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Hu's characterization of metric completeness revisited

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Abstract

In this note we show the somewhat surprising fact that the proof of the 'if part' of the distinguished characterizations of metric completeness due to Kirk, and Suzuki and Takahashi, respectively, can be deduced in a straightforward manner from Hu's theorem that a metric space is complete if and only if any Banach contraction on bounded and closed subsets thereof has a fixed point. We also take advantage of this approach to easily deduce a characterization of metric completeness via fixed point theorems for $\alpha - \psi$-contractive mappings.

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2010 MSC: 54H25, 54E50, 47H10.

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1. Introduction

In a simple but appealing way, Hu presented in [9] the first characterization of complete metric spaces via fixed point methods. Later, several important and renowned characterizations of metric completeness have been obtained with the help of fixed point results. For instance, Kirk proved in [15] that the validity of the celebrated Caristi fixed point theorem [6] provides a necessary and sufficient condition to a metric space be complete, while Subrahmanyam [22] reached similar conclusions with respect to Kannan's fixed point theorem [11]. In this setting are also remarkable the contributions of Park [18], Suzuki and Takahashi [24], and more recently the characterization given by Suzuki [23] by means of a weak version of the Banach contraction principle. Usually, the procedure to prove the 'if part', i.e., the sufficiency, of these results consists in assuming the existence of a non-convergent Cauchy sequence in the given metric space and then constructing a suitable self mapping on it that is free of fixed points but verifies the corresponding contraction condition. In general, this technique is quite arduous (see for instance the proof of Theorem 4 of [24]). Our

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aim in this paper is to show the somewhat surprising fact that the proof of the 'if part' of the results of Kirk, and Suzuki and Takahashi can be directly deduced in a nice fashion from Hu's theorem. We also take advantage of this approach to easily deduce a characterization of metric completeness via fixed point theorems for $\alpha - \psi$-contractive mappings.

2. Hu's theorem revisited

In his paper [9], Hu proved that a metric space is complete if and only if every Banach contraction on any of its closed subsets has a fixed point. Regarding Hu's theorem, recall that Connell [7] had earlier given an example of a non-complete metric space for which every Banach contraction has a fixed point.

An easy examination of Hu's proof shows that his theorem can be reformulated as follows.

Theorem 2.1. *A metric space is complete if and only if every Banach contraction on any of its bounded and closed subsets has a fixed point.*

Caristi proved in [6] his famous theorem that every Caristi mapping on a complete metric space has a fixed point, where a self mapping T of a metric space (X, d) is a Caristi mapping provided that there exists a lower semicontinuous function $\varphi : X \to [0, \infty)$ such that $d(x, Tx) \leq \varphi(x) - \varphi(Tx)$ for all $x \in X$.

As we point out in Section 1, Kirk showed in [15] that the validity of Caristi's theorem actually characterizes the metric completeness. Thus, we get the following well-known and outstanding result.

Theorem 2.2. *(Caristi-Kirk's theorem) A metric space is complete if and only if every Caristi mapping on it has a fixed point.*

Remark 2.3. *We proceed to deduce the 'if part' of Theorem 2.2 from Theorem 2.1.*

Let (X, d) be a metric space for which every Caristi's mapping has a fixed point, and let T be a Banach contraction on a bounded and closed subset C of (X, d). Then there exist constants $M > 0$ and $c \in (0, 1)$ such that $d(x, y) \leq M$ and $d(Tx, Ty) \leq cd(x, y)$ for all $x, y \in C$.

Choose an $a \in C$ and define a self mapping S of X by $Sx = Tx$ if $x \in C$ and $Sx = a$ if $x \in X \backslash C$.

Put $r = 1/(1 - c)$ and construct a function $\varphi : X \to [0, \infty)$ as $\varphi(x) = rd(x, Sx)$ if $x \in C$, and $\varphi(x) = rM + d(x, a)$ if $x \in X \backslash C$.

We show that φ is lower semicontinuous on (X, d). Indeed, let $(x_n)_{n \in \mathbb{N}}$ be a sequence in X and $x \in X$ such that $d(x, x_n) \to 0$ as $n \to \infty$. We consider two cases.

Case 1. $x \in X \backslash C$. Since C is closed there exists $n_0 \in \mathbb{N}$ such that $x_n \in X \backslash C$ for all $n \geq n_0$. So, from the definition of φ and the triangle inequality, we deduce that $\varphi(x) \leq \varphi(x_n) + d(x, x_n)$ for all $n \geq n_0$.

Case 2. $x \in C$. If $x_n \in C$ we get, by the triangle inequality and the fact that $d(Sx, Sx_n) \leq cd(x, x_n)$, that $\varphi(x) \leq \varphi(x_n) + (r + rc)d(x, x_n)$. If $x_n \in X \backslash C$ we get $\varphi(x) \leq rM < \varphi(x_n)$.

Therefore, in both cases, for any $\varepsilon > 0$ there is $n_\varepsilon \in \mathbb{N}$ such that $\varphi(x) < \varepsilon + \varphi(x_n)$ for all $n \geq n_\varepsilon$.

We conclude that φ is lower semicontinuous on (X, d).

Finally, if $x \in C$ we deduce that

$$d(x, Sx) = (r - cr)d(x, Sx) \leq rd(x, Sx) - rd(Sx, S^2x) = \varphi(x) - \varphi(Sx);$$

and if $x \in X \backslash C$,

$$d(x, Sx) = d(x, a) \leq rM + d(x, a) - rd(a, Sa) = \varphi(x) - \varphi(Sx).$$

We have shown that S is a Caristi's mapping on (X, d), so it has a fixed point z which obviously belongs to C. Hence z is a fixed point of T. So (X, d) is complete by Theorem 2.1.

In the seminal paper [10], Kada, Suzuki and Takahashi introduced and analyzed the notion of w-distance in the framework of metric spaces. They showed that this concept provides an efficient instrument to improve various crucial theorems as Ekeland's variational principle and its 'equivalent' Caristi's fixed point theorem, as well as Takahashi's nonconvex minimization theorem.

Let us recall that a w-distance on a metric space (X, d) is a function $w : X \times X \to [0, \infty)$ satisfying the following conditions for any $x, y, z \in X$:

(w1) $w(x, y) \leq w(x, z) + w(z, y)$;

(w2) $w(x, \cdot) : X \to [0, \infty)$ is a lower semicontinuous function;

(w3) for each $\varepsilon > 0$ there is $\delta > 0$ such that $w(x, y) \leq \delta$ and $w(x, z) \leq \delta$ imply $d(y, z) \leq \varepsilon$.

Obviously, every metric d on a set X is a w-distance on the metric space (X, d) (see, for instance, [10, 24] for more interesting examples of w-distances). At this point it is appropriate to recall that the notion of w-distance has been extended and applied by several authors in different contexts, mainly in the realm of fixed point theory (for details, see e.g. [1, 2, 3, 12, 16, 17] and the references therein). In this direction, and following [24], by a weakly contractive mapping on a metric space (X, d) we mean a self mapping T of X such that there exist a w-distance w on (X, d)) and a constant $r \in (0, 1)$ satisfying $w(Tx, Ty) \leq rw(x, y)$, for all $x, y \in X$.

With the help of the above concept, Suzuki and Takahashi ([24, Theorem 4]) proved the following relevant result.

Theorem 2.4. *(Suzuki-Takahashi's theorem) A metric space is complete if and only if every weakly contractive mapping on it has a fixed point.*

Remark 2.5. *We proceed to deduce the 'if part' of Theorem 2.4 from Theorem 2.1.*

Let (X, d) be a metric space for which every weakly contractive mapping has a fixed point, and let T be a Banach contraction on a bounded and closed subset C of (X, d). Then there exist constants $M > 0$ and $c \in (0, 1)$, with $M > c$, such that $d(x, y) \leq M$ and $d(Tx, Ty) \leq cd(x, y)$ for all $x, y \in C$.

Choose an $a \in C$ and define a self mapping S of X by $Sx = Tx$ if $x \in C$ and $Sx = a$ if $x \in X \backslash C$.

Construct a function $w : X \times X \to [0, \infty)$ as $w(x, y) = d(x, y)$ if $x, y \in C$, and $w(x, y) = M/c$ otherwise. Note that $w(x, y) \leq M/c$ for all $x, y \in X$.

We show that w is a w-distance on (X, d).

*In fact, condition **(w1)** is clearly satisfied.*

*To show that condition **(w2)** holds, fix an $x \in X$ and let $(y_n)_{n\in\mathbb{N}}$ be a sequence in X and $y \in X$ such that $d(y_n, y) \to 0$ as $n \to \infty$.*

If $x \in X \backslash C$ or there is a subsequence $(y_{n(k)})_{k\in\mathbb{N}}$ of $(y_n)_{n\in\mathbb{N}}$ such that $y_{n(k)} \in X \backslash C$ for all $k \in \mathbb{N}$, we deduce that $w(x, y) \leq M/c = w(x, y_{n(k)})$ for all $k \in \mathbb{N}$. Otherwise, we get that $y \in C$ because C is closed and, thus, $w(x, y) = d(x, y)$, and $w(x, y_n) = d(x, y_n)$ eventually.

Therefore, the function $w(x, \cdot) : X \to [0, \infty)$ is lower semicontinuous.

*To show that condition **(w3)** holds, choose an arbitrary $\varepsilon > 0$. Let $\delta = \min\{\varepsilon/2, 1\}$, and suppose that $w(x, y) \leq \delta$ and $w(x, z) \leq \delta$. Then $w(x, y) \leq 1$ and $w(x, z) \leq 1$. Since $M/c > 1$, we deduce that $x, y, z \in C$. Thus $w(x, y) = d(x, y)$ and $w(x, z) = d(x, z)$. Consequently $d(y, z) \leq 2\delta \leq \varepsilon$.*

Finally, if $x, y \in C$ we get

$$w(Sx, Sy) = w(Tx, Ty) = d(Tx, Ty) \leq cd(x, y) = cw(x, y).$$

If $x \in C$ and $y \in X \backslash C$ (the case $x \in X \backslash C$ and $y \in C$ is identical by the symmetry of w), we get

$$w(Sx, Sy) = w(Tx, a) = d(Tx, a) \leq M = cw(x, y),$$

and if $x, y \in X \backslash C$ we get

$$w(Sx, Sy) = w(a, a) = d(a, a) = 0.$$

We have proved that S is a weakly contractive mapping on (X, d). By assumption, it has a fixed point z which obviously belongs to C. Hence z is a fixed point of T. So (X, d) is complete by Theorem 2.1.

In their remarkable paper [21], Samet, Vetro and Vetro introduced the notion of $\alpha-\psi$-contractive mapping and obtained several interesting and general fixed point theorems by using such a notion. Since then the question of obtaining fixed point results involving $\alpha-\psi$-contractive mappings has been the subject of a prominent research (see e.g. [4, 5, 8, 13, 14, 19] and the references therein).

According to [21], a self mapping T of a set X is said to be α-admissible if there is a function $\alpha : X\times X \to [0,\infty)$ such that, for each $x, y \in X$, $\alpha(x, y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$.

A self mapping T of a metric space (X, d) is called an $\alpha-\psi$-contractive mapping if

$$\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y)),$$

for all $x, y \in X$, where $\alpha : X \times X \to [0,\infty)$ is a function, and $\psi : [0,\infty) \to [0,\infty)$ is a function satisfying that $\sum_{n=0}^{\infty}\psi^n(t) < \infty$ for each $t \geq 0$.

Then, Samet, Vetro and Vetro proved in [21, Theorem 2.2] the following general result.

Theorem 2.6. *(Samet-Vetro-Vetro) Let (X, d) be a complete metric space and $T : X \to X$ be an $\alpha-\psi$-contractive mapping satisfying the following conditions:*

(a) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(b) T is α-admissible;

(c) for any sequence $(x_n)_{n\in\mathbb{N}}$ in Xsatisfying $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and such that $d(x_n, x) \to 0$ for some $x \in X$ it follows that $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.

Then T has a fixed point.

In the sequel, a self mapping of a metric space satisfying conditions (a), (b) and (c) above will be said to be an (α, ψ)-SVV-contractive mapping. If T is an (α, ψ)-SVV-contractive mapping on a metric space (X, d) such that $\psi(t) = rt$ ($0 < r < 1$, constant) for all $t \geq 0$, we say that T is an (α, r)-SVV-contractive mapping. Then, we have the following.

Theorem 2.7. *For a metric space (X,d) the following conditions are equivalent.*

(1) (X,d) is complete;

(2) every (α, ψ)-SVV-contractive mapping on (X,d) has a fixed point;

(3) every (α, r)-SVV-contractive mapping on (X,d) has a fixd point.

Proof. Proof. (1) $\Longrightarrow$ (2). Apply Theorem 2.6.

(2) $\Longrightarrow$ (3). It is obvious.

(3) $\Longrightarrow$ (1). Let T be a Banach contraction on a bounded and closed subset C of (X, d). Then, there is a constant $r \in (0, 1)$ such that $d(Tx, Ty) \leq rd(x, y)$ for all $x, y \in C$. Fix $a \in C$ and define a self mapping S of X as $Sx = Tx$ for all $x \in C$, and $Sx = a$ otherwise.

Now let $\alpha : X \times X \to [0,\infty)$ given by $\alpha(x, y) = 1$ whenever $x, y \in C$, and $\alpha(x, y) = 0$ otherwise.

It is clear that S is an (α, r)-contractive mapping on (X, d).

Furthermore, we have the following three facts:

(f1) for any $x_0 \in C$, $\alpha(x_0, Sx_0) = \alpha(x_0, Tx_0) = 1$. So, condition (a) of Theorem 2.6 is fulfilled.

(f2) S is α-admissible because condition $\alpha(x, y) \geq 1$ implies that $x, y \in C$, and hence $\alpha(Sx, Sy) = \alpha(Tx, Ty) = 1$. So, condition (b) of Theorem 2.6 is fulfilled.

(f3) If $(x_n)_{n\in\mathbb{N}}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, and $d(x_n, x) \to 0$ for some $x \in X$,then $x_n \in C$ for all $n \in \mathbb{N}$, and hence $x \in C$ (recall that C is a closed subset of (X, d)). Consequently $\alpha(x_n, x) = 1$ for all $n \in \mathbb{N}$. So, condition (c) of Theorem 2.6 is fulfilled.

We have shown that S is an (α, r)-SVV-contractive mapping on (X, d). From our assumption it follows that S has a fixed point z which obviously belongs to C. Therefore z is a fixed point of T. By Theorem 2.1, we conclude that (X, d) is complete. □

Remark 2.8. *Variants of Theorem 2.7 whose proofs do not invoke Theorem 2.1, may be found in [19, Theorem and Corollary 1] and in [20, Theorem 6 and Corollary 2].*

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