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On Quasi 2-Crossed Modules for Lie Algebras and Functorial Relations

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Abstract — In this paper, we have introduced the category of quasi 2-crossed modules for Lie algebras and we have constructed a pair of adjoint functors between this category and that of 2-crossed modules Lie algebras.

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1. Introduction

Crossed modules have been introduced for Lie algebras by Kassel and Loday in [6], as well as they initially originate in Whitehead's work for groups,[10]. It is known that the notion of crossed modules modelling homotopy 2-type has become an important tool in various contexts. Some of related works with crossed modules of Lie algebras are [2], [8], and [9]. The notion of 2-crossed modules of groups based on that of crossed modules has been introduced by Conduche [3] as an algebraic models of homotopy 3-types. In [5], Ellis has also presented the Lie algebra version of that for getting the equivalence between the category of 2-crossed modules and that of simplicial Lie algebras with Moore complex of length 2. Akça and Arvasi apply higher order Peiffer elements in simplicial Lie algebras to the Lie 2-crossed module in [1].

In this paper, we invented the concept of quasi 2-crossed modules of Lie algebras. In [4], Carrasco and Porter have mentioned this notion for group cases. We have also intend to use it to work on functorial relations, similar to how algebraic models of homotopy 2-types are used. We will see that the roles of quasi 2-crossed modules in Lie algebras and those of pre-crossed modules are similar except dimensionally.

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2. Preliminaries

2.1. Crossed modules of Lie algebras

If Y and Z are two Lie algebras, then a left Lie algebra action of Z on Y is a k-bilinear map

$$\begin{array}{ccc} Z \times Y & \longrightarrow & Y \\ (z,y) & \longmapsto & z * y, \end{array}$$

that satisfies the following two axioms:

L1) $z * [y, y'] = [z * y, y'] + [y, z * y']$,

L2) $[z, z'] * y = z * (z' * y) - z' * (z * y)$

for each $z, z' \in Z$ and each $y, y' \in Y$.

A pre-crossed module over Lie algebras (Y, Z, ∂) is given by a Lie homomorphism $\partial : Y \to Z$, together with a left Lie algebra action of Z on Y such that the condition

XMod$_L$1 $\partial(z * y) = [z, \partial(y)]$ is satisfied for each $z \in Z$ and each $y \in Y$.

A crossed module over Lie algebras (Y, Z, ∂) is a pre-crossed module satisfying, in addition "Peiffer identity" condition:

XMod$_L$2 $\partial(y) * y' = [y, y']$

for all $y, y' \in Y$.

Example 2.1. An inclusion map $i : I \longrightarrow Z$ is a crossed module where I is any ideal of a Lie algebra Z. Conversely given any crossed module $\partial : I \to Z$, one can easily verify that $\partial(Y) = I$ is an ideal in Z.

Example 2.2. Any Z-module Y can be considered as a Lie algebra with zero multiplication, and then $\mathbf{0} : Y \to Z$ is a crossed module by $\mathbf{0}(y) * y' = 0y' = [y, y']$ and $\mathbf{0}(z * y) = 0 = [z, \mathbf{0}(y)]$, for all $y, y' \in Y, z \in Z$.

Example 2.3. A Lie k-algebra morphism

$$\begin{array}{ccccccc} \mu : & S & \to & Der(S) & & & \\ & s & \mapsto & \mu(s) = \mu_s : & S & \to & S \\ & & & & s' & \mapsto & \mu_s(s') = [s, s'] \end{array}$$

with the action of Der(S) on S given as

$$\begin{array}{ccc} Der(S) \times S & \to & S \\ (d, s) & \mapsto & d * s = d(s) \end{array}$$

is a crossed module where $Der(S)$ is a set of derivations of S,i.e.

$$Der(S) = \{d \mid d : S \to S,\ d([s_1, s_2]) = [s_1, ds_2] + [ds_1, s_2]\ s_1, s_2 \in S\}.$$

(See for detail [7].)

A crossed module morphism $f : (Y, Z, \partial) \to (Y', Z', \partial')$ is a pair $(f_1 : Y \to Y', f_0 : Z \to Z')$ of Lie algebra morphisms, making the diagram below commutative:

$$\begin{array}{ccc} Y & \xrightarrow{\partial} & Z \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_0} \\ Y' & \xrightarrow[\partial']{} & Z' \end{array}$$

also preserving action of Z on Y.

Although the following discussion may be found in various algebraic cases, we include it here since we will need to generalize it later.

If $\partial : M \to P$ is a pre-crossed module of Lie algebras then $\overline{\partial} : M/\overline{M} \to P$ given by $\overline{\partial}([m]) = \partial(m)$ is a crossed module where $\overline{M}$ is the ideal generated by the elements $[m, m'] - \partial(m) * m'$, for $m, m' \in M$. It is not difficult to see that following equations are satisfied

$$\overline{\partial}([m]) * [m'] = \partial(m) * [m'] = [\partial(m) * m'] = [[m, m']] = [[m], [m']]$$

$$\overline{\partial}(p * [m]) = \overline{\partial}([p * m]) = \partial(p * m) = [p, \partial(m)].$$

For any pre-crossed module morphism $(f_1, f_0) : (M, P, \partial) \to (M', P', \partial')$, we get the crossed module morphism $(\overline{f}_1, f_0) : (M/\overline{M}, P, \partial) \to (M'/\overline{M}', P', \partial')$, where $\overline{f}_1([m]) = [f_1(m)], m \in M$. Since

$$\begin{aligned} f_1([m, m'] - \partial(m) * m') &= f_1([m, m']) - f_1(\partial(m) * m') \\ &= [f_1(m), f_1(m')] - f_0(\partial(m)) * f_1(m') \\ &= [f_1(m), f_1(m')] - \partial'(f_1(m)) * f_1(m') \in \overline{M}' \end{aligned}$$

$\overline{f}_1$ is well-defined morphism. Thus, it can be given a functor

$$F : PXMOD \to XMOD$$

defined as $F((M, P, \partial)) = (M'/\overline{M}', P', \partial')$ on object and as $F((f_1, f_0)) = (\overline{f}_1, f_0)$ on morphism.

Furthermore, it is clear that there is forgetful functor $G : XMOD \to PXMOD$ and the functor F is left adjoint to G.

2.2. 2-Crossed Modules of Lie algebras

In this section, we recall the definition of 2-crossed modules over Lie algebras given [5].

A pair of Lie homomorphisms $L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} P$ with an action of P on M and L, and a bilinear function $\{,\} : M \times M \to L$ such that below axioms are satisfied for every $m, m', m'' \in M$, $l, l' \in L$ and $p \in P$ are defined as a 2-crossed module of Lie algebras

1. $\partial_1 \partial_2 = 0$
2. $\partial_2({}^p l) = {}^p(\partial_2 l)$, $\partial_1({}^p m) = [p, \partial_1(m)]$

3. $\partial_2\{m,m^{'}\} = {}^{(\partial_1 m)}m^{'} - [m,m^{'}]$
4. $\{\partial_2 l, \partial_2 l^{'}\} = [l,l^{'}]$
5. $\{\partial_2 l, m\} + \{m, \partial_2 l\} = {}^{\partial_1 m} l$
6. ${}^{p}\{m,m^{'}\} = \{{}^{p}m, m^{'}\} + \{m, {}^{p}m^{'}\}$
7. $\{[m,m^{'}],m^{''}\} = {}^{\partial_1 m}\{m^{'},m^{''}\} + \{m,[m^{'},m^{''}]\} - {}^{\partial_1 m^{'}}\{m,m^{'}\} - \{m^{'},[m,m^{''}]\}$
8. $\{m,[m^{'},m^{''}]\} = {}^{\partial_1 m^{'}}\{m,m^{''}\} - {}^{\partial_1 m^{''}}\{m,m^{'}\} - \{m^{'}, {}^{\partial_1 m}m^{''} - [m,m^{''}]\} + \{m^{''}, {}^{\partial_1 m}m^{'} - [m,m^{'}]\}$

It is denoted by $(L,M,P,\partial_2,\partial_1,\{,\})$. If the below diagram is commutative

$$\begin{array}{ccccccc} M\times M & \xrightarrow{\{,\}} & L & \xrightarrow{\partial_2} & M & \xrightarrow{\partial_1} & P \\ \downarrow{\scriptstyle f_1\times f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} \\ M'\times M' & \xrightarrow[\{,\}^{'}]{} & L' & \xrightarrow[\partial_2']{} & M' & \xrightarrow[\partial_1']{} & P' \end{array}$$

that is, the equations

$$\partial_1{}^{'} f_1 = f_0\partial_1$$

$$\partial_2{}^{'} f_2 = f_1\partial_2,$$

$$f_2\{,\} = \{,\}^{'}(f_1,f_1)$$

are satisfied and

$$f_1({}^{p}m) = {}^{f_0(p)}f_1(m)$$

$$f_2({}^{p}l) = {}^{f_0(p)}f_2(l)$$

then a triple (f_2,f_1,f_0) is called by the morphism of between 2-crossed modules $(L,M,P,\partial_2,\partial_1,\{,\})$ and $(L^{'},M^{'},P',\partial_2',\partial_1',\{,\}')$.

As a result, the category of 2-crossed modules is obtained, with 2-crossed modules as objects and morphisms between them as morphisms and it is denoted by L2XMOD.

When the morphisms f_1 and f_0 above are the identity, we will get a subcategory $L2XMOD/(M,P)$, the category of 2-crossed modules, over fixed pre-crossed module $\partial_1: M \to P$.

2.3. Quasi 2-Crossed Modules of Lie Algebras

A quasi 2-crossed module of Lie algebras is a sequence $L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} P$ of Lie algebra homomorphisms together with a bilinear map $\{,\}: M\times M \to L$ satisfying the below axioms

LQ2X1) $\partial_1\partial_2 = 0$

LQ2X2) $\partial_2({}^{p}l) = {}^{p}(\partial_2 l)$, $\partial_1({}^{p}m) = [p,\partial_1(m)]$

LQ2X3) ${}^{p}\{m_0,m_1\} = \{{}^{p}m_0, m_1\} + \{m_0, {}^{p}m_1\}$

LQ2X4) $\partial_2\{m_0,m_1\} = {}^{\partial_1 m_0}m_1 - [m_0,m_1]$

LQ2X5) $\{m_0,[m_1,m_2]\} = {}^{\partial_1 m_1}\{m_0,m_2\} - {}^{\partial_1 m_2}\{m_0,m_1\} - \{m_1, {}^{\partial_1}m_2 - [m_0,m_2]\}$
$+\{m_2, {}^{\partial_1 m_0}m_1 - [m_0,m_1]\}$

LQ2X6) $\{[m_0,m_1],m_2\} = {}^{\partial_1 m_0}\{m_1,m_2\} + \{m_0,[m_1,m_2]\} - {}^{\partial_1 m_1}\{m_1,m_2\}$
$-\{m_1,[m_0,m_2]\}$

LQ2X7) $[\{m_0,m_1\}, {}^{\partial_1 m_0}(m_1 \lhd l)] = \{{}^{\partial_1 m_0}[m_1,\partial_2 l], \partial_2\{m_0,m_1\}\}$

for all $m, m_0, m_1, m_2 \in M$ and $l \in L$. Also, the action on L of M is given by

$$m \lhd l = {}^{\partial_1 m}l - \{m, \partial_2 l\} = \{\partial_2 l, m\}.$$

We get the category $LQ2XMOD$ quasi 2-crossed modules of Lie algebras by defining whose morphisms similar to that of L2XMOD and it is obtained a subcategory $L2XMOD/(M,P)$ with base ÃÂ $\partial_1 : M \to P$, [7].

Proposition 2.4. Every 2-crossed module is a quasi 2-crossed module.

Proof.
Let $(L, M, P, \partial_2, \partial_1, \{,\})$ be a 2-crossed module. To complete the proof, just axiom 7 has to be verified.

$$\begin{aligned}
[\{m_0,m_1\}, {}^{\partial m_0}(m_1 \lhd l)] &= \partial_2\{m_0,m_1\} \lhd {}^{\partial m_0}\{\partial_2 l, m\} \\
&= \partial_2\{m_0,m_1\} \lhd \{{}^{\partial m_0}\partial_2 l, m\} + \{\partial_2 l, {}^{\partial m_0}m_1\} \\
&= \partial_2\{m_0,m_1\} \lhd \{{}^{\partial m_0}\partial_2 l, m\} + \partial_2\{m_0,m_1\} \lhd \{\partial_2 l, {}^{\partial m_0}m_1\} \\
&= \left\{\partial_2\left(\{{}^{\partial m_0}\partial_2 l, m\}\right), \partial_2\{m_0,m_1\}\right\} + \\
&\quad \left\{\partial_2\left(\{\partial_2 l, {}^{\partial m_0}m_1\}\right), \partial_2\{m_0,m_1\}\right\} \\
&= \left\{{}^{(\partial_1({}^{\partial m_0}\partial_2 l))}m - [{}^{\partial m_0}\partial_2 l, m]\right\} + \\
&\quad \left\{{}^{(\partial_1(\partial_2 l))}({}^{\partial m_0}m_1) - [\partial_2 l, {}^{\partial m_0}m_1]\right\} \\
&= \left\{{}^{[\partial m_0, \partial_1(\partial_2 l)]}m - [{}^{\partial m_0}\partial_2 l, m], {}^{\partial m_0}m_1 - [m_0,m_1]\right\} + \\
&\quad \{0 - [\partial_2 l, {}^{\partial m_0}m_1], {}^{\partial m_0}m_1 - [m_0,m_1]\} \\
&= \{-[{}^{\partial m_0}\partial_2 l, m], {}^{\partial m_0}m_1 - [m_0,m_1]\} + \\
&\quad \{-[\partial_2 l, {}^{\partial m_0}m_1], {}^{\partial m_0}m_1 - [m_0,m_1]\} \\
&= \{[m_1, {}^{\partial m_0}\partial_2 l] + [{}^{\partial m_0}m_1, \partial_2 l], \partial_2\{m_0,m_1\}\} \\
&= \{{}^{\partial m_0}[m_1,\partial_2 l], \partial_2\{m_0,m_1\}\}
\end{aligned}$$

for all $m, m_0, m_1 \in M$ and $l \in L$.

Proposition 2.5. If $(L, M, P, \partial_2, \partial_1, \{,\})$ is a Lie quasi 2-crossed module, then ideal $\overline{L}$ generated by the elements of the type

$$m * l = {}^{\partial_1 m}l - \{m, \partial_2 l\} - \{\partial_2 l, m\}$$

$$l \circledast l' = [l, l'] - \{\partial_2 l, \partial_2 l'\}$$

is a P-invariant ideal in Lie algebra L, for all $l, l' \in L$, $m \in M$.

Proof.

$$
\begin{aligned}
{}^{p}(m*l) &= {}^{p}({}^{\partial_1 m}l-\{m,\partial_2 l\}-\{\partial_2 l,m\}) \\
&= {}^{p}({}^{\partial_1 m}l)-{}^{p}\{m,\partial_2 l\}-{}^{p}\{\partial_2 l,m\} \\
&= \{p,\partial_1 m\}\cdot l+(\partial_1 m)\cdot(p\cdot l)-\{{}^{p}m,\partial_2 l\}-\{m,\partial_2({}^{p}l)\}-\{\partial_2({}^{p}l),m\}-\{\partial_2 l,{}^{p}m\} \\
&= m*{}^{p}l+{}^{\partial_1({}^{p}m)}l-\{p_m,\partial_2 l\}-\{\partial_2 l,{}^{p}m\} \\
&= m*{}^{p}l+({}^{p}m*l)\in\overline{L}
\end{aligned}
$$

for $p\in P$, $m\in M$, $l\in L$, and we also get

$$
\begin{aligned}
{}^{p}(l\circledast l') &= {}^{p}([l,l']-\{\partial_2 l,\partial_2 l'\}) \\
&= [l,{}^{p}l']-\{{}^{p}\partial_2 l,\partial_2 l'\}-\{\partial_2 l,{}^{p}\partial_2 l'\} \\
&= [{}^{p}l,l']+[l,{}^{p}l']-\{\partial_2({}^{p}l),\partial_2 l'\}-\{\partial_2 l,\partial_2({}^{p}l')\} \\
&= [{}^{p}l,l']-\{\partial_2({}^{p}l),\partial_2 l'\}+[l,{}^{p}l']-\{\partial_2 l,\partial_2({}^{p}l')\} \\
&= ({}^{p}l\circledast l')+(l\circledast {}^{p}l')\in\overline{L}
\end{aligned}
$$

for $p\in P$, $l,l'\in L$

Theorem 2.6. Let $(L,M,P,\partial_2,\partial_1,\{,\})$ be a Lie quasi 2-crossed module and $\overline{L}$ be as in previous proposition. Then $(L/\overline{L},M,P,\overline{\partial},\partial_1,\overline{\{,\}})$ is a 2-crossed module where $\overline{\partial}:L/\overline{L}\longrightarrow M$, is given by $\overline{\partial}(l+\overline{L})=\partial_2 l$ and $\overline{\{,\}}:M\times M\longrightarrow L/\overline{L}$ is defined by $\overline{\{,\}}(m_1,m_2)=\{m_1,m_2\}+\overline{L}$ for $l\in L$ and $m_1,m_2\in M$,respectively.

Proof.

$$
\begin{aligned}
\partial_2(m*l) &= \partial_2({}^{\partial_1 m}l-\{m,\partial_2 l\}-\{\partial_2 l,m\}) \\
&= \partial_2({}^{\partial_1 m}l)-\partial_2(\{m,\partial_2 l\})-\partial_2(\{\partial_2 l,m\}) \\
&= {}^{\partial_1 m}\partial_2 l-{}^{\partial_1 m}\partial_2 l+[m,\partial_2 l]-{}^{\partial_1(\partial_2 l)}m+[\partial_2 l,m] \\
&= 0+[m,\partial_2 l]-{}^{0}m-[m,\partial_2 l] \\
&= 0
\end{aligned}
$$

and

$$
\begin{aligned}
\partial_2(l\circledast l') &= \partial_2([l,l']-\{\partial l,\partial l'\}) \\
&= \partial_2([l,l'])-\partial_2(\{\partial l,\partial l'\}) \\
&= [\partial_2 l,\partial_2 l']-{}^{\partial_1(\partial_2 l)}\partial_2 l'-[\partial_2 l,\partial_2 l'] \\
&= 0
\end{aligned}
$$

for all $m\in M, l,l'\in L$, that is $\partial_2(\overline{L})=0$. Thus

$$
\begin{aligned}
\overline{\partial}:\quad L/\overline{L} &\rightarrow M \\
l+\overline{L} &\mapsto \overline{\partial}(l+\overline{L})=\partial_2 l
\end{aligned}
$$

is well-defined. It is seen that some of the axioms of the 2-crossed module are verified.

$$
\begin{aligned}
\overline{\{\overline{\partial}(l+\overline{L}),\overline{\partial}(l'+\overline{L})\}} &= \overline{\{\partial_2 l,\partial_2 l'\}} \\
&= \{\partial_2 l,\partial_2 l'\}+\overline{L} \\
&= [l,l']+\overline{L} \qquad (\because \{\partial_2 l,\partial_2 l'\}-[l,l']\in\overline{L})
\end{aligned}
$$

$$\begin{aligned}
\overline{\{\overline{\partial}(l+\overline{L}),m\}}+\overline{\{m,\overline{\partial}(l+\overline{L})\}} &= \overline{\{\partial_2 l,m\}}+\overline{\{m,\partial_2 l\}} \\
&= \{\partial_2 l,m\}+\overline{L}+\{m,\partial_2 l\}+\overline{L} \\
&= {}^{\partial_1 m}l+\overline{L} \\
&= {}^{\partial_1 m}(l+\overline{L})
\end{aligned}$$

$$\begin{aligned}
\overline{\partial}\,\overline{\{m_1,m_2\}} &= \overline{\partial}(\{m_1,m_2\}+\overline{L}) \\
&= \partial_2(\{m_1,m_2\}) \\
&= {}^{\partial_1 m_1}m_2-[m_1,m_2]
\end{aligned}$$

for all $m,m_1,m_2 \in M,\ l+\overline{L},l'+\overline{L} \in L/\overline{L}$. The validity of other axioms can be seen similarly. Therefore we have following result:

Corollary 2.7. There is (F,G) adjoint functor pair,

$$LQ2XMOD \underset{G}{\overset{F}{\rightleftarrows}} L2XMOD.$$

Proof.

Let $\mathscr{L}=(L,M,P,\partial_2,\partial_1,\{,\})$ and $\mathscr{L}'=(L',M',P',\partial_2',\partial_1',\{,\}')$ be two Lie quasi 2-crossed module and (f_2,f_1,f_0) be morphism between them. The functor

$$F: LQ2XMOD \to L2XMOD$$

is given by $F(\mathscr{L})=(L/\overline{L},M,P,\overline{\partial},\partial_1,\overline{\{,\}}),F(\mathscr{L}')=(L'/\overline{L'},M',P',\overline{\partial'},\partial_1',\overline{\{,\}'})$ and $F(f_2,f_1,f_0)=(f_2^*,f_1,f_0)$ where $f_2^*(l+\overline{L})=f_2(l)+\overline{L'}$. We have

$$\begin{aligned}
f_2(m*l) &= f_2({}^{\partial_1 m}l-\{m,\partial_2 l\}-\{\partial_2 l,m\}) \\
&= f_2({}^{\partial_1 m}l)-f_2(\{m,\partial_2 l\})-f_2(\{\partial_2 l,m\}) \\
&= {}^{f_0(\partial_1 m)}f_2(l)-\{f_1(m),f_1(\partial_2 l)\}'-\{f_1(\partial_2 l),f_1(m)\}' \\
&= {}^{\partial_1' f_1(m)}f_2(l)-\{f_1(m),\partial_2'(f_2(l))\}'-\{\partial_2'(f_2(l)),f_1(m)\}' \\
&= f_1(m)*f_2(l)\in\overline{L'}
\end{aligned}$$

and

$$\begin{aligned}
f_2(l_1\circledast l_2) &= f_2([l_1,l_2']-\{\partial_2 l,\partial_2 l'\}) \\
&= f_2[l_1,l_2]-f_2(\{\partial_2 l_1,\partial_2 l_2\}) \\
&= [f_2 l_1,f_2 l_2]-(f_2(\{\})(\partial_2 l_1,\partial_2 l_2)) \\
&= [f_2 l_1,f_2 l_2]-\{\}'(f_1,f_2)(\partial_2 l_1,\partial_2 l_2) \\
&= [f_2 l_1,f_2 l_2]-\{f_1\partial_2 l_1,f_1\partial_2 l_2\}' \\
&= [f_2 l_1,f_2 l_2]-\{\partial_2' f_2 l_1,\partial_2' f_2 l_2\}' \\
&= f_2(l_1)*f_2(l_2)\in\overline{L'}
\end{aligned}$$

for $m*l$ and $l_1\circledast l_2\in\overline{L}$, and so $f_2(\overline{L})\subseteq\overline{L'}$.

The morphism $f_2^*: L/\overline{L}\to L'/\overline{L'}$ given by $f_2^*(l+\overline{L})=f_2(l)+\overline{L'}$ is well-defined, since $f_2(l_1-l_2)\in f_2(\overline{L})\subseteq\overline{L'}$ for $l_1-l_2\in\overline{L}$.

We have

$$\begin{aligned}\overline{\partial_2'}(f_2^*(l+\overline{L})) &= \overline{\partial_2'}((f_2 l)+\overline{L'}) \\ &= \partial_2'(f_2 l) \\ &= f_1(\partial_2(l)) \\ &= f_1(\overline{\partial}(l+\overline{L}))\end{aligned}$$

and also

$$\partial_1' f_1 = f_0 \partial_1$$

since (f_2, f_1, f_0) is a morphism of quasi 2-crossed modules of Lie algebras. Therefore we get following commutative diagram:

$$\begin{array}{ccccc} L/\overline{L} & \xrightarrow{\overline{\partial}} & M & \xrightarrow{\partial_1} & P \\ \downarrow{\scriptstyle f_2^*} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} \\ L'/\overline{L'} & \xrightarrow[\overline{\partial}_2']{} & M' & \xrightarrow[\partial_1']{} & P' \end{array}$$

Furthermore we have below equations:

$$\begin{aligned} f_2^*\overline{\{,\}}(m_1, m_2) &= f_2^*(\{m_1, m_2\}+\overline{L}) \\ &= f_2(\{m_1, m_2\})+\overline{L} \\ &= \{f_1(m_1), f_1(m_2)\}' \\ &= \{,\}(f_1, f_1)(m_1, m_2)\end{aligned}$$

$$\begin{array}{ccccccc} M\times M & \xrightarrow{\{,\}} & L/\overline{L} & \xrightarrow{\overline{\partial}} & M & \xrightarrow{\partial_1} & P \\ \downarrow{\scriptstyle f_1\times f_1} & & \downarrow{\scriptstyle f_2^*} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} \\ M'\times M' & \xrightarrow[\{,\}']{} & L'/\overline{L'} & \xrightarrow[\overline{\partial}_2']{} & M' & \xrightarrow[\partial_1']{} & P' \end{array}$$

Thus (f_2^*, f_1, f_0) is a morphism of 2-crossed modules, as seen above.

For $\mathscr{K} = (K, N, Q, \partial_2', \partial_1', \{,\}')$ and $(f, f_1, f_0) : F(\mathscr{L}) \to \mathscr{K} \in Mor(L2XMOD)$, the morphism $(f q_{_L}, f_1, f_0) : \mathscr{L} \to \mathscr{K}$ is in $Mor(LQ2XMOD)$, where $q_{_L} : L \to L/\overline{L}$. Conversely, for $(f_2, f_1, f_0) : \mathscr{L} \to G(\mathscr{K}) \in Mor(LQ2XMOD)$,

$$(f_2^*, f_1, f_0) : (L/\overline{L}, M, P, \overline{\partial}, \partial, \overline{\{,\}}) \to (K, N, Q, \partial_2', \partial_1', \{,\}')$$

is a morphism in $Mor(L2XMOD)$. Thus, we get the bijection

$$L2XMOD(F(\mathscr{L}), \mathscr{K}) \cong LQ2XMOD(\mathscr{L}, G(\mathscr{K}))$$

such that this family of bijections is natural in $\mathscr{L}$ and $\mathscr{K}$. Clearly; for $h : (h_2, h_1, h_0) = \mathscr{L}' \to \mathscr{L} \in Mor(LQ2XMOD)$, we have following commutative diagram

$$\begin{array}{ccc} L2XMOD(F(\mathscr{L}), \mathscr{K}) & \xrightarrow{\eta_{\mathscr{L},\mathscr{K}}} & LQ2XMOD(\mathscr{L}, G(\mathscr{K})) \\ \downarrow{\scriptstyle F(h)^* = -\circ F(h)} & & \downarrow{\scriptstyle -\circ h = h^*} \\ L2XMOD(F(\mathscr{L}'), \mathscr{K}) & \xrightarrow[\eta_{\mathscr{L}',\mathscr{K}}]{} & LQ2XMOD(\mathscr{L}', G(\mathscr{K})) \end{array}$$

since

$$\begin{aligned} f_2 h_2^* q_{_L}(l') &= f_2 h_2^*(l' + \overline{L'}) \\ &= f_2(h_2(l') + \overline{L}) \\ &= f_2(q_{_L}(h_2(l'))), \end{aligned}$$

and for $k : (k_2, k_1, k_0) = \mathscr{K} \to \mathscr{K}' \in Mor(LX_2MOD)$, we get commutative diagram

$$\begin{array}{ccc} L2XMOD(F(\mathscr{L}), \mathscr{K}) & \xrightarrow{\eta_{\mathscr{L},\mathscr{K}}} & LQ2XMOD(\mathscr{L}, G(\mathscr{K})) \\ {\scriptstyle k^* = k\circ -}\downarrow & & \downarrow{\scriptstyle G(k)\circ - = G(k)^*} \\ L2XMOD(F(\mathscr{L}), \mathscr{K}') & \xrightarrow[\eta_{\mathscr{L},\mathscr{K}'}]{} & LQ2XMOD(\mathscr{L}, G(\mathscr{K}')) \end{array}$$

because of

$$(k_2 f_2) q_{_L} = k_2 (f_2 q_{_L}).$$

Hence, it is concluded that there is an adjunction between LQ2XMOD and L2XMOD.

3. Conclusion

In this paper, the category of quasi 2-crossed modules for Lie algebras has been introduced, and an adjunction between this category and that of 2-crossed modules for Lie algebras is constructed. It is concluded that this category has a similar role to that of pre-crossed modules in corresponding adjunction to their 1-dimensional analogous.

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Author Contributions

All authors contributed equally to this work. They all read and approved the final version of the manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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