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Co-Hopf Space Structure on Closure Spaces

Sibel Demiralp

Department of Mathematics, Faculty of Science and Letter, Kastamonu University, 37000, Kastamonu, Turkey.

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Abstract. By constructing Hopf costructures on closure spaces via homotopy, we give the concepts of closure Hopf cospace (CH-cospace) and closure Hopf cogroup (CH-cogroup). We then prove that retract and deformation retract of a CH-cospace are also a CH-cospace. We construct a Hopf costructure on a set with the help of the quotient closure operator. We also show that a closure space with the same homotopy type as a CH-cogroup is itself a CH-cogroup. We prove the existence of a covariant functor between the homotopy category of the pointed closure spaces ($\mathcal{CHC}$) and the category of groups and homomorphisms.

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1. Introduction

[4] Closure space is defined by a closure operator $\upsilon : P(X) \to P(X)$ satisfies

(c_1) $\upsilon(\emptyset) = \emptyset$,

(c_2) $A \subseteq \upsilon(A)$ for all $A \in P(X)$,

(c_3) $\upsilon(A \cup B) = \upsilon(A) \cup \upsilon(B)$ for all $A, B \in P(X)$.

Then, (X, υ) is called a closure space. If υ additionally satisfies the axiom (c_4) $\upsilon(\upsilon(A)) = \upsilon(A)$, then υ is called a topological closure operator. In this case (X, υ) is called a topological space.

A lot of topological notions, like continuity, separation axioms, compactness, are studied in closure spaces [3,8,10]. In [12], homotopy concept is defined in closure spaces. In this study, co-Hopf structures are defined in closure spaces.

If (X, υ) is a closure space, the closure operator υ satisfies the condition (c_5) $A \subseteq B \Rightarrow \upsilon(A) \subseteq \upsilon(B)$. If for all $A \in P(X)$, $\upsilon(A) = A$, then A is called a closed set. If $\upsilon(X - A) = X - A$, then A is called an open set.
A closure operator υ is called discrete if $\upsilon(A) = A$, for all $A \in P(X)$, and called trivial if $\upsilon(A) = X$, for all $A \in P(X)$, $A \neq \emptyset$.

If there are two or more closure spaces, we use the notation υ_X for the closure operator on X.
If $Y \subseteq X$, and (X, υ_X) is a closure space, then Y is a closure space with the closure operator $\upsilon_Y(A) = \upsilon_X(A) \cap Y$, for all $A \subseteq Y$.

Email address: sdemiralp@kastamonu.edu.tr (S. Demiralp)

A map $f : (X, \upsilon_X) \to (Y, \upsilon_Y)$ is said to be continuous iff $f(\upsilon_X(A)) \subseteq \upsilon_Y(f(A))$ for all $A \subseteq X$, f is called closed iff $f(\upsilon_X(A)) = \upsilon_Y(f(A))$.

Example 1.1. Let $X = \mathbb{N}$ and $\upsilon : P(X) \to P(X)$ be defined as $\upsilon(\{n\}) = \{n, n+1\}$ and for all $A \subseteq X$,

$$\upsilon(A) = \begin{cases} \emptyset & , \text{if } A = \emptyset \\ \bigcup\{\upsilon(\{a\}) \mid a \in A\} & , \text{if } A \neq \emptyset. \end{cases}$$

Let show υ is a closure operator on $\mathbb{N}$. (c_1) and (c_2) are clear by definition of υ.

$$\begin{aligned} \upsilon(A \cup B) &= \bigcup\{\upsilon(\{x\}) \mid x \in A \cup B\} \\ &= \left(\bigcup\{\upsilon(\{x\}) \mid x \in A\}\right) \cup \left(\bigcup(\{\upsilon(\{x\}) \mid x \in B\})\right) \\ &= \upsilon(A) \cup \upsilon(B). \end{aligned}$$

Therefore, υ is a closure operator on X. Let $A = \{1, 2, 3\}$. Then, $\upsilon(A) = \{1, 2, 3, 4\} \neq \upsilon(\upsilon(A)) = \{1, 2, 3, 4, 5\}$. Therefore, υ is not topological.

Let (X, υ) be a closure space and $\alpha : X \to Y$ be a onto map. Then $\upsilon_\alpha : P(Y) \to P(Y)$ defined as $\upsilon_\alpha(B) = \alpha\upsilon\alpha^{-1}(B)$ is a closure operator on Y, called as quotient closure operator induced by υ.

Example 1.2. Let $X = \{a, b, c, d\}$ and define a closure operator υ on X such that

$$\upsilon(\emptyset) = \emptyset, \upsilon(\{a\}) = \{a, c\}, \upsilon(\{b\}) = \{b\}, \upsilon(\{c\}) = \{c, d\}, \upsilon(\{d\}) = \{a, d\}, \upsilon(\{a, b\}) = \{a, b, c\}$$

$$\upsilon(\{a, c\}) = \upsilon(\{a, d\}) = \upsilon(\{c, d\}) = \upsilon(\{a, c, d\}) = \{a, c, d\}, \upsilon(\{b, c\}) = \{b, c, d\}$$

$$\upsilon(\{b, d\}) = \{a, b, d\}, \upsilon(\{a, b, c\}) = \upsilon(\{a, b, d\}) = \upsilon(\{b, c, d\}) = \upsilon(X) = X.$$

Let $Y = \{1, 2, 3\}$ and $\alpha : X \to Y$ be defined as $\alpha(a) = \alpha(c) = 1$, $\alpha(b) = 2$, $\alpha(d) = 3$. The quotient closure operator υ_α is defined as

$$\upsilon_\alpha(\emptyset) = \emptyset, \upsilon_\alpha(\{1\}) = \upsilon_\alpha(\{3\}) = \upsilon_\alpha(\{1, 3\}) = \{1, 3\},\ \upsilon_\alpha(\{2\}) = \{2\}, \upsilon_\alpha(\{1, 2\}) = \upsilon_\alpha(\{2, 3\}) = \upsilon_\alpha(Y) = Y.$$

Let $\beta : X \to Y$ be defined as $\beta(a) = \beta(b) = 1, \beta(c) = 2, \beta(d) = 3$. The quotient closure operator υ_β is defined as

$$\upsilon_\beta(\emptyset) = \emptyset, \upsilon_\beta(\{1\}) = \{1, 2\}, \upsilon_\beta(\{2\}) = \{2, 3\}, \upsilon_\beta(\{3\}) = \{1, 3\}, \upsilon_\beta(\{1, 2\}) = \upsilon_\beta(\{1, 3\}) = \upsilon_\beta(\{2, 3\}) = \upsilon_\beta(\{Y\}) = Y.$$

Now define $\gamma : X \to Y$ such that $\gamma(a) = \gamma(b) = 1, \gamma(c) = 3$. Then,

$$\upsilon_\gamma(\{2\}) = \gamma\upsilon\gamma^{-1}(\{2\}) = \gamma\upsilon(\emptyset) = \gamma(\emptyset) = \emptyset.$$

Therefore, υ_γ is not a closure operator on Y, since γ is not an onto map.

Lemma 1.3. *[12] Let (X, υ) be a closure space, $\alpha : X \to Y$ be an onto map. Then, the closure operator υ_α induced by υ is the finest closure operator on Y makes α continuous.*

Definition 1.4. A set $W \subseteq X$ is called a neighbourhood of A iff $W \subseteq X - \upsilon(X - A)$. The set of all neighbourhood of A is denoted by $\mathcal{V}_A$.

A closure operator on is defined in [12] by the help of neighbourhood as following:

Definition 1.5. [12] Let (X, υ) be a closure space, $Y \subset X$ and $\mathcal{B} \subset P(X)$. If

i) $A \in \mathcal{V}_Y$, for all $A \in \mathcal{B}$,
ii) For all $U \in \mathcal{V}_Y$ there exists $A \in \mathcal{B}$ such that $B \subset U$,

then $\mathcal{B}$ is called a base of the neighbourhood system $\mathcal{V}_Y$.
Let $\mathcal{S}_Y \subset P(X)$ and $\mathcal{V}_Y$ is a neighbourhood system of Y. If all finite intersections of elements of $\mathcal{S}_Y$ is a base for $\mathcal{V}_Y$, then $\mathcal{S}_Y$ is called a subbase for $\mathcal{V}_Y$.

Theorem 1.6. *[4] Let $\prod_{i\in I} X_\alpha$ be the cartesian product of the closure spaces $(X_i, \upsilon_i)_{i\in I}$. For each $x \in \prod_{i\in I} X_i$, let*

$$\mathcal{V}_x = \{\pi_j^{-1}(V) : j \in I, V \subset X_j \text{ a neighborhood of } \pi_j(x) \in X_j\},$$

where $\pi_j : \prod_{i\in I} X_i \to (X_j, \upsilon_j)$ is the projection map. Then, there exist a unique closure structure on $\prod_{\alpha\in I} X_\alpha$ such that $\mathcal{V}_x$ is a subbase for each $x \in \prod_{\alpha\in I} X_\alpha$.

Definition 1.7. Let (X, x_0, υ_X) and (Y, y_0, υ_Y) be pointed closure spaces. The wedge sum of X and Y is

$$X \vee Y = X \times \{y_0\} \cup \{x_0\} \times Y \subset X \times Y.$$

The wedge sum of pointed closure spaces (X, x_0, υ_X) and (Y, y_0, υ_Y) is a pointed closure space with the base point (x_0, y_0) and the closure operator $\upsilon_{X\vee Y}$, defined as

$$\upsilon_{X\vee Y}(u) = \upsilon_{X\times Y}(u) \cap (X \vee Y),$$

for all $u \in P(X \times Y)$.
If $f : (X, x_0) \to Z$,$g : (Y, y_0) \to Z$ and $h : Y \to W$, then $(f, g) : X \vee Y \to Z$ is a map defined as

$$(f,g)(x,y) = \begin{cases} f(x) & \text{if } \; y = y_0 \\ g(y) & \text{if } \; x = x_0 \end{cases}$$

and $f \vee h : X \vee Y \to Z \vee W$ is a map defined as $(f \vee h)(x, y) = (f(x), h(y))$.

2. Closure H-Cospaces

The concepts of Hopf space and Hopf cospace have been studied by many researchers on different spaces. In [2, 5, 6, 9],the concept of hopf space is examined and in [7], the concept of Hopf cospace examined in digital spaces. Adhikari and Rahaman [1] defined generalized topological monoid as a generalization of Hopf group. Park defined the concept of subgroup in Hopf spaces [11]. In this part, we define closure Hopf cospace with the help of homotopy and investigate some properties of closure Hopf cospaces.

Homotopy on closure spaces defined in [12] as following: Continuous functions $f, g : (X, \upsilon_X) \to (Y, \upsilon_Y)$ are called homotopic, denoted by $f \simeq g$, if there exists a continuous map

$$F : (X \times I, \upsilon_\Pi) \to (Y, \upsilon_Y)$$

such that $F|_{X\times\{0\}} = f$ and $F|_{X\times\{1\}} = g$, where $I = [0, 1]$ and (X, υ_X) and (Y, υ_Y) are closure spaces and υ_Π is the closure operator on $X \times I$. Then, H is called homotopy between f and g.

The homotopy relation " $\simeq$ " is an equivalence relation. We use $[f] = \{ g \mid f \simeq g, \; g : (X, \upsilon_X) \to (Y, \upsilon_Y)\}$ to denote of homotopy class of f, and $[(X, \upsilon_X); (Y, \upsilon_Y)] = \{ [f] \mid f : (X, \upsilon_X) \to (Y, \upsilon_Y)\}]$ to denote the set of all homotopy classes of the functions from (X, υ_X) to (Y, υ_Y).

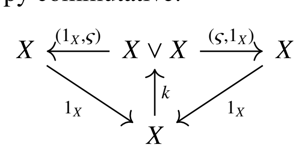
If the continuous functions $g, h : (X, \upsilon_X) \to (Y, \upsilon_Y)$ are homotopic with the homotopy G, then $f \circ g \simeq f \circ h$ with the homotopy $F = f \circ G$ for any continuous function $f : (Y, \upsilon_Y) \to (Z, \upsilon_Z)$.

Let (X, υ_X) be a closure space and $x_0 \in X$ be a point. Then, (X, x_0, υ_X) is called a pointed closure space and x_0 is called base point of (X, x_0, υ_X).

Definition 2.1. Let (X, x_0, υ_X) be a pointed closure space and $k : X \to X \vee X$ be a continuous comultiplication, $\varsigma : X \to X$ be a constant function such that $\varsigma(x) = x_0$ for all $x \in X$. Then, (X, x_0, υ_X) is called as a closure H-cospace (CH-cospace) if

$$(\varsigma, 1_X) \circ k \simeq 1_X \simeq (1_X, \varsigma) \circ k.$$

This means the following diagram is homotopy commutative:



Also ς is called homotopy identity of (X, x_0, υ_X).

In the case of more than one CH-cospace, we use the notations k_X and ς_X for the continuous comultiplication and homotopy identity of the CH-cospace (X, x_0, υ_X) to avoid confusion.

Theorem 2.2. *Let (X, x_0, υ_X) and (Y, y_0, υ_Y) be CH-cospaces. Then, $X \vee Y$ is a CH-cospace.*

Proof. Let $P : X \vee Y \to X \vee Y$ be defined as $P(x, y_0) = (y_0, x)$ and $P(x_0, y) = (y, x_0)$ for all $x \in X$ and $y \in Y$.
Define $k_{X\vee Y} : X \vee Y \to (X \vee Y) \vee (X \vee Y)$ such that

$$k_{X\vee Y} = (1_X \vee P \vee 1_Y) \circ (k_X \vee k_Y).$$

Then,

$$\begin{aligned} \Big((\varsigma_{X\vee Y}, 1_{X\vee Y}) \circ k_{X\vee Y}\Big) &= (\varsigma_{X\vee Y}, 1_{X\vee Y}) \circ (1_X \vee P \vee 1_Y) \circ (k_X \vee k_Y) \\ &\simeq \Big((\varsigma_X, 1_X) \vee (\varsigma_Y, 1_Y)\Big) \circ (k_X \vee k_Y) \\ &\simeq \Big((\varsigma_X, 1_X) \circ k_X\Big) \vee \Big((\varsigma_Y, 1_Y) \circ k_Y\Big) \\ &\simeq 1_X \vee 1_Y = 1_{X\vee Y}. \end{aligned}$$

In a similar way $(1_{X\vee Y}, \varsigma_{X\vee Y}) \circ k_{X\vee Y} \simeq 1_{X\vee Y}$. Therefore, $X \vee Y$ is a CH-cospace with the base point (x_0, y_0) and the comultiplication $k_{X\vee Y}$. □

To examine the relationship between the retract or weak retract of a CH-cospace and the CH-cospace, let us first give the definitions of retract and weak retrack.

Definition 2.3. Let (A, υ_A) be a subspace of a closure space (X, υ_X). Then,

* (A, υ_A) is called a retract of (X, υ_X) if there exists a map $r : (X, \upsilon_X) \to (A, \upsilon_A)$ such that $r(x) = x$, for all $x \in X$.
* (A, υ_A) is called weak retract of (X, υ_X) if $r \circ i \simeq 1_A$, for the inclusion map $i : (A, \upsilon_A) \hookrightarrow (X, \upsilon_X)$.

Therefore, every retract of a closure space is a weak retract of it.

Theorem 2.4. *Let (X, x_0, υ_X) is a CH-cospace and (Z, z_0, υ_Z) be a weak retract of X. Then, (Z, z_0, υ_Z) is a CH-cospace.*

Proof. Let r be the retraction. Let $k_Z = (r \vee r) \circ k_X \circ i$ and $(\varsigma_Z, 1_Z) : Z \vee Z \to Z$ be defined as the following composition:

$$Z \vee Z \xrightarrow{i\vee i} X \vee X \xrightarrow{(\varsigma_X,1_X)} X \xrightarrow{r} Z$$

Then,

$$\begin{aligned} (\varsigma_Z, 1_Z) \circ k_Z &= r \circ (\varsigma_X, 1_X) \circ (i \vee i) \circ (r \vee r) \circ k_X \circ i \\ &= r \circ (\varsigma_X, 1_X) \circ (i \circ r) \vee (i \circ r) \circ k_X \circ i \\ &\simeq r \circ (\varsigma_X, 1_X) \circ 1_{X\vee X} \circ k_X \circ i \\ &= r \circ (\varsigma_X, 1_X) \circ k_X \circ i \\ &\simeq r \circ 1_X \circ i \\ &= r \circ i \simeq 1_Z. \end{aligned}$$

Now let $(1_Z, \varsigma_Z) : Z \vee Z \to Z$ be the following composition:

$$Z \vee Z \xrightarrow{i\vee i} X \vee X \xrightarrow{(1_X,\varsigma_X)} X \xrightarrow{r} Z.$$

Then,

$$\begin{aligned} (1_Z, \varsigma_Z) \circ k_Z &= r \circ (1_X, \varsigma_X) \circ (i \vee i) \circ (r \vee r) \circ k_X \circ i \\ &= r \circ (1_X, \varsigma_X) \circ (i \circ r) \vee (i \circ r) \circ k_X \circ i \\ &\simeq r \circ (1_X, \varsigma_X) \circ 1_{X\vee X} \circ k_X \circ i \\ &= r \circ (1_X, \varsigma_X) \circ k_X \circ i \\ &\simeq r \circ 1_X \circ i \\ &= r \circ i \simeq 1_Z. \end{aligned}$$

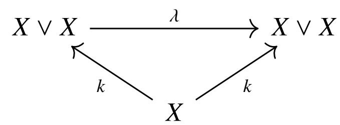
Consequently, (Z, z_0, υ_Z) is a CH-cospace. □

Definition 2.5. Let (Z, z_0, υ_Z) be a retract of the closure space (X, x_0, υ_X). If there exists a homotopy such that $i \circ r \simeq 1_X$ for the inclusion map i and the retraction r, then (Z, z_0, υ_Z) is called deformation retract of (X, x_0, υ_X).

If (Z, z_0, υ_Z) is a deformation retract of (X, x_0, υ_X), then it is retract of (X, x_0, υ_X). So we have the following corollary:

Corollary 2.6. *A retract (deformation retract) of a CH-cospace is itself a CH-cospace.*

Definition 2.7. Let (X, x_0, υ_X) be a CH-cospace. If there exists a map $\lambda : X \vee X \to X \vee X$ defined as $\lambda(a, b) = (b, a)$ such that $\lambda \circ k \simeq k$, that is the following diagram homotopy commutative:



then, the comultiplication k is called homotopy abelian and (X, x_0, υ_X) is called abelian CH-cospace.

Theorem 2.8. *Let (X, x_0, υ_X) be an abelian CH-cospace and (Y, y_0, υ_Y) be a weak retract of it. Then, (Y, y_0, υ_Y) is also an abelian CH-cospace.*

Proof. Since (X, x_0, υ_X) is an abelian CH-cospace, $\lambda \circ k_X \simeq k_X$ for a map $\lambda : X \vee X \to X \vee X$, $\lambda(a, b) = (b, a)$. Then, (Y, y_0, υ_Y) is a CH-cospace with the comultiplication $k_Y = (i \vee i) \circ k_X \circ r$, by Theorem 2.4. Let λ_Y be the following composition:

$$Y \vee Y \xrightarrow{i \vee i} X \vee X \xrightarrow{\lambda} X \vee X \xrightarrow{r \vee r} Y \vee Y.$$

Then,

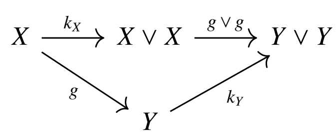
$$\begin{aligned} \lambda_Y \circ k_Y &= (r \vee r) \circ \lambda \circ (i \vee i) \circ (r \vee r) \circ k_X \circ i \\ &= (r \vee r) \circ \lambda \circ (i \circ r) \vee (i \circ r) \circ k_X \circ i \\ &\simeq (r \vee r) \circ \lambda \circ 1_{X \vee X} \circ k_X \circ i \\ &= (r \vee r) \circ \lambda \circ k_X \circ i \\ &= k_Y. \end{aligned}$$

Therefore, (Y, y_0, υ_Y) is an abelian CH-cospace. □

Definition 2.9. Let (X, x_0, υ_X) and (Y, y_0, υ_Y) be CH-cospaces. A function

$$g : (X, x_0, \upsilon_X) \to (Y, y_0, \upsilon_Y)$$

is called co-H-homomorphism if $(g \vee g) \circ k_X \simeq k_Y \circ g$, that is the following diagram homotopy commutative:



Theorem 2.10. *Composition of two co-H-homomorphisms is a co-H-homomorphism.*

Proof. Let $f : (X, x_0, \upsilon_X) \to (Y, y_0, \upsilon_Y)$ and $g : (Y, y_0, \upsilon_Y) \to (Z, z_0, \upsilon_Z)$ be co-H-homomorphisms. Then,

$$k_Y \circ f \simeq (f \vee f) \circ k_X \quad \text{and} \quad k_Z \circ g \simeq (g \vee g) \circ k_Y.$$

We obtain $(g \circ f) \vee (g \circ f) \circ k_X = (g \vee g) \circ (f \vee f) \circ k_X \simeq (g \vee g) \circ k_Y \circ f \simeq k_Z \circ (g \circ f)$. □

Theorem 2.11. *Let (X, x_0, υ_X) be a CH-cospace and (Y, y_0, υ_Y) be a deformation retract of (X, x_0, υ_X). Then the inclusion map and the retraction are co-H-homomorphisms.*

Proof. Let $i : (Y, y_0, \upsilon_Y) \hookrightarrow (X, x_0, \upsilon_X)$ be the inclusion and $r : (X, x_0, \upsilon_X) \longrightarrow (Y, y_0, \upsilon_Y)$ be the retraction. Define $k_Y = (r \vee r) \circ k_X \circ i$. Then

$$\begin{aligned} k_Y \circ i &= (r \vee r) \circ k_X \circ (i \circ r)) \\ &\simeq (r \vee r) \circ k_X \circ 1_X \\ &= (r \vee r) \circ k_X. \end{aligned}$$

This proves that the inclusion map i is a co-H-homomorphism. Since

$$\begin{aligned}(i \vee i) \circ k_Z &= (i \vee i) \circ (r \vee r) \circ k_X \circ i \\ &= (i \circ r) \vee (i \circ r) \circ k_X \circ i \\ &\simeq 1_{X \vee X} \circ k_X \circ i \\ &= k_X \circ i,\end{aligned}$$

the retraction r is a co-H-homomorphism. □

The following theorem shows that a Hopf co-structure can be constructed on a set with the quotient closure operator induced by a closure operator of a CH-cospace.

Theorem 2.12. *Let (X, x_0, υ) be a CH-cospace, (Z, z_0) be a pointed space and α be a surjective mapping from (X, x_0, υ) to (Z, z_0). Then (Z, z_0) is a CH-cospace.*

Proof. We know (Z, z_0) is a closure space with the quotient closure operator $\upsilon_\alpha = \alpha \circ \upsilon \circ \alpha^{-1}$. Now let define a comultiplication on Z with the help of comultiplication of X.
Let k_Y be the following composition:

$$Z \xrightarrow{\alpha^{-1}} X \xrightarrow{k_X} X \vee X \xrightarrow{\alpha \vee \alpha} Z \vee Z$$

and $(1_Y, \varsigma_Y), (\varsigma_Y, 1_Y) : Y \vee Y \to Y$ be defined as the following compositions, respectively:

$$Z \vee Z \xrightarrow{\alpha^{-1} \vee \alpha^{-1}} X \vee X \xrightarrow{(1_X, \varsigma_X)} X \xrightarrow{\alpha} Z,$$

$$Z \vee Z \xrightarrow{\alpha^{-1} \vee \alpha^{-1}} X \vee X \xrightarrow{(\varsigma_X, 1_X)} X \xrightarrow{\alpha} Z$$

Then,

$$\begin{aligned}(1_Y, \varsigma_Y) \circ k_Y &= \alpha \circ (1_X, \varsigma_X) \circ (\alpha^{-1} \vee \alpha^{-1}) \circ (\alpha \vee \alpha) \circ k_X \circ \alpha^{-1} \\ &= \alpha \circ (1_X, \varsigma_X) \circ (\alpha^{-1} \circ \alpha) \vee (\alpha^{-1} \circ \alpha) \circ k_X \circ \alpha^{-1} \\ &\simeq \alpha \circ (1_X, \varsigma_X) \circ 1_{X \vee X} \circ k_X \circ \alpha^{-1} \\ &= \alpha \circ (1_X, \varsigma_X) \circ k_X \circ \alpha^{-1} \\ &\simeq \alpha \circ 1_X \circ \alpha^{-1} = \alpha \circ \alpha^{-1} \simeq 1_Y,\end{aligned}$$

$$\begin{aligned}(\varsigma_Y, 1_Y) \circ k_Y &= \alpha \circ (\varsigma_X, 1_X) \circ (\alpha^{-1} \vee \alpha^{-1}) \circ (\alpha \vee \alpha) \circ k_X \circ \alpha^{-1} \\ &= \alpha \circ (\varsigma_X, 1_X) \circ (\alpha^{-1} \circ \alpha) \vee (\alpha^{-1} \circ \alpha) \circ k_X \circ \alpha^{-1} \\ &\simeq \alpha \circ (\varsigma_X, 1_X) \circ 1_{X \vee X} \circ k_X \circ \alpha^{-1} \\ &= \alpha \circ (\varsigma_X, 1_X) \circ k_X \circ \alpha^{-1} \\ &\simeq \alpha \circ 1_X \circ \alpha^{-1} = \alpha \circ \alpha^{-1} \simeq 1_Y\end{aligned}$$

Consequently (Y, y_0, υ_Y) is a CH-cospace. □

Theorem 2.13. *Let (X, x_0, υ_X) be a CH-cospace and (Y, y_0, υ_Y) has the same homotopy type with (X, x_0, υ_X). Then, (Y, y_0, υ_Y) is a CH-cospace.*

Proof. Proof is similar to Theorem 2.4, take $k_Y = (g \vee g) \circ k_X \circ f$ and $(\varsigma_Y, 1_Y)$, $(1_Y, \varsigma_Y)$ as the composition of

$$Y \vee Y \xrightarrow{g \vee g} X \vee X \xrightarrow{(\varsigma_X, 1_X)} X \xrightarrow{f} Y,$$

$$Y \vee Y \xrightarrow{g \vee g} X \vee X \xrightarrow{(1_X, \varsigma_X)} X \xrightarrow{f} Y$$

respectively, where $f : X \to Y, g : Y \to X$ are homotopy equivalences. □

3. Closure H-Cogroup

This section defines the concept of CH-cogroup and examines some of its properties.

Definition 3.1. Let (X, x_0, υ_X) be a CH-cospace. If the following diagram is homotopy commutative:

$$\begin{array}{ccc} X & \xrightarrow{\ k\ } & X \vee X \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle 1_X \vee k} \\ X \vee X & \xrightarrow{\ k \vee 1_X\ } & X \vee X \vee X \end{array}$$

then, k is called homotopy associative.

$$k \text{ is homotopy commutative } \iff (1_X \vee m) \circ k \simeq (k \vee 1_X) \circ k.$$

A continuous function $\delta : X \to X$ is called homotopy inverse of k if each composite map

$$X \xrightarrow{\ k\ } X \vee X \xrightarrow{\ (1_X, \delta)\ } X,$$

$$X \xrightarrow{\ k\ } X \vee X \xrightarrow{\ (\delta, 1_X)\ } X$$

is homotopic to homotopy identity $\varsigma : X \longrightarrow X$. A CH-cogroup is a CH-cospace which has a homotopy associative comultiplication and homotopy inverse.

Theorem 3.2. *Weak retract of a CH-cogroup is a CH-cogroup.*

Proof. Let (X, x_0, υ_X) be a CH-cogroup and (Y, y_0, υ_Y) be a weak retract of (X, x_0, υ_X). Let $k_Y = (r \vee r) \circ k_X \circ i$ be continuous comultiplication of (Y, y_0, υ_Y). Then, (Y, y_0, υ_Y) is a CH-cospace by Theorem 2.13.

$$\begin{aligned}
(1_Y \vee k_Y) \circ k_Y &= \Big(1_Y \vee \big((r \vee r) \circ k_X \circ i\big)\Big) \circ \Big((r \vee r) \circ k_X \circ i\Big) \\
&\simeq \Big((r \circ i) \vee \big((r \vee r) \circ k_X \circ i\big)\Big) \circ \Big((r \vee r) \circ k_X \circ i\Big) \\
&\simeq (r \vee r \vee r) \circ (1_X \vee k_X) \circ (i \vee i) \circ (r \vee r) \circ k_X \circ i \\
&= (r \vee r \vee r) \circ (1_X \vee k_X) \circ \Big((i \circ r) \vee (i \circ r)\Big) \circ k_X \circ i \\
&\simeq (r \vee r \vee r) \circ (1_X \vee k_X) \circ 1_{X \vee X} \circ k_X \circ i \\
&= (r \vee r \vee r) \circ (1_X \vee k_X) \circ k_X \circ i \\
&\simeq (r \vee r \vee r) \circ (k_X \vee 1_X) \circ k_X \circ i \\
&= (r \vee r \vee r) \circ (k_X \vee 1_X) \circ 1_{X \vee X} \circ k_X \circ i \\
&\simeq (r \vee r \vee r) \circ (k_X \vee 1_X) \circ \Big((i \circ r) \vee (i \circ r)\Big) \circ k_X \circ i \\
&\simeq \Big(\big((r \vee r) \circ k_X \circ i\big) \vee (r \circ i)\Big) \circ \Big((r \vee r) \circ k_X \circ i\Big) \\
&= \Big(\big((r \vee r) \circ k_X \circ i\big) \vee 1_Y\Big) \circ \Big((r \vee r) \circ k_X \circ i\Big) \\
&= (k_Y \vee 1_Y) \circ k_Y.
\end{aligned}$$

Therefore, k_Y is homotopy associative. Let δ_X be the homotopy inverse of (X, x_0, υ_X) and $\delta_Y = r \circ \delta \circ i$. Then,

$$\begin{aligned}
(\delta_Y, 1_Y) \circ k_Y &= (r \circ \delta_X \circ i, 1_Y) \circ \big((r \vee r) \circ k_X \circ i\big) \\
&= \big((r \circ \delta_X \circ i \circ r) \vee r\big) \circ (k_X \circ i) \\
&\simeq \big((r \circ \delta_X) \vee r\big) \circ (k_X \circ i) \\
&= r \circ ((\delta_X, 1_X) \circ k_X\big) \circ i \\
&\simeq r \circ \big((1_X, \delta_X) \circ k_X\big) \circ i \\
&= (r \vee (r \circ \delta_X)) \circ (k_X \circ i) \\
&\simeq \big(r \vee (r \circ \delta_X \circ i \circ r)\big) \circ (k_X \circ i) \\
&= (1_Y, r \circ \delta_X \circ i) \circ \big((r \vee r) \circ k_X \circ i\big) \\
&= (1_Y, \delta_Y) \circ k_Y.
\end{aligned}$$

So (Y, y_0, υ_Y) has a homotopy inverse. Consequently, (Y, y_0, υ_Y) is a CH-cogroup. □

Corollary 3.3. *Let (X, x_0, υ_X) be a CH-cogroup and (Y, y_0, υ_Y) has the same homotopy type with (X, x_0, υ_X). Then, (Y, y_0, υ_Y) is a CH-cogroup.*

Proof. Take $i = g : Y \to X$ and $r = h : X \to Y$ in Theorem 3.2 as homotopy equivalences and take $k_Y = (h \circ h) \circ k_X \circ g$. □

Theorem 3.4. *Let (X, x_0, υ_X) and (Y, y_0, υ_Y) have the same homotopy type. If (X, x_0, υ_X) is an abelian CH-cogroup, then (Y, y_0, υ_Y) also an abelian CH-cogroup.*

Proof. Let g and h are homotopy equivalences. By Corollary 3.3, (Y, y_0, υ_Y) is a CH-cogroup with the comultiplication $k_Y = (g \vee g) \circ k_X \circ h$. Since k_X is homotopy commutative, then there exists a map

$$\lambda_X : X \vee X \Rightarrow X \vee X,\ \lambda_X(a, b) = (b, a)$$

such that $\lambda_X \circ k_X \simeq k_X$. Let $\lambda_Y : Y \vee Y \to Y \vee Y$ be defined as $\lambda_Y(a', b') = (b', a')$ for all $a', b' \in Y$. Then,

$$\lambda_Y \circ k_Y = \lambda_Y \circ (g \vee g) \circ k_X \circ h = (g \vee g) \circ \lambda_X \circ k_X \circ h \simeq (g \vee g) \circ k_X \circ h = k_Y.$$

So k_Y is homotopy commutative. □

Theorem 3.2 gives these results: A deformation retract of a CH-cogroup is also a CH-cogroup and a deformation retract of an abelian CH-cogroup is also abelian CH-cogroup.

Theorem 3.5. *Let (X, x_0, υ_X) be a CH-cogroup. The set $[(X, x_0, \upsilon_X); (Y, y_0, \upsilon_Y)]$ is a group for every pointed closure space (Y, y_0, υ_Y). If (X, x_0, υ_X) is an abelian CH-cogroup, then $[(X, x_0, \upsilon_X); (Y, y_0, \upsilon_Y)]$ is abelian.*

Proof. Define

$$\Delta : [(X, x_0, \upsilon_X); (Y, y_0, \upsilon_Y)] \times [(X, x_0, \upsilon_X); (Y, y_0, \upsilon_Y)] \to [(X, x_0, \upsilon_X); (Y, y_0, \upsilon_Y)]$$

as the homotopy class of the following composition:

$$X \xrightarrow{k_X} X \vee X \xrightarrow{(f,g)} Y$$

for all $[f], [g] \in [(X, x_0, \upsilon_X); (Y, y_0, \upsilon_Y)]$. Let $[f_1] = [g_1]$ and $[f_2] = [g_2]$). Then,

$$\big((f_1, f_2) \circ k_X\big)(x) = (f_1, f_2)(x, x_0) = f_2(x)$$

$$\big((f_1, f_2) \circ k_X\big)(x) = (f_1, f_2)(x_0, x) = f_2(x)$$

$$\big((g_1, g_2) \circ k_X\big)(x) = (g_1, g_2)(x, x_0) = g_1(x)$$

$$\big((g_1, g_2) \circ k_X\big)(x) = (g_1, g_2)(x_0, x) = g_2(x).$$

Therefore, $\Delta([f_1],[f_2]) = [(f_1, f_2)\circ k_X] = [(g_1, g_2)\circ k_X] = \Delta([g_1],[g_2])$. So Δ is well defined.
Let $\varepsilon : X \to Y, \varepsilon(x) = y_0$, for all $x \in X$. Then,

$$\Delta([g],[\varepsilon]) = [(f,\varepsilon)\circ k_X] = [g\circ(1_X,\varsigma)\circ k_X] = [g\circ 1_X] = [g]$$

for any $[g] \in [(Y, y_0, \upsilon_Y);(X, x_0, \upsilon_X)]$. We get $\Delta([\varepsilon],[g]) = [g]$ by a similar way. So $[\varepsilon]$ is the unit element of $[(Y, y_0, \upsilon_Y);(X, x_0, \upsilon_X)]$ for Δ.
Let $[1]$ be the unit function of $[(X, x_0, \upsilon_X);(Y, y_0, \upsilon_Y)]$. Let show Δ is associative:

$$\begin{aligned}
\Delta\circ([1]\times\Delta)([f],([g],[h])) &= \Delta([f],\Delta([g],[h])) = \Delta\big([f],[(g,h)\circ k_X]\big)\\
&= [(f,(g,h)\circ k_X)\circ k_X]\\
&= [(f,(g,h)\circ(k_X\vee 1_X)\circ k_X]\\
&= [(f,(g,h)\circ(1_X\vee k_X)\circ k_X]\\
&= [(f,g)\circ k_X, h)\circ k_X]\\
&= \Delta([(f,g)\circ k_X],[h]) = \Delta(\Delta([f],[g]),[h])\\
&= \Delta\circ(\Delta\times[1])(([f],[g]),[h]).
\end{aligned}$$

Let δ be the homotopy inverse of (X, x_0, υ_X). For any $[f] \in [(X, x_0, \upsilon_X);(Y, y_0, \upsilon_Y)]$,

$$\Delta([f],[f\circ\delta]) = [(f, f\circ\delta)\circ k_X] = [f\circ(1_X,\delta)\circ k_X] = [f\circ\varsigma] = [e].$$

Similarly $\Delta([f\circ\delta],[f]) = [e]$. Therefore, $[f\circ\delta]$ is the homotopy inverse of $[f]$. Finally, let k_X be abelian. Then,

$$\Delta([f],[g]) = [(f,g)\circ k_X] = [(g,f)\circ k_X] = \Delta([g],[f]).$$

□

The category whose objects are pointed closure spaces and the set of morphisms

$$hom((X, x_0, \upsilon_X),(Y, y_0, \upsilon_Y)) = [(X, x_0, \upsilon_X),(Y, y_0, \upsilon_Y)]$$

is called the homotopy category of the pointed closure spaces, denoted $\mathcal{CHC}$. The composition of morphisms is the operation Δ that defined as in Theorem 3.5.

Theorem 3.6. *Let (X, x_0, υ_X) be a CH-cogroup. There exists a covariant functor from $\mathcal{CHC}$ to the category of groups and homomorphisms.*

Proof. Define Υ_X from $\mathcal{CHC}$ to the category of sets and functions such that associates to an object (Y, y_0, υ_Y) the set $\Upsilon_X(Y, y_0, \upsilon_Y) = [(X, x_0, \upsilon_X),(Y, y_0, \upsilon_Y)]$ and to a morphism $[g]$ the function

$$\Upsilon_X([g]) = g_* : [(X, x_0, \upsilon_X),(Z, z_0, \upsilon_Z)] \to [(X, x_0, \upsilon_X),(Y, y_0, \upsilon_Y)], g_*([f]) = [g\circ f],$$

where $[g] \in [(Z, z_0, \upsilon_Z),(Y, y_0, \upsilon_Y),]$. Let $[f],[h] \in [(X, x_0, \upsilon_X),(Z, z_0, \upsilon_Z)]$.

$$\begin{aligned}
g_*(\Delta([f],[h])) &= g_*([(h,f)\circ k_X])\\
&= [g\circ((h,f)\circ k_X)]\\
&= [(g\circ h, g\circ f)\circ k_X]\\
&= \Delta([g\circ h],[g\circ f])\\
&= \Delta(g_*([h]), g_*([f])).
\end{aligned}$$

Therefore, g_* is a homomorphism. By the Theorem 3.5, $\Upsilon_X(Y, y_0, \upsilon_Y) = [(X, x_0, \upsilon_X),(Y, y_0, \upsilon_Y)]$ is a group with the binary operation Δ . Let show that Υ_X is a covariant functor.
Let $[1_Y] \in [(Y, y_0, \upsilon_Y),(Y, y_0, \upsilon_Y)]$ be the unit morphism of $\mathcal{CHC}$. Then,

$$\Upsilon_X([1_Y] = (1_Y)_* : [(X, x_0, \upsilon_X),(Y, y_0, \upsilon_Y)] \to [(X, x_0, \upsilon_X),(Y, y_0, \upsilon_Y)]$$

and for any morphism $[f] \in [(X, x_0, \upsilon_X),(Y, y_0, \upsilon_Y)]$, $(1_X)_*([f]) = [f\circ 1_X] = [f]$. So $\Upsilon_X([1_X])$ is the unit morphism.

Let $[f] \in [(Z, z_0, \upsilon_Z), (Y, y_0, \upsilon_Y)]$ and $[g] \in [(W, w_0, \upsilon_W), (Z, z_0, \upsilon_Z)]$. Then,

$$\begin{aligned} \Upsilon_X([f \circ g])([h]) &= [(f \circ g) \circ h] = [f \circ (g \circ h)] \\ &= \Upsilon_X([f])([g \circ h)] \\ &= \Upsilon_X([f])(\Upsilon_X([g])([h])) \\ &= (\Upsilon_X([f]) \circ \Upsilon_X([g]))([h]) \end{aligned}$$

for any morphism $[h] \in [(X, x_0, \upsilon_X), (W, w_0, \upsilon_W)]$. Then, $\Upsilon_X([f \circ g]) = \Upsilon([f]) \circ \Upsilon_X([g])$, so Υ_X is a convariant functor. □

By Theorem 3.5 and 3.6, we get that result: There exists a covariant functor from $\mathcal{CHC}$ to the category of abelian groups and homomorphisms.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The author has read and agreed to the published version of the manuscript.

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