omega))^2} \\
u_{7,8}(x,t) &= \tfrac{m\csc h^2(\omega)}{1\pm\coth(\omega)} \\
v_{7,8}(x,t) &= -\tfrac{m^2\csc h^2(\omega)}{1\pm\coth(\omega)} - \tfrac{m^2\csc h^4(\omega)}{2(1\pm\coth(\omega))^2} \\
u_{9,10}(x,t) &= -m(1\pm\tanh(\omega)) \\
v_{9,10}(x,t) &= m^2(1\pm\tanh(\omega)) - \tfrac{1}{2}m^2(1\pm\tanh(\omega))^2 \\
u_{11,12}(x,t) &= -m(1\pm\coth(\omega)) \\
v_{11,12}(x,t) &= m^2(1\pm\coth(\omega)) - \tfrac{1}{2}m^2(1\pm\coth(\omega))^2 \\
u_{13,14}(x,t) &= \tfrac{12a_1m^2e^{\pm2\omega}}{-9a_1m^2+(e^{\pm2\omega}-3a_1m)^2} \\
v_{13,14}(x,t) &= -\tfrac{12a_1m^3e^{\pm2\omega}}{-9a_1m^2+(e^{\pm2\omega}-3a_1m)^2} - \tfrac{12a_1m^4e^{\pm2\omega}}{(-9a_1m^2+(e^{\pm2\omega}-3a_1m)^2)^2}
\end{aligned}$$

where

$$\omega = \frac{\sqrt{3}m}{2}(x + \frac{mt^p}{p})$$

Similarly, using the solution given by (2.14), the analytical solutions $u(x,t)$ of the DLW equation given by (2.7) are obtained as:

$$\begin{aligned}
u_{15,16}(x,t) &= -2m + \tfrac{4m\sec h^2(\omega)}{4-(1\pm\tanh(\omega))^2} \\
u_{17,18}(x,t) &= -2m - \tfrac{4m\csc h^2(\omega)}{4-(1\pm\coth(\omega))^2} \\
u_{19,20}(x,t) &= -2m + \tfrac{m\sec h^2(\omega)}{1\pm\tanh(\omega)} \\
u_{21,22}(x,t) &= -2m - \tfrac{m\csc h^2(\omega)}{1\pm\coth(\omega)} \\
u_{23,24}(x,t) &= -2m + m(1\pm\tanh(\omega)) \\
u_{25,26}(x,t) &= -2m + m(1\pm\coth(\omega)) \\
u_{27,28}(x,t) &= -2m + \tfrac{12a_1m^2e^{\pm2\omega}}{-9a_1m^2+(e^{\pm2\omega}-3a_1m)^2}
\end{aligned}$$

By using analytical solutions $u(x,t)$ in the equation given by (2.10), analytical solutions $v(x,t)$ are obtained. From the solution set given by (2.15),

$$u_{29,30}(x,t) = -m + \frac{m\cosh(\frac{\omega}{2}\sec(\frac{\omega}{2}))}{\sqrt{2}}$$

the analytical solution is obtained. Similarly, by substituting the obtained analytical solution in the equation given by (2.10), the analytical solution $v(x,t)$ is obtained. The analytical solution

$$u_{29,30}(x,t) = -m + \frac{m\cosh(\frac{\omega}{2}\sec(\frac{\omega}{2}))}{\sqrt{2}}$$

is obtained from the solution set given in (2.15). Using the analytical solution obtained in (2.10), the analytical solution $v(x,t)$ is obtained

Table 1. [2] analytical solutions of differential equation in (2.2) where $\epsilon = \pm 1$ and $\Delta = b^2 - 4ac$.

No	$\mathbf{z}(\xi)$
1	$\frac{-ab\sec h^2(\frac{\sqrt{a}}{2}\xi)}{b^2-ac(1-\epsilon\tanh(\frac{\sqrt{a}}{2}\xi))^2},\ a>0$
2	$\frac{ab\csc h^2(\frac{\sqrt{a}}{2}\xi)}{b^2-ac(1+\epsilon\coth(\frac{\sqrt{a}}{2}\xi))^2},\ a>0$
3	$\frac{2ab\sec h(\sqrt{a}\xi)}{\epsilon\sqrt{\Delta}-b\sec h(\sqrt{a}\xi)},\ a>0,\ \Delta>0$
4	$\frac{2ab\sec(\sqrt{-a}\xi)}{\epsilon\sqrt{\Delta}-b\sec(\sqrt{-a}\xi)},\ a<0,\ \Delta>0$
5	$\frac{2ab\csc h(\sqrt{a}\xi)}{\epsilon\sqrt{-\Delta}-b\csc h(\sqrt{a}\xi)},\ a>0,\ \Delta<0$
6	$\frac{2ab\csc(\sqrt{-a}\xi)}{\epsilon\sqrt{\Delta}-b\csc(\sqrt{-a}\xi)},\ a<0,\ \Delta>0$
7	$\frac{-a\sec h^2(\frac{\sqrt{a}}{2}\xi)}{b+2\epsilon\sqrt{ac}\tanh(\frac{\sqrt{a}}{2})},\ a>0,\ c>0$
8	$\frac{-a\sec^2(\frac{\sqrt{-a}}{2}\xi)}{b+2\epsilon\sqrt{-ac}\tanh(\frac{\sqrt{-a}}{2})},\ a<0,\ c>0$
9	$\frac{a\csc h^2(\frac{\sqrt{a}}{2}\xi)}{b+2\epsilon\sqrt{ac}\coth(\frac{\sqrt{a}}{2})},\ a>0,\ c>0$
10	$\frac{-a\csc^2(\frac{\sqrt{-a}}{2}\xi)}{b+2\epsilon\sqrt{-ac}\cot(\frac{\sqrt{-a}}{2})},\ a<0,\ c>0$
11	$-\frac{a}{b}(1+\epsilon\tanh(\frac{\sqrt{a}}{2})\xi),\ a>0,\ \Delta=0$
12	$-\frac{a}{b}(1+\epsilon\coth(\frac{\sqrt{a}}{2})\xi),\ a>0,\ \Delta=0$
13	$\frac{4ae^{\epsilon\sqrt{a}\xi}}{(e^{\epsilon\sqrt{a}\xi}-b)^2-4ac},\ a>0$
14	$\frac{\pm 4ae^{\epsilon\sqrt{a}\xi}}{1-4ace^{2\epsilon\sqrt{a}\xi}},\ a>0,\ b=0$

4. Conclusion

In this study, exact solutions for the conformable fractional order nonlinear DLW equation are obtained by using the auxiliary equation method.

The considered equation is converted into a nonlinear differential equation with the help of wave transform. Then, the analytical solutions are obtained by writing the power of the $z(\xi)$ terms instead of the obtained differential equation. It is seen with the help of Mathematica program that the obtained analytical solutions provided the conformable fractional order nonlinear DLW equation.

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The author(s) declared that they comply with the scientific, ethical, and citation rules of Journal of Universal Mathematics in all processes of the study and that they do not make any falsification on the data collected. Besides, the author(s) declared that Journal of Universal Mathematics and its editorial board have no responsibility for any ethical violations that may be encountered and this study has not been evaluated in any academic publication environment other than Journal of Universal Mathematics.

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(Ahmet Mucahid Goktas) Department of Mathematics, Faculty of Science and Arts, Kutahya Dumlupinar University, Kutahya, Turkey
Email address: a.m.goktas.ist@gmail.com

(Koray Yilmaz) Department of Mathematics, Faculty of Science and Arts, Kutahya Dumlupinar University, Kutahya, Turkey
Email address: koray.yilmaz@dpu.edu.tr

(Orkun Tasbozan) Department of Mathematics, Faculty of Science and Arts, Hatay Mustafa Kemal University, Hatay, Turkey
Email address: otasbozan@mku.edu.tr