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Research Article

A study on the tangent bundle with the vertical generalized Berger type deformed Sasaki metric

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Abstract

In this paper, we introduce a vertical generalized Berger type deformed Sasaki metric on the tangent bundle TM over an anti-paraKähler manifold as a new natural metric. Firstly, we investigate the Levi-Civita connection of this metric and then we calculate all forms of the Riemannian curvature tensors. Also, we present some results concerning curvature properties. Finally, we study the geometry of φ-unit tangent bundle equipped with the vertical generalized Berger type deformed Sasaki metric.

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1. Introduction

On the tangent bundle of a Riemannian manifold with a (pseudo-)Riemannian metric, different Riemannian and pseudo-Riemannian metrics can be constructed by using the natural lifts of a (pseudo-)Riemannian metric. That is exactly why metrics are referred to as natural metrics. The Sasaki metric on the tangent bundle, the most well-known of these metrics, was developed by Sasaki [11]. The geometry of (co)tangent bundle or tensor bundle equipped with the Sasaki metric has been studied by many authors such as Yano and Ishihara [15], Dombrowski [6], Salimov, Gezer and Akbulut [9] etc. Problems studied on the (co)tangent or tensor bundle with the Sasaki metric often result in the base manifold being locally flat. This is the reason why some authors have attempted to search for different metrics on the tangent bundle which are different deformations of the Sasaki metric. In this direction, some authors defined and studied some natural metrics which are called as Cheeger-Gromoll metric [7, 8, 12] or Kaluza-Klein metric [3] or Berger type deformed Sasaki metric [4, 13] or more generally g-natural metrics [1, 2]. For deformations of the Sasaki metric, we also refer to [4, 5, 16–22].

The main idea in this note consists of a new type deformation of Sasaki metric on the tangent bundle over an anti-paraKähler manifold. Firstly, we introduce the vertical

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generalized Berger type deformed Sasaki metric on the tangent bundle TM over an anti-paraKähler manifold (M^{2m}, φ, g) and investigate the Levi-Civita connection of this metric (Theorem 3.6, Lemma 3.7 and Proposition 3.9). Then, we establish all formulas of the Riemannian curvature tensors (Theorem 4.1) and also for the Ricci curvature (Proposition 4.3 and Corollary 4.4) and characterize the sectional curvature (Theorem 4.5 and Lemma 4.6) and also the scalar curvature (Theorem 4.7). In the last section, we study the geometry of φ-unit tangent bundle equipped with vertical generalized Berger type deformed Sasaki metric. Also, we present the formulas of the Levi-Civita connection (Theorem 5.2) and all formulas of the Riemannian curvature tensors (Theorem 5.3).

2. Preliminaries

Let TM be the tangent bundle over an m-dimensional Riemannian manifold (M^m, g) and the natural projection $\pi : TM \to M$. A local chart $(U, x^i)_{i=\overline{1,m}}$ on M^m induces a local chart $(\pi^{-1}(U), x^i, u^i)_{i=\overline{1,m}}$ on TM. Denote by Γ_{ij}^k the Christoffel symbols of g and by ∇ the Levi-Civita connection of g.

The Levi Civita connection ∇ induces a natural splitting of the tangent space $T_{(x,u)}TM$ into its vertical and horizontal subspaces

$$T_{(x,u)}TM = V_{(x,u)}TM \oplus H_{(x,u)}TM, \tag{2.1}$$

where

$$V_{(x,u)}TM \quad = \quad Ker(d\pi_{(x,u)}) = \{\xi^i \frac{\partial}{\partial u^i}|_{(x,u)},\ \xi^i \in \mathbb{R}\} \tag{2.2}$$

and

$$H_{(x,u)}TM \quad = \quad \{\xi^i \frac{\partial}{\partial x^i}|_{(x,u)} - \xi^i u^j \Gamma_{ij}^k \frac{\partial}{\partial u^k}|_{(x,u)},\ \xi^i \in \mathbb{R}\}. \tag{2.3}$$

Note that the map $\xi \to {}^H\xi = \xi^i \frac{\partial}{\partial x^i}|_{(x,u)} - \xi^i u^j \Gamma_{ij}^k \frac{\partial}{\partial u^k}|_{(x,u)}$ is an isomorphism between the vector spaces T_xM and $H_{(x,u)}TM$. Similarly, the map $\xi \to {}^V\xi = \xi^i \frac{\partial}{\partial u^i}|_{(x,u)}$ is an isomorphism between the vector spaces T_xM and $V_{(x,u)}TM$. Obviously, each tangent vector $Z \in T_{(x,u)}TM$ can be written in the form $Z = {}^HX + {}^VY$, where $X, Y \in T_xM$ are uniquely determined vectors.

Let $X = X^i \frac{\partial}{\partial x^i}$ be a local vector field on M^m. The vertical and the horizontal lifts of X are defined by

$$\begin{aligned} {}^VX &= X^i \frac{\partial}{\partial u^i}, \\ {}^HX &= X^i (\frac{\partial}{\partial x^i} - u^j \Gamma_{ij}^k \frac{\partial}{\partial u^k}). \end{aligned} \tag{2.4}$$

We have ${}^H(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial x^i} - u^j \Gamma_{ij}^k \frac{\partial}{\partial u^k}$ and ${}^V(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial u^i}$, then $({}^H(\frac{\partial}{\partial x^i}), {}^V(\frac{\partial}{\partial x^i}))_{i=\overline{1,m}}$ is a local adapted frame on TTM.

In particular, if U be a constant local vector field on each fiber T_xM such that $(U = u)$, the vertical lift U^V is called the canonical vertical vector field or Liouville vector field on TM.

The bracket operations of vertical and horizontal vector fields are given by the formulas [6, 15]

$$\left\{ \begin{array}{l} \left[{}^HX, {}^HY\right] = {}^H[X,Y] - {}^V(R(X,Y)u), \\ \left[{}^HX, {}^VY\right] = {}^V(\nabla_X Y), \\ \left[{}^VX, {}^VY\right] = 0 \end{array} \right. \tag{2.5}$$

for all vector fields X and Y on M^m, where R is the Riemannian curvature tensor of g defined by

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

3. Vertical generalized Berger type deformed Sasaki metric

Let M be a $2m$-dimensional Riemannian manifold with a Riemannian metric g. An almost paracomplex manifold is an almost product manifold (M^{2m}, φ), $\varphi^2 = id$, such that the two eigenbundles T^+M and T^-M associated to the two eigenvalues $+1$ and -1 of φ, respectively, have the same rank.

A Riemannian metric g is said to be an anti-paraHermitian metric if

$$g(\varphi X, \varphi Y) = g(X, Y) \tag{3.1}$$

or equivalently (purity condition), (B-metric)[10]

$$g(\varphi X, Y) = g(X, \varphi Y) \tag{3.2}$$

for all vector fields X, Y on M^{2m}.

If (M^{2m}, φ) is an almost paracomplex manifold with an anti-paraHermitian metric g, then the triple (M^{2m}, φ, g) is said to be an almost anti-paraHermitian manifold (an almost B-manifold)[10]. Moreover, (M^{2m}, φ, g) is said to be anti-paraKähler manifold (B-manifold)[10] if φ is parallel with respect to the Levi-Civita connection ∇ of g, i.e., $(\nabla\varphi = 0)$.

As is well known, the anti-paraKähler condition $(\nabla\varphi = 0)$ is equivalent to paraholomorphicity of the anti-paraHermitian metric g, i.e., $(\phi_\varphi g) = 0$, where ϕ_φ is the Tachibana operator [10, 14].

It is well known that if (M^{2m}, φ, g) is a anti-paraKähler manifold, the Riemannian curvature tensor is pure [10] and

$$\begin{cases} R(\varphi Y, Z) &= R(Y, \varphi Z) = R(Y, Z)\varphi = \varphi R(Y, Z), \\ R(\varphi Y, \varphi Z) &= R(Y, Z), \end{cases} \tag{3.3}$$

for all vector fields Y, Z on M^{2m}.

Definition 3.1. Let (M^{2m}, φ, g) be an almost anti-paraHermitian manifold and TM be its tangent bundle. A fiber-wise generalized Berger type deformation of the Sasaki metric on TM is defined by

$$\begin{aligned} \widetilde{G}({}^HX, {}^HY) &= g(X, Y), \\ \widetilde{G}({}^VX, {}^HY) &= \widetilde{G}({}^HX, {}^VY) = 0, \\ \widetilde{G}({}^VX, {}^VY) &= g(X, Y) + f g(X, \varphi u) g(Y, \varphi u), \end{aligned}$$

for all vector fields X, Y on M^{2m}, where $f : M \to [0, +\infty[$ is a positive smooth function on M^{2m}.

If $f = 0$, $\widetilde{G}$ is the Sasaki metric [11].

If $f = \delta^2$, $\widetilde{G}$ is the Berger type deformed Sasaki metric [4].

Remark 3.2. Let U be a constant local vector field on each fiber T_xM, such that $U_x = u$ and $(x, u) \in TM$, then for any vector field X on M^{2m} we have

$$\widetilde{G}({}^VX, {}^V(\varphi U)) = (1 + f g(u, u)) g(X, \varphi u).$$

Subsequently, we consider $f \neq 0$, $\lambda = 1 + f r^2$, where $r^2 = g(u, u) = |u|^2$ and $|.|$ denotes the norm with respect to g.

Lemma 3.3 ([1]). *Let (M^{2m}, g) be a Riemannian manifold and $\rho : \mathbb{R} \to \mathbb{R}$ a smooth function. Then we have the followings*

(1) ${}^HX(\rho(r^2)) = 0$,
(2) ${}^VX(\rho(r^2)) = 2\rho'(r^2) g(X, u)$,
(3) ${}^HX g(Y, u) = g(\nabla_X Y, u)$,
(4) ${}^VX g(Y, u) = g(X, Y)$

for all vector fields X, Y on M^{2m}, where $r^2 = g(u, u)$.

Lemma 3.4. *Let (M^{2m},φ,g) be an anti-paraKähler manifold. Then, we have the followings*

(1) ${}^{H}X(g(u,\varphi u))=0$,
(2) ${}^{V}X(g(u,\varphi u))=2g(X,\varphi u)$,
(3) ${}^{H}X(g(Y,\varphi u))=g(\nabla_X Y,\varphi u)$,
(4) ${}^{V}X(g(Y,\varphi u))=g(X,\varphi Y)$,
(5) ${}^{H}(\varphi U)(g(Y,\varphi u))=g(\nabla_{\varphi U}Y,\varphi u)$,
(6) ${}^{V}(\varphi U)(g(Y,\varphi u))=g(Y,U)$

for all vector fields X,Y on M^{2m}.

Proof. From (2.4), we obtain

$$
\begin{aligned}
(1)\ {}^{H}X(g(u,\varphi u)) &= X^i\frac{\partial}{\partial x^i}(g(u,\varphi u)) - u^s\Gamma^i_{sk}X^k\frac{\partial}{\partial u^i}(g(u,\varphi u))\\
&= X^i\frac{\partial}{\partial x^i}(g_{lj}u^l\varphi^j_t u^t) - u^s\Gamma^i_{sk}X^k\frac{\partial}{\partial u^i}(g_{lj}u^l\varphi^j_t u^t)\\
&= Xg(U,\varphi U) - g_{ij}u^s\Gamma^i_{sk}X^k\varphi^j_t u^t - g_{lj}u^l\varphi^j_i u^s\Gamma^i_{sk}X^k\\
&= Xg(U,\varphi U) - g(\nabla_X U,\varphi U) - g(U,\varphi\nabla_X U)\\
&= 0.\\
(2)\ {}^{V}X(g(u,\varphi u)) &= X^i\frac{\partial}{\partial u^i}(g(u,\varphi u))\\
&= X^i\frac{\partial}{\partial u^i}(g_{lj}u^l\varphi^j_t u^t)\\
&= X^i(g_{ij}\varphi^j_t u^t + g_{lj}u^l\varphi^j_i)\\
&= g(X,\varphi u) + g(u,\varphi X)\\
&= 2g(X,\varphi u).
\end{aligned}
$$

$$
\begin{aligned}
(3)\ {}^{H}X(g(Y,\varphi u)) &= X^i\frac{\partial}{\partial x^i}(g_{lj}Y^l\varphi^j_t u^t) - u^s\Gamma^i_{sk}X^k\frac{\partial}{\partial u^i}(g_{lj}Y^l\varphi^j_t u^t)\\
&= X(g(Y,\varphi U)) - u^s\Gamma^i_{sk}X^k g_{lj}Y^l\varphi^j_t\delta^t_i\\
&= g(\nabla_X Y,\varphi U) + g(Y,\nabla_X(\varphi U)) - g_{lj}Y^l\varphi^j_i u^s\Gamma^i_{sk}X^k\\
&= g(\nabla_X Y,\varphi u) + g(Y,\nabla_X(\varphi U)) - g(Y,\varphi(\nabla_X U))\\
&= g(\nabla_X Y,\varphi u).\\
(4)\ {}^{V}X(g(Y,\varphi u)) &= X^i\frac{\partial}{\partial u^i}(g_{lj}Y^l\varphi^j_k u^k) = X^i g_{lj}Y^l\varphi^j_k\delta^k_i = g_{lj}Y^l\varphi^j_i X^i\\
&= g(Y,\varphi X)\\
&= g(X,\varphi Y).
\end{aligned}
$$

□

Lemma 3.5. *Let (M^{2m},φ,g) be an anti-paraKähler manifold and $(TM,\widetilde{G})$ its tangent bundle equipped with the vertical generalized Berger type deformed Sasaki metric. We have the followings*

$$
\begin{aligned}
(1)\ {}^{H}X\left(\widetilde{G}({}^{H}Y,{}^{H}Z)\right) &= X(g(Y,Z)),\\
(2)\ {}^{V}X\left(\widetilde{G}({}^{H}Y,{}^{H}Z)\right) &= 0,\\
(3)\ {}^{H}X\left(\widetilde{G}({}^{V}Y,{}^{V}Z)\right) &= \widetilde{G}({}^{V}(\nabla_X Y),{}^{V}Z) + \widetilde{G}({}^{V}Y,{}^{V}(\nabla_X Z))\\
&\quad + X(f)g(Y,\varphi u)g(Z,\varphi u),\\
(4)\ {}^{V}X\left(\widetilde{G}({}^{V}Y,{}^{V}Z)\right) &= f\left(g(X,\varphi Y)g(Z,\varphi u) + g(Y,\varphi u)g(X,\varphi Z)\right)
\end{aligned}
$$

for all vector fields X, Y *and* Z *on* M^{2m}.

Proof. The results directly follow from Definition 3.1 and Lemma 3.4. □

We shall calculate the Levi-Civita connection $\widetilde{\nabla}$ of TM with the vertical generalized Berger type deformed Sasaki metric $\widetilde{G}$. This connection is characterized by the Koszul formula

$$\begin{aligned} 2\widetilde{G}(\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y},\widetilde{Z}) &= \widetilde{X}\left(\widetilde{G}(\widetilde{Y},\widetilde{Z})\right)+\widetilde{Y}\left((\widetilde{G}(\widetilde{Z},\widetilde{X})\right)-\widetilde{Z}\left(\widetilde{G}(\widetilde{X},\widetilde{Y})\right) \\ &\quad +\widetilde{G}(\widetilde{Z},[\widetilde{X},\widetilde{Y}])+\widetilde{G}(\widetilde{Y},[\widetilde{Z},\widetilde{X}])-\widetilde{G}(\widetilde{X},[\widetilde{Y},\widetilde{Z}]) \end{aligned} \tag{3.4}$$

for all vector fields $\widetilde{X}, \widetilde{Y}$ and $\widetilde{Z}$ on TM.

Theorem 3.6. *Let* (M^{2m}, φ, g) *be an anti-paraKähler manifold and* $(TM, \widetilde{G})$ *its tangent bundle equipped with the vertical generalized Berger type deformed Sasaki metric. Then we have the followings*

$$\begin{aligned} 1.\ \widetilde{\nabla}_{^HX}{}^HY &= {}^H(\nabla_XY)-\frac{1}{2}{}^V(R(X,Y)u), \\ 2.\ \widetilde{\nabla}_{^HX}{}^VY &= \frac{1}{2}{}^H(R(u,Y)X)+{}^V(\nabla_XY)+\frac{1}{2\lambda}X(f)g(Y,\varphi u)^V(\varphi U), \\ 3.\ \widetilde{\nabla}_{^VX}{}^HY &= \frac{1}{2}{}^H(R(u,X)Y)+\frac{1}{2\lambda}Y(f)g(X,\varphi u)^V(\varphi U), \\ 4.\ \widetilde{\nabla}_{^VX}{}^VY &= -\frac{1}{2}g(X,\varphi u)g(Y,\varphi u)^H(grad f)+\frac{f}{\lambda}g(X,\varphi Y)^V(\varphi U) \end{aligned}$$

for all vector fields X, Y *on* M^{2m}*, where* ∇ *is the Levi-Civita connection and* R *is its Riemannian curvature tensor of* (M^{2m}, φ, g).

Proof. In the proof, we will use (3.3), Koszul formula (3.4) and Lemma 3.5.
1. With direct calculations we get

$$\begin{aligned} 2\widetilde{G}(\widetilde{\nabla}_{^HX}{}^HY,{}^HZ) &= {}^HX\left(\widetilde{G}({}^HY,{}^HZ)\right)+{}^HY\left(\widetilde{G}({}^HZ,{}^HX)\right) \\ &\quad -{}^HZ\left(\widetilde{G}({}^HX,{}^HY)\right)+\widetilde{G}({}^HZ,[{}^HX,{}^HY]) \\ &\quad +\widetilde{G}({}^HY,[{}^HZ,{}^HX])-\widetilde{G}({}^HX,[{}^HY,{}^HZ]) \\ &= Xg(Y,Z)+Yg(Z,X)-Zg(X,Y)+g(Z,[X,Y]) \\ &\quad +g(Y,[Z,X])-g(X,[Y,Z]) \\ &= 2g(\nabla_XY,Z) \\ &= 2\widetilde{G}({}^H(\nabla_XY),{}^HZ) \end{aligned}$$

and

$$\begin{aligned} 2\widetilde{G}(\widetilde{\nabla}_{^HX}{}^HY,{}^VZ) &= {}^HX\left(\widetilde{G}({}^HY,{}^VZ)\right)+{}^HY\left(\widetilde{G}({}^VZ,{}^HX)\right) \\ &\quad -{}^VZ\left(\widetilde{G}({}^HX,{}^HY)\right)+\widetilde{G}({}^VZ,[{}^HX,{}^HY]) \\ &\quad +\widetilde{G}({}^HY,[{}^VZ,{}^HX])-\widetilde{G}({}^HX,[{}^HY,{}^VZ]) \\ &= \widetilde{G}({}^VZ,[{}^HX,{}^HY]) \\ &= -\widetilde{G}({}^V(R(X,Y)u),{}^VZ). \end{aligned}$$

Also, we have

$$\widetilde{\nabla}_{^HX}{}^HY = {}^H(\nabla_XY)-\frac{1}{2}{}^V(R(X,Y)u).$$

2. Calculations similar to that above give

$$\begin{aligned}
2\widetilde{G}(\widetilde{\nabla}_{{}^{H}X}{}^{V}Y, {}^{H}Z) &= {}^{H}X\left(\widetilde{G}({}^{V}Y, {}^{H}Z)\right) + {}^{V}Y\left(\widetilde{G}({}^{H}Z, {}^{H}X)\right) \\
&\quad - {}^{H}Z\left(\widetilde{G}({}^{H}X, {}^{V}Y)\right) + \widetilde{G}({}^{H}Z, [{}^{H}X, {}^{V}Y]) \\
&\quad + \widetilde{G}({}^{V}Y, [{}^{H}Z, {}^{H}X]) - \widetilde{G}({}^{H}X, [{}^{V}Y, {}^{H}Z]) \\
&= \widetilde{G}({}^{V}Y, [{}^{H}Z, {}^{H}X]) \\
&= -\widetilde{G}({}^{V}(R(Z,X)u), {}^{V}Y) \\
&= -g(R(Z,X)u, Y) - fg(Y, \varphi u)g(R(Z,X)u, \varphi u) \\
&= \widetilde{G}({}^{H}(R(u,Y)X), {}^{H}Z).
\end{aligned}$$

In here

$$-g(R(Z,X)u, Y) = g(R(u,Y)X, Z) = \tilde{g}({}^{H}(R(u,Y)X), {}^{H}Z).$$

Since the Riemannian curvature tensor field is pure with respect to φ, then we have

$$g(R(Z,X)u, \varphi u) = g(R(\varphi Z, X)u, u) = 0.$$

We calculate

$$\begin{aligned}
2\widetilde{G}(\widetilde{\nabla}_{{}^{H}X}{}^{V}Y, {}^{V}Z) &= {}^{H}X\left(\widetilde{G}({}^{V}Y, {}^{V}Z)\right) + {}^{V}Y\left(\widetilde{G}({}^{V}Z, {}^{H}X)\right) \\
&\quad - {}^{V}Z\left(\widetilde{G}({}^{H}X, {}^{V}Y)\right) + \widetilde{G}({}^{V}Z, [{}^{H}X, {}^{V}Y]) \\
&\quad + \widetilde{G}({}^{V}Y, [{}^{V}Z, {}^{H}X]) - \widetilde{G}({}^{H}X, [{}^{V}Y, {}^{V}Z]) \\
&= {}^{H}X\widetilde{G}({}^{V}Y, {}^{V}Z) + \widetilde{G}({}^{V}Z, [{}^{H}X, {}^{V}Y]) + \widetilde{G}({}^{V}Y, [{}^{V}Z, {}^{H}X]) \\
&= \widetilde{G}({}^{V}(\nabla_X Y), {}^{V}Z) + \widetilde{G}({}^{V}Y, {}^{V}(\nabla_X Z)) \\
&\quad + X(f)g(Y, \varphi u)g(Z, \varphi u) \\
&\quad + \widetilde{G}({}^{V}Z, {}^{V}(\nabla_X Y)) - \widetilde{G}({}^{V}Y, {}^{V}(\nabla_X Z)) \\
&= 2\widetilde{G}({}^{V}(\nabla_X Y), {}^{V}Z) + \frac{1}{\lambda}X(f)g(Y, \varphi u)\widetilde{G}({}^{V}(\varphi U), {}^{V}Z).
\end{aligned}$$

So, we see that

$$\widetilde{\nabla}_{{}^{H}X}{}^{V}Y = \frac{1}{2}{}^{H}(R(u,Y)X) + {}^{V}(\nabla_X Y) + \frac{1}{2\lambda}X(f)g(Y, \varphi u){}^{V}(\varphi U).$$

The other formulas are obtained by similar calculations. □

Lemma 3.7. *Let (M^{2m}, φ, g) be an anti-paraKähler manifold and $(TM, \widetilde{G})$ its tangent bundle equipped with the vertical generalized Berger type deformed Sasaki metric. Then we have*

$$\begin{aligned}
1.\ \widetilde{\nabla}_{{}^{H}X}{}^{V}(\varphi U) &= \frac{\lambda - 1}{2f\lambda}X(f){}^{V}(\varphi U), \\
2.\ \widetilde{\nabla}_{{}^{V}(\varphi U)}{}^{H}X &= \frac{\lambda - 1}{2f\lambda}X(f){}^{V}(\varphi U), \\
3.\ \widetilde{\nabla}_{{}^{V}X}{}^{V}(\varphi U) &= \frac{1-\lambda}{2f}g(X, \varphi u){}^{H}(grad f) + {}^{V}(\varphi X) + \frac{f}{\lambda}g(X, u){}^{V}(\varphi U), \\
4.\ \widetilde{\nabla}_{{}^{V}(\varphi U)}{}^{V}X &= \frac{1-\lambda}{2f}g(X, \varphi u){}^{H}(grad f) + \frac{f}{\lambda}g(X, u){}^{V}(\varphi U), \\
5.\ \widetilde{\nabla}_{{}^{V}(\varphi U)}{}^{V}(\varphi U) &= -\frac{(\lambda - 1)^2}{2f^2}{}^{H}(grad f) + \frac{f}{\lambda}g(u, \varphi u){}^{V}(\varphi U) + {}^{V}U
\end{aligned}$$

for any vector field X on M^{2m}.

Definition 3.8. Let (M,g) be a Riemannian manifold and $F: TM \to TM$ be a smooth bundle endomorphism of TM. Then the vertical and horizontal vector fields ${}^VF$ and ${}^HF$ are respectively defined on TM by

$$\begin{array}{rccc} {}^VF: & TM & \to & TTM \\ & (x,u) & \mapsto & {}^V(Fu)_x \end{array}, \quad \begin{array}{rccc} {}^HF: & TM & \to & TTM \\ & (x,u) & \mapsto & {}^H(Fu)_x. \end{array}$$

Locally we have

$$ {}^V(Fu) = u^j\, {}^V(F(\frac{\partial}{\partial x^j})), \tag{3.5}$$

$$ {}^H(Fu) = u^j\, {}^H(F(\frac{\partial}{\partial x^j})). \tag{3.6}$$

Proposition 3.9. *Let (M^{2m},φ,g) be an anti-paraKähler manifold and $(TM,\widetilde{G})$ its tangent bundle equipped with the vertical generalized Berger type deformed Sasaki metric. Then we have the following formulas*

$$\begin{aligned}
1.\ \widetilde{\nabla}_{{}^HX}{}^H(Fu) &= {}^H((\nabla_X F)u) - \frac{1}{2}{}^V(R(X,Fu)u),\\
2.\ \widetilde{\nabla}_{{}^HX}{}^V(Fu) &= \frac{1}{2}{}^H(R(u,Fu)X) + {}^V((\nabla_X F)u) + \frac{1}{2\lambda}X(f)g(Fu,\varphi u)\,{}^V(\varphi U),\\
3.\ \widetilde{\nabla}_{{}^VX}{}^H(Fu) &= {}^H(FX) + \frac{1}{2}{}^H(R(u,X)Fu) + \frac{1}{2\lambda}g(X,\varphi u)g(Fu,grad f)\,{}^V(\varphi U),\\
4.\ \widetilde{\nabla}_{{}^VX}{}^V(Fu) &= -\frac{1}{2}g(X,\varphi u)g(Fu,\varphi u)\,{}^H(grad f) + {}^V(FX)\\
&\quad + \frac{f}{\lambda}g(\varphi X,Fu)\,{}^V(\varphi U)
\end{aligned}$$

for any vector field X on M^{2m}.

Proof. The results come directly from Theorem 3.6. □

4. Curvatures of vertical generalized Berger type deformed Sasaki metric

We shall calculate the Riemannian curvature tensor $\widetilde{R}$ of TM with the vertical generalized Berger type deformed Sasaki metric $\widetilde{G}$. The Riemannian curvature tensor is characterized by the formula

$$\widetilde{R}(\widetilde{X},\widetilde{Y})\widetilde{Z} = \widetilde{\nabla}_{\widetilde{X}}\widetilde{\nabla}_{\widetilde{Y}}\widetilde{Z} - \widetilde{\nabla}_{\widetilde{Y}}\widetilde{\nabla}_{\widetilde{X}}\widetilde{Z} - \widetilde{\nabla}_{[\widetilde{X},\widetilde{Y}]}\widetilde{Z} \tag{4.1}$$

for all vector fields $\widetilde{X},\widetilde{Y}$ and $\widetilde{Z}$ on TM.

Theorem 4.1. *Let (M^{2m},φ,g) be an anti-paraKähler manifold and $(TM,\widetilde{G})$ its tangent bundle equipped with the vertical generalized Berger type deformed Sasaki metric. We have the following formulas*

$$\begin{aligned}
\widetilde{R}({}^HX,{}^HY){}^HZ &= {}^H(R(X,Y)Z) + \frac{1}{2}{}^H(R(u,R(X,Y)u)Z)\\
&\quad + \frac{1}{4}{}^H(R(u,R(X,Z)u)Y) - \frac{1}{4}{}^H(R(u,R(Y,Z)u)X)\\
&\quad + \frac{1}{2}{}^V((\nabla_Z R)(X,Y)u),
\end{aligned} \tag{4.2}$$

$$\begin{aligned}
\widetilde{R}({}^HX,{}^VY){}^HZ &= \frac{1}{2}{}^H((\nabla_X R)(u,Y)Z) - \frac{1}{4}{}^V(R(X,R(u,Y)Z)u)\\
&\quad + \frac{1}{2}{}^V(R(X,Z)Y) + \frac{1}{2\lambda}Hess_f(X,Z)g(Y,\varphi u)\,{}^V(\varphi U)\\
&\quad + \frac{1-\lambda}{4f\lambda^2}X(f)Z(f)g(Y,\varphi u)\,{}^V(\varphi U) - \frac{f}{2\lambda}g(R(X,Z)Y,\varphi u)\,{}^V(\varphi U),
\end{aligned} \tag{4.3}$$

$$\begin{aligned}\widetilde{R}(^HX,{}^HY)^VZ &= \frac{1}{2}{}^H((\nabla_X R)(u,Z)Y) - \frac{1}{2}{}^H((\nabla_Y R)(u,Z)X) \\ &\quad -\frac{1}{4}{}^V(R(X,R(u,Z)Y)u) + \frac{1}{4}{}^V(R(Y,R(u,Z)X)u) \\ &\quad +{}^V(R(X,Y)Z) - \frac{f}{\lambda}g(R(X,Y)Z,\varphi u)^V(\varphi U), \end{aligned} \tag{4.4}$$

$$\begin{aligned}\widetilde{R}(^HX,{}^VY)^VZ &= -\frac{1}{4}{}^H(R(u,Y)R(u,Z)X) - \frac{1}{2}g(Y,\varphi u)g(Z,\varphi u)^H(\nabla_X gradf) \\ &\quad -\frac{1}{2}{}^H(R(Y,Z)X) + \frac{\lambda-1}{4f\lambda}X(f)g(Y,\varphi u)g(Z,\varphi u)^H(gradf) \\ &\quad -\frac{1}{2\lambda}X(f)g(Z,\varphi u)^V(\varphi Y) + \frac{1}{2\lambda^2}X(f)\widetilde{G}(^V(\varphi Y),{}^VZ)^V(\varphi U) \\ &\quad -\frac{1}{4\lambda}g(Y,\varphi u)g(R(u,Z)X,gradf)^V(\varphi U) \\ &\quad +\frac{1}{4}g(Y,\varphi u)g(Z,\varphi u)^V(R(X,gradf)u), \end{aligned} \tag{4.5}$$

$$\begin{aligned}\widetilde{R}(^VX,{}^VY)^HZ &= \frac{1}{4}{}^H(R(u,X)R(u,Y)Z) - \frac{1}{4}{}^H(R(u,Y)R(u,X)Z) \\ &\quad +{}^H(R(X,Y)Z) + \frac{1}{2\lambda}Z(f)(g(Y,\varphi u)^V(\varphi X) - g(X,\varphi u)^V(\varphi Y)) \\ &\quad +\frac{f}{2\lambda^2}Z(f)(g(X,\varphi u)g(Y,u) - g(X,u)g(Y,\varphi u))^V(\varphi U) \\ &\quad +\frac{1}{4\lambda}g(X,\varphi u)g(R(u,Y)Z,gradf)^V(\varphi U) \\ &\quad -\frac{1}{4\lambda}g(Y,\varphi u)g(R(u,X)Z,gradf)^V(\varphi U), \end{aligned} \tag{4.6}$$

$$\begin{aligned}\widetilde{R}(^VX,{}^VY)^VZ &= \frac{1}{2\lambda}(g(X,\varphi u)g(Y,\varphi Z) - g(Y,\varphi u)g(X,\varphi Z))^H(gradf) \\ &\quad +\frac{1}{4}g(X,\varphi u)g(Z,\varphi u)^H(R(u,Y)gradf) \\ &\quad -\frac{1}{4}g(Y,\varphi u)g(Z,\varphi u)^H(R(u,X)gradf) \\ &\quad +\frac{f^2}{\lambda^2}(g(Y,u)g(X,\varphi Z) - g(X,u)g(Y,\varphi Z))^V(\varphi U) \\ &\quad +\frac{f}{\lambda}(g(Y,\varphi Z)^V(\varphi X) - g(X,\varphi Z)^V(\varphi Y)) \end{aligned} \tag{4.7}$$

for all vector fields X, Y and Z on M^{2m}, where $Hess_f(X,Z) = g(\nabla_X gradf, Z)$.

Proof. In the proof, we will use Theorem 3.6 and Lemma 3.7, Proposition 3.9,
1) Let $F : TM \to TM$ be the bundle endomorphism given by $Fu = R(Y,Z)u$. The direct calculations give

$$\begin{aligned}\widetilde{\nabla}_{^HX}\widetilde{\nabla}_{^HY}{}^HZ &= \widetilde{\nabla}_{^HX}(^H(\nabla_Y Z) - \frac{1}{2}{}^V(Fu)) \\ &= {}^H(\nabla_X\nabla_Y Z) - \frac{1}{2}{}^V(R(X,\nabla_Y Z)u) - \frac{1}{4}{}^H(R(u,R(Y,Z)u)X) \\ &\quad -\frac{1}{2}{}^V(\nabla_X(R(Y,Z)u)) + \frac{1}{2}{}^V(R(Y,Z)(\nabla_X U)), \end{aligned} \tag{4.8}$$

from which, with permutation of X by Y, we get

$$\widetilde{\nabla}_{^HY}\widetilde{\nabla}_{^HX}{}^HZ = {}^H(\nabla_Y\nabla_X Z) - \frac{1}{2}{}^V(R(Y,\nabla_X Z)u) - \frac{1}{4}{}^H(R(u,R(X,Z)u)Y) - \frac{1}{2}{}^V(\nabla_Y(R(X,Z)u)) + \frac{1}{2}{}^V(R(X,Z)(\nabla_Y U)). \quad (4.9)$$

Also, we find

$$\begin{aligned}\widetilde{\nabla}_{[^HX,^HY]}{}^HZ &= \widetilde{\nabla}_{^H[X,Y]}{}^HZ - \widetilde{\nabla}_{^V(R(X,Y)u)}{}^HZ \\ &= {}^H(\nabla_{[X,Y]}Z) - \frac{1}{2}{}^V(R([X,Y],Z)u) \\ &\quad -\frac{1}{2}{}^H(R(u,R(X,Y)u)Z). \end{aligned} \quad (4.10)$$

From the formulas (4.8), (4.9) and (4.10) we get

$$\begin{aligned}\widetilde{R}({}^HX,{}^HY){}^HZ &= {}^H(R(X,Y)Z) + \frac{1}{2}{}^H(R(u,R(X,Y)u)Z) \\ &\quad +\frac{1}{4}{}^H(R(u,R(X,Z)u)Y) - \frac{1}{4}{}^H(R(u,R(Y,Z)u)X) \\ &\quad +\frac{1}{2}{}^V((\nabla_Y R)(X,Z)u) - \frac{1}{2}{}^V((\nabla_X R)(Y,Z)u).\end{aligned}$$

Using the second Bianchi identity

$$(\nabla_X R)(Y,Z) + (\nabla_Y R)(Z,X) + (\nabla_Z R)(X,Y) = 0,$$

we obtain the formula (4.2).

2) Let $F : TM \to TM$ be the bundle endomorphism given by $Fu = R(u,Y)Z$. Hence we obtain

$$\begin{aligned}\widetilde{\nabla}_{^HX}\widetilde{\nabla}_{^VY}{}^HZ &= \widetilde{\nabla}_{^HX}\Big(\frac{1}{2}{}^H(Fu) + \frac{1}{2\lambda}Z(f)g(Y,\varphi u)^V(\varphi U)\Big) \\ &= \frac{1}{2}{}^H(\nabla_X(R(u,Y)Z)) - \frac{1}{2}{}^H(R(\nabla_X U,Y)Z) \\ &\quad -\frac{1}{4}{}^V(R(X,R(u,Y)Z)u) + \frac{1-\lambda}{4f\lambda^2}X(f)Z(f)g(Y,\varphi u)^V(\varphi U) \\ &\quad +\frac{1}{2\lambda}X(Z(f))g(Y,\varphi u)^V(\varphi U) + \frac{1}{2\lambda}Z(f)g(\nabla_X Y,\varphi u)^V(\varphi U).\end{aligned}$$

Let $F : TM \to TM$ be the bundle endomorphism given by $Fu = R(X,Z)u$. We get

$$\begin{aligned}\widetilde{\nabla}_{^VY}\widetilde{\nabla}_{^HX}{}^HZ &= \widetilde{\nabla}_{^VY}\Big({}^H(\nabla_X Z) - \frac{1}{2}{}^V(Fu)\Big) \\ &= \frac{1}{2}{}^H(R(u,Y)\nabla_X Z) + \frac{1}{2\lambda}(\nabla_X Z)(f)g(Y,\varphi u)^V(\varphi U) \\ &\quad -\frac{1}{2}{}^V(R(X,Z)Y) - \frac{f}{2\lambda}g(R(X,Z)u,\varphi Y)^V(\varphi U).\end{aligned}$$

Also

$$\begin{aligned}\widetilde{\nabla}_{[^HX,^VY]}{}^HZ &= \widetilde{\nabla}_{^V(\nabla_X Y)}{}^HZ \\ &= \frac{1}{2}{}^H(R(u,\nabla_X Y)Z) + \frac{1}{2\lambda}Z(f)g(\nabla_X Y,\varphi u)^V(\varphi U),\end{aligned}$$

which gives the formula (4.3).
3) Applying the formula (4.3) and 1^{st} Bianchi identity

$$\widetilde{R}({}^HX,{}^HY){}^VZ = \widetilde{R}({}^HX,{}^VZ){}^HY - \widetilde{R}({}^HY,{}^VZ){}^HX,$$

we find

$$\begin{aligned}\widetilde{R}({}^HX,{}^VZ){}^HY &= \frac{1}{2}{}^H((\nabla_X R)(u,Z)Y) - \frac{1}{4}{}^V(R(X,R(u,Z)Y)u)\\ &+\frac{1}{2}{}^V(R(X,Y)Z) + \frac{1}{2\lambda}Hess_f(X,Y)g(Z,\varphi u)^V(\varphi U)\\ &+\frac{1-\lambda}{4f\lambda^2}X(f)Y(f)g(Z,\varphi u)^V(\varphi U) - \frac{f}{2\lambda}g(R(X,Y)Z,\varphi u)^V(\varphi U)\end{aligned}$$

and

$$\begin{aligned}\widetilde{R}({}^HY,{}^VZ){}^HX &= \frac{1}{2}{}^H((\nabla_Y R)(u,Z)X) - \frac{1}{4}{}^V(R(Y,R(u,Z)X)u)\\ &+\frac{1}{2}{}^V(R(Y,X)Z) + \frac{1}{2\lambda}Hess_f(Y,X)g(Z,\varphi u)^V(\varphi U)\\ &+\frac{1-\lambda}{4f\lambda^2}Y(f)X(f)g(Z,\varphi u)^V(\varphi U) + \frac{f}{2\lambda}g(R(Y,X)Z,\varphi u)^V(\varphi U),\end{aligned}$$

which gives the formula (4.4).

The other formulas are obtained by similar calculations. We omit them to avoid repetition. □

Proposition 4.2. *Let (M^{2m},φ,g) be an anti-paraKähler manifold and $(TM,\widetilde{G})$ its tangent bundle equipped with the vertical generalized Berger type deformed Sasaki metric. If $(TM,\widetilde{G})$ is flat, then (M^{2m},φ,g) is flat.*

Proof. It is easy to see from the formula (4.2). If we assume that $\widetilde{R}=0$ and calculate the Riemannian curvature tensor for three horizontal vector fields at $(x,0)$ we get

$$\widetilde{R}_{(x,0)}({}^HX,{}^HY){}^HZ = {}^H(R_x(X,Y)Z) = 0.$$

□

Now let $(x,u)\in TM$ with $u\neq 0$ and $\{E_i\}_{i=\overline{1,2m}}$ be a local orthonormal frame on (M^{2m},φ,g) at x, such that $E_1=\dfrac{u}{|u|}$, then

$$\{F_i={}^HE_i, F_{2m+1}=\frac{1}{\sqrt{\lambda}}{}^V(\varphi E_1), F_{2m+j}={}^V(\varphi E_j)\}_{i=\overline{1,2m},j=\overline{2,2m}} \tag{4.11}$$

is a local orthonormal frame at (x,u) on TM.

Proposition 4.3. *Let (M^{2m},φ,g) be an anti-paraKähler manifold and $(TM,\widetilde{G})$ its tangent bundle equipped with the vertical generalized Berger type deformed Sasaki metric. If Ric (resp., $\widetilde{Ric}$) denotes the Ricci curvature of (M^{2m},φ,g) (resp. $(TM,\widetilde{G})$), then we have*

$$\begin{aligned} i)\ \widetilde{Ric}({}^HX,{}^HY) &= Ric(X,Y) - \frac{1}{2}\sum_{a=1}^{2m} g(R(E_a,X)u,R(E_a,Y)u)\\ &+\frac{1-\lambda}{2f\lambda}Hess_f(X,Y) + \frac{(1-\lambda)^2}{4f^2\lambda^2}X(f)Y(f),\end{aligned} \tag{4.12}$$

$$\begin{aligned} ii)\ \widetilde{Ric}({}^HX,{}^VY) &= \frac{1}{2}\sum_{a=1}^{2m} g((\nabla_{E_a}R)(u,Y)X,E_a) + \frac{\lambda-1}{4f\lambda}g(R(u,Y)X,grad f)\\ &-\frac{f}{2\lambda^2}X(f)g(Y,\varphi u)g(u,\varphi u) - \frac{1}{2\lambda^2}X(f)g(Y,u)\\ &+\frac{1}{2\lambda}X(f)g(Y,\varphi u)\sum_{a=1}^{2m} g(E_a,\varphi E_a),\end{aligned} \tag{4.13}$$

$$iii)\ \widetilde{Ric}({}^VX, {}^VY) = \frac{1}{4}\sum_{a=1}^{2m} g(R(u,X)E_a, R(u,Y)E_a) - \frac{1}{2}g(X,\varphi u)g(Y,\varphi u)\Delta(f) + \frac{\lambda-1}{4f\lambda}g(X,\varphi u)g(Y,\varphi u)|gradf|^2 + \frac{f^2}{\lambda^2}g(X,u)g(Y,u) - \frac{f^2}{\lambda^2}g(u,\varphi u)g(X,\varphi Y) - \frac{f}{\lambda}g(X,Y) + \frac{f}{\lambda}g(X,\varphi Y)\sum_{a=1}^{2m} g(E_a,\varphi E_a). \quad (4.14)$$

Proof. We will use the local orthonormal frame (4.11) on TM.
$i)$ From the formula (4.2), we have

$$\widetilde{Ric}({}^HX, {}^HY) = \sum_{a=1}^{2m} \widetilde{G}(\widetilde{R}({}^HE_a, {}^HX){}^HY, {}^HE_a) + \frac{1}{\lambda}\widetilde{G}(\widetilde{R}({}^V(\varphi E_1), {}^HX){}^HY, {}^V(\varphi E_1)) + \sum_{a=2}^{2m} \widetilde{G}(\widetilde{R}({}^V(\varphi E_a), {}^HX){}^HY, {}^V(\varphi E_a))$$

$$= \sum_{a=1}^{2m} \Big(g(R(E_a,X)Y,E_a) + \frac{1}{2}g(R(u,R(E_a,X)u)Y,E_a) + \frac{1}{4}g(R(u,R(E_a,Y)u)X,E_a) \Big) + \frac{1}{2\lambda}Hess_f(X,Y)g(\varphi E_1,\varphi u)\widetilde{G}({}^V(\varphi u), {}^V(\varphi E_1)) + \frac{1-\lambda}{4f\lambda^2}X(f)Y(f)\widetilde{G}({}^V(\varphi u), {}^V(\varphi E_1)) + \frac{1}{4}\sum_{a=2}^{2m} \widetilde{G}({}^V(R(X,R(u,\varphi E_a)Y)u), {}^V(\varphi E_a))$$

$$= Ric(X,Y) - \frac{3}{4}\sum_{a=1}^{2m} g(R(E_a,X)u, R(E_a,Y)u) + \frac{1-\lambda}{2f\lambda}Hess_f(X,Y) + \frac{(1-\lambda)^2}{4f^2\lambda^2}X(f)Y(f) + \frac{1}{4}\sum_{a=1}^{2m} g(R(u,E_a)X, R(u,E_a)Y)$$

and from

$$\sum_{a=1}^{2m} g(R(E_a,X)u, R(E_a,Y)u) = \sum_{a=1}^{2m} g(R(u,E_a)X, R(u,E_a)Y),$$

we get

$$\widetilde{Ric}({}^HX, {}^HY) = Ric(X,Y) - \frac{1}{2}\sum_{a=1}^{2m} g(R(E_a,X)u, R(E_a,Y)u) + \frac{1-\lambda}{2f\lambda}Hess_f(X,Y) + \frac{(1-\lambda)^2}{4f^2\lambda^2}X(f)Y(f).$$

The other formulas are obtained by similar calculations. □

Proposition 4.4. *Let (M^{2m}, φ, g) be an anti-paraKähler manifold. $(TM, \widetilde{G})$ is an Einstein manifold if and only if (M^{2m}, φ, g) is flat and $f = 0$.*

Proof. If we suppose that $(TM, \widetilde{G})$ is μ-Einstein, then

$$\widetilde{Ric}(\widetilde{X}, \widetilde{Y}) = \mu \widetilde{G}(\widetilde{X}, \widetilde{Y}).$$

Using the formula (4.14), we obtain $f = \mu = 0$ and $R = \widetilde{R} = 0$. □

In the following, let $\widetilde{Q}(V, W)$ denote the square of the area of the parallelogram with sides V and W for vector fields V, W on TM given by

$$\widetilde{Q}(V, W) = \widetilde{G}(V, V)\widetilde{G}(W, W) - \widetilde{G}(V, W)^2. \tag{4.15}$$

If $p \in TM$, V_p and W_p are linearly independent,

$$\widetilde{K}(V_p, W_p) = \frac{\widetilde{G}(\widetilde{R}(V, W)W, V)}{\widetilde{Q}(V_p, W_p)} \tag{4.16}$$

is the sectional curvature of the plane spanned by V_p and W_p.

Let $\widetilde{K}({}^HX, {}^HY)$, $\widetilde{K}({}^HX, {}^VY)$ and $\widetilde{K}({}^VX, {}^VY)$ denote the sectional curvature of the plane spanned by $\{{}^HX, {}^HY\}$, $\{{}^HX, {}^VY\}$ and $\{{}^VX, {}^VY\}$ on $(TM, \widetilde{G})$ respectively, where X, Y are vector fields on M^{2m}.

Theorem 4.5. *Let (M^{2m}, φ, g) be an anti-paraKähler manifold and $(TM, \widetilde{G})$ its tangent bundle equipped with the vertical generalized Berger type deformed Sasaki metric. Then the sectional curvature $\widetilde{K}$ satisfies the following equations*

$$\begin{aligned}
(1)\ \widetilde{K}({}^HX, {}^HY) &= K(X, Y) - \frac{3}{4(|X|^2|Y|^2 - g(X, Y)^2)}|R(X, Y)u|^2,\\
(2)\ \widetilde{K}({}^HX, {}^VY) &= \frac{1}{|X|^2(|Y|^2 + fg(Y, \varphi u)^2)}\big(\frac{1}{4}|R(u, Y)X|^2\\
&\quad -\frac{1}{2}g(Y, \varphi u)^2 Hess_f(X, X) + \frac{\lambda - 1}{4f\lambda}X(f)^2 g(Y, \varphi u)^2\big),\\
(3)\ \widetilde{K}({}^VX, {}^VY) &= \frac{f}{\lambda}\frac{(g(X, \varphi X)g(Y, \varphi Y) - g(X, \varphi Y)^2)}{\widetilde{Q}({}^VX, {}^VY)},
\end{aligned}$$

where K denotes the sectional curvature of (M^{2m}, φ, g).

Proof. On one hand we have

$$\begin{aligned}
i)\ \widetilde{Q}({}^HX, {}^HY) &= \widetilde{G}({}^HX, {}^HX)\widetilde{G}({}^HY, {}^HY) - \widetilde{G}({}^HX, {}^HY)^2\\
&= g(X, X)g(Y, Y) - g(X, Y)^2\\
&= |X|^2|Y|^2 - g(X, Y)^2,
\end{aligned}$$

$$\begin{aligned}
ii)\ \widetilde{Q}({}^HX, {}^VY) &= \widetilde{G}({}^HX, {}^HX)\widetilde{G}({}^VY, {}^VY) - \widetilde{G}({}^HX, {}^VY)^2\\
&= g(X, X)\big(g(Y, Y) + fg(Y, u)g(Y, \varphi u)\big)\\
&= |X|^2(|Y|^2 + fg(Y, \varphi u)^2),
\end{aligned}$$

$$\begin{aligned}
iii)\ \widetilde{Q}({}^VX, {}^VY) &= \widetilde{G}({}^VX, {}^VX)\widetilde{G}({}^VY, {}^VY) - \widetilde{G}({}^VX, {}^VY)^2\\
&= (|X|^2 + fg(X, \varphi u)^2)(|Y|^2 + fg(Y, \varphi u)^2)\\
&\quad -\big(g(X, Y) + fg(X, \varphi u)g(X, \varphi u)\big)^2\\
&= |X|^2|Y|^2 + f|X|^2 g(Y, \varphi u)^2 + f|Y|^2 g(X, \varphi u)^2\\
&\quad -g(X, Y)^2 - 2fg(X, Y)g(X, \varphi u)g(Y, \varphi u).
\end{aligned}$$

On the other hand, we have the followings.

i) From the formula (4.2), we get

$$\begin{aligned}\widetilde{G}(\widetilde{R}({}^{H}X,{}^{H}Y){}^{H}Y,{}^{H}X) &= g(R(X,Y)Y,X)+\frac{1}{2}g(R(u,R(X,Y)u)Y,X)\\ &\quad +\frac{1}{4}g(R(u,R(X,Y)u)Y,X)\\ &= g(R(X,Y)Y,X)-\frac{3}{4}|R(X,Y)u|^2.\end{aligned}$$

ii) From the formula (4.5), we get

$$\begin{aligned}\widetilde{G}(\widetilde{R}({}^{H}X,{}^{V}Y){}^{V}Y,{}^{H}X) &= -\frac{1}{4}g(R(u,Y)R(u,Y)X,X)\\ &\quad -\frac{1}{2}g(Y,\varphi u)^2 g(\nabla_X grad f,X)\\ &\quad +\frac{\lambda-1}{4f\lambda}X(f)g(Y,\varphi u)^2 g(X,grad f)\\ &= \frac{1}{4}|R(u,Y)X|^2-\frac{1}{2}g(Y,\varphi u)^2 Hess_f(X,X)\\ &\quad +\frac{\lambda-1}{4f\lambda}X(f)^2 g(Y,\varphi u)^2.\end{aligned}$$

iii) The result immediately follows from the formula (4.7) that

$$\begin{aligned}\widetilde{G}(\widetilde{R}({}^{V}X,{}^{V}Y){}^{V}Y,{}^{V}X) &= \frac{f^2}{\lambda^2}g(Y,u)g(X,\varphi Y)\widetilde{G}((\varphi U)^V,{}^{V}X)\\ &\quad -\frac{f^2}{\lambda^2}g(X,u)g(Y,\varphi Y)\widetilde{G}((\varphi U)^V,{}^{V}X)\\ &\quad +\frac{f}{\lambda}g(Y,\varphi Y)\widetilde{G}((\varphi X)^V,{}^{V}X)\\ &\quad -\frac{f}{\lambda}g(X,\varphi Y)\widetilde{G}((\varphi Y)^V,{}^{V}X)\\ &= \frac{f}{\lambda}(g(X,\varphi X)g(Y,\varphi Y)-g(X,\varphi Y)^2).\end{aligned}$$

The division of $\widetilde{G}(\widetilde{R}(X^i,Y^j)Y^j,X^i)$ by $\widetilde{Q}(X^i,Y^j)$ for $i,j\in\{H,V\}$ gives the result. □

Lemma 4.6. *Let (M^{2m},φ,g) be an anti-paraKähler manifold and $(TM,\widetilde{G})$ its tangent bundle equipped with the vertical generalized Berger type deformed Sasaki metric. Let (x,u) be a point on TM and $(F_a)_{a=\overline{1,4m}}$ be an orthonormal basis for the tangent space $T_{(x,u)}TM$ as above (4.11). Then the sectional curvature $\widetilde{K}$ satisfies the following equations*

$$\begin{aligned}\widetilde{K}(F_i,F_j) &= K(E_i,E_j)-\frac{3}{4}|R(E_i,E_j)u|^2,\\ \widetilde{K}(F_i,F_{2m+1}) &= \frac{1-\lambda}{2f\lambda}Hess_f(E_i,E_i)+\frac{(1-\lambda)^2}{4f^2\lambda^2}E_i(f)^2,\\ \widetilde{K}(F_i,F_{2m+l}) &= \frac{1}{4}|R(u,E_l)E_i|^2,\\ \widetilde{K}(F_{2m+k},F_{2m+1}) &= \frac{f^2}{\lambda^2(\lambda-1)}(g(E_k,\varphi E_k)g(u,\varphi u)-g(E_k,\varphi u)^2),\\ \widetilde{K}(F_{2m+k},F_{2m+l}) &= \frac{f}{\lambda}(g(E_k,\varphi E_k)g(E_l,\varphi E_l)-g(E_l,\varphi E_k)^2)\end{aligned}$$

for $i,j=\overline{1,2m},\ i\neq j$ and $k,l=\overline{2,2m},\ k\neq l$.

Proof. The results directly come from Theorem 4.5 and the formula (4.11). □

We now consider the scalar curvature $\widetilde{\sigma}$ of $(TM, \widetilde{G})$, with standard calculations we have the following result.

Theorem 4.7. *Let (M^{2m}, φ, g) be an anti-paraKähler manifold and $(TM, \widetilde{G})$ its tangent bundle equipped with the vertical generalized Berger type deformed Sasaki metric. If σ (resp. $\widetilde{\sigma}$) denotes the scalar curvature of (M^{2m}, φ, g) (resp. $(TM, \widetilde{G})$), then we have*

$$\begin{aligned} \widetilde{\sigma} &= \sigma - \frac{1}{4}\sum_{i,j=1}^{2m} |R(E_i,E_j)u|^2 + \frac{1-\lambda}{f\lambda}\Delta(f) + \frac{(1-\lambda)^2}{2f^2\lambda^2}|grad f|^2 \\ &\quad + \frac{f}{\lambda}A^2 + \frac{2f^2}{\lambda^2(\lambda-1)} g(u,\varphi u)A - \frac{(\lambda-2)f^3}{\lambda^2(\lambda-1)^2} g(u,\varphi u)^2 \\ &\quad - \frac{(2+(m-2)\lambda)f}{\lambda^2}, \end{aligned} \tag{4.17}$$

where $A = \sum_{i=2}^{2m} g(E_i, \varphi E_i)$ and $\{E_i\}_{i=\overline{1,2m}}$ is a local orthonormal frame on M^{2m}.

Proof. Let (x, u) be a point on TM and $(F_a)_{a=\overline{1,4m}}$ be an orthonormal basis for the tangent space $T_{(x,u)}TM$ as above (4.11). We have

$$\begin{aligned} \widetilde{\sigma} &= \sum_{\substack{i,j=1\\ i\neq j}}^{2m} \widetilde{K}(F_i,F_j) + 2\sum_{i,j=1}^{2m} \widetilde{K}(F_i,F_{2m+j}) + \sum_{\substack{i,j=1\\ i\neq j}}^{2m} \widetilde{K}(F_{2m+i},F_{2m+j}) \\ &= \sum_{\substack{i,j=1\\ i\neq j}}^{2m} \widetilde{K}(F_i,F_j) + 2\sum_{i=1}^{2m} \widetilde{K}(F_i,F_{2m+1}) + 2\sum_{i=1,j=2}^{2m} \widetilde{K}(F_i,F_{2m+j}) \\ &\quad + 2\sum_{i=2}^{2m} \widetilde{K}(F_{2m+i},F_{2m+1}) + \sum_{\substack{i,j=2\\ i\neq j}}^{2m} \widetilde{K}(F_{2m+i},F_{2m+j}). \end{aligned}$$

Using Lemma 4.6, we have

$$\begin{aligned} \widetilde{\sigma} &= \sum_{\substack{i,j=1\\ i\neq j}}^{2m} K(E_i,E_j) - \frac{3}{4}\sum_{\substack{i,j=1\\ i\neq j}}^{2m} |R(E_i,E_j)u|^2 + \frac{1-\lambda}{f\lambda}\sum_{i=1}^{2m} Hess_f(E_i,E_i) \\ &\quad + \frac{(1-\lambda)^2}{2f^2\lambda^2}\sum_{i=1}^{2m} E_i(f)^2 + \frac{1}{2}\sum_{i,j=1}^{2m} |R(u,E_j)E_i|^2 \\ &\quad + \frac{2f^2 g(u,\varphi u)}{\lambda^2(\lambda-1)}\sum_{i=2}^{2m} g(E_i,\varphi E_i) - \frac{2f^2}{\lambda^2(\lambda-1)}\sum_{i=2}^{2m} g(E_i,\varphi u)^2 \\ &\quad + \frac{f}{\lambda}\sum_{\substack{i,j=2\\ i\neq j}}^{2m} g(E_i,\varphi E_i)g(E_j,\varphi E_j) - \frac{f}{\lambda}\sum_{\substack{i,j=2\\ i\neq j}}^{2m} g(E_i,\varphi E_j)^2. \end{aligned}$$

In order to simplify this last expression, we put $A = \sum_{i=2}^{2m} g(E_i, \varphi E_i)$, and we are using

$$\sum_{i=1}^{2m} Hess_f(E_i,E_i) = \Delta(f), \quad \sum_{i=1}^{2m} E_i(f)^2 = |grad f|^2,$$

$$\sum_{i=2}^{2m} g(E_i,\varphi u)^2 = \frac{\lambda-1}{f} - \frac{f}{\lambda-1} g(u,\varphi u)^2, \quad \sum_{i,j=2}^{2m} g(E_i,\varphi E_i)g(E_j,\varphi E_j) = A^2,$$

$$\sum_{i,j=2}^{2m} g(E_i,\varphi E_j)^2 = m-2+\frac{f^2}{(\lambda-1)^2}g(u,\varphi u)^2,$$

$$\sum_{i,j=1}^{2m} |R(u,E_j)E_i|^2 = \sum_{i,j=1}^{2m} |R(E_i,E_j)u|^2.$$

Hence, we get

$$\begin{aligned}\widetilde{\sigma} &= \sigma - \frac{1}{4}\sum_{i,j=1}^{2m}|R(E_i,E_j)u|^2 + \frac{1-\lambda}{f\lambda}\Delta(f) + \frac{(1-\lambda)^2}{2f^2\lambda^2}|gradf|^2 + \frac{f}{\lambda}A^2 \\ &\quad + \frac{2f^2}{\lambda^2(\lambda-1)}g(u,\varphi u)A - \frac{(\lambda-2)f^3}{\lambda^2(\lambda-1)^2}g(u,\varphi u)^2 - \frac{(2+(m-2)\lambda)f}{\lambda^2}.\end{aligned}$$

□

5. Vertical generalized Berger type deformed Sasaki metric on φ-unit tangent bundle $T_1^{\varphi}M$

The φ-tangent sphere bundle of radius $r>0$ over an anti-paraKähler manifold (M^{2m},φ,g), is the hypersurface

$$T_r^{\varphi}M = \{(x,u)\in TM,\ g(u,\varphi u)=r^2\}.$$

When $r=1$, $T_1^{\varphi}M$ is called the φ-unit tangent (sphere) bundle:

$$T_1^{\varphi}M = \{(x,u)\in TM,\ g(u,\varphi u)=1\}. \tag{5.1}$$

If we set

$$\begin{aligned} F: TM &\to \mathbb{R} \\ (x,u) &\mapsto F(x,u)=g(u,\varphi u)-1,\end{aligned}$$

then the hypersurface $T_1^{\varphi}M$ is given by

$$T_1^{\varphi}M = \{(x,u)\in TM, \quad F(x,u)=0\}$$

and ${}^{\widetilde{G}}gradf$ (the gradient of F with respect to $\widetilde{G}$) is a normal vector field to $T_1^{\varphi}M$. From Lemma 3.4, for any vector field X on M^{2m}, we get

$$\widetilde{G}({}^HX, {}^{\widetilde{G}}gradf) = {}^HX(F) = {}^HX(g(u,\varphi u)-1)=0,$$

$$\widetilde{G}({}^VX, {}^{\widetilde{G}}gradf) = {}^VX(F) = {}^VX(g(u,\varphi u)-1)=2g(X,\varphi u)=\frac{2}{\lambda}\widetilde{G}({}^VX, {}^V(\varphi U)).$$

So

$${}^{\widetilde{G}}gradf = \frac{2}{\lambda}{}^V(\varphi U).$$

Then the unit normal vector field to $T_1^{\varphi}M$ is given by

$$\mathcal{N} = \frac{{}^{\widetilde{G}}gradf}{\sqrt{\widetilde{G}({}^{\widetilde{G}}gradf, {}^{\widetilde{G}}gradf)}} = \frac{{}^V(\varphi U)}{\sqrt{\widetilde{G}({}^V(\varphi U), {}^V(\varphi U))}} = \sqrt{\frac{f}{\lambda(\lambda-1)}}\,{}^V(\varphi U),$$

where $\lambda = 1+fg(u,u)$.

The tangential lift ${}^TX$ with respect to $\widetilde{G}$ of a vector $X\in T_xM$ to $(x,u)\in T_1^{\varphi}M$ is the tangential projection of the vertical lift of X to (x,u) with respect to $\mathcal{N}$, that is,

$${}^TX = {}^VX - \widetilde{G}_{(x,u)}({}^VX,\mathcal{N}_{(x,u)})\mathcal{N}_{(x,u)} = {}^VX - \frac{f}{\lambda-1}g_x(X,\varphi u){}^V(\varphi U)_{(x,u)}.$$

For the sake of notational clarity, we will use $\bar{X} = X - \frac{f}{\lambda-1}g(X,\varphi u)\varphi U$, then ${}^TX = {}^V\bar{X}$.

From the above, we get the direct sum decomposition

$$T_{(x,u)}TM = T_{(x,u)}T_1^{\varphi}M \oplus span\{\mathcal{N}_{(x,u)}\}, \tag{5.2}$$

where $(x,u) \in T_1^{\varphi}M$.

Indeed, if $W \in T_{(x,u)}TM$, they exist $X, Y \in T_xM$, such that

$$\begin{aligned} W &= {}^HX + {}^VY \\ &= {}^HX + {}^TY + \widetilde{G}_{(x,u)}({}^VY, \mathcal{N}_{(x,u)})\mathcal{N}_{(x,u)}. \end{aligned}$$

The tangent space $T_{(x,u)}T_1^{\varphi}M$ of $T_1^{\varphi}M$ at (x,u) is spanned by vectors of the form ${}^HX$ and ${}^TY$. Hence

$$T_{(x,u)}T_1^{\varphi}M = \{{}^HX + {}^TY \,/\, X, Y \in T_xM\}.$$

Given a vector field X on M^{2m}, the tangential lift ${}^TX$ of X is given by

$${}^TX_{(x,u)} = \left({}^VX - \widetilde{G}({}^VX, \mathcal{N})\mathcal{N}\right)_{(x,u)} = {}^VX_{(x,u)} - \frac{f}{\lambda - 1} g_x(X_x, \varphi u)^V(\varphi U)_{(x,u)}. \tag{5.3}$$

For any vector field X on M^{2m}, we have the followings

(1) $\widetilde{G}({}^HX, \mathcal{N}) = 0$,
(2) $\widetilde{G}({}^TX, \mathcal{N}) = 0$,
(3) ${}^TX = {}^VX$ if and only if $g(X, \varphi u) = 0$,
(4) ${}^T(\varphi U) = 0$,
(5) $g(\bar{X}, \varphi u) = 0$.

Definition 5.1. Let (M^{2m}, φ, g) be an anti-paraKähler manifold and $(TM, \widetilde{G})$ its tangent bundle equipped with the vertical generalized Berger type deformed Sasaki metric. The Riemannian metric $\widehat{G}$ on $T_1^{\varphi}M$, induced by $\widetilde{G}$, is completely determined by the identities

$$\begin{aligned} \widehat{G}({}^HX, {}^HY) &= g(X, Y), \\ \widehat{G}({}^TX, {}^HY) &= \widehat{G}({}^HX, {}^TY) = 0, \\ \widehat{G}({}^TX, {}^TY) &= g(X, Y) - \frac{f}{\lambda - 1} g(X, \varphi u) g(Y, \varphi u). \end{aligned}$$

We shall calculate the Levi-Civita connection $\widehat{\nabla}$ of $T_1^{\varphi}M$ with the vertical generalized Berger type deformed Sasaki metric $\widehat{G}$. This connection is characterized by the formula

$$\widehat{\nabla}_{\widehat{X}}\widehat{Y} = \widetilde{\nabla}_{\widehat{X}}\widehat{Y} - \widetilde{G}(\widetilde{\nabla}_{\widehat{X}}\widehat{Y}, \mathcal{N})\mathcal{N} \tag{5.4}$$

for all vector fields $\widehat{X}$ and $\widehat{Y}$ on $T_1^{\varphi}M$.

Theorem 5.2. *Let (M^{2m}, φ, g) be an anti-paraKähler manifold and $(T_1^{\varphi}M, \widehat{G})$ its φ-unit tangent bundle equipped with the vertical generalized Berger type deformed Sasaki metric. Then we have the following formulas*

$$\begin{aligned} 1.\ \widehat{\nabla}_{{}^HX}{}^HY &= {}^H(\nabla_X Y) - \frac{1}{2}{}^T(R(X, Y)u), \\ 2.\ \widehat{\nabla}_{{}^HX}{}^TY &= \frac{1}{2}{}^H(R(u, Y)X) + {}^T(\nabla_X Y), \\ 3.\ \widehat{\nabla}_{{}^TX}{}^HY &= \frac{1}{2}{}^H(R(u, X)Y), \\ 4.\ \widehat{\nabla}_{{}^TX}{}^TY &= \frac{f^2}{(\lambda - 1)^2} g(X, \varphi u) g(Y, \varphi u)^T U - \frac{f}{\lambda - 1} g(Y, \varphi u)^T(\varphi X) \end{aligned}$$

for all vector fields X, Y on M^{2m}, where ∇ is the Levi-Civita connection and R is the Riemannian curvature tensor of (M^{2m}, φ, g).

Proof. In the proof, we will use Theorem 3.6, Lemma 3.7 and the formula (5.4).
1. By direct calculation, we have

$$\begin{aligned}\widehat{\nabla}_{{}^HX}{}^HY &= \widetilde{\nabla}_{{}^HX}{}^HY - \widetilde{G}(\widetilde{\nabla}_{{}^HX}{}^HY,\mathcal{N})\mathcal{N}\\ &= {}^H(\nabla_XY) - \frac{1}{2}{}^V(R(X,Y)u) - \widetilde{G}(-\frac{1}{2}{}^V(R(X,Y)u),\mathcal{N})\mathcal{N}\\ &= {}^H(\nabla_XY) - \frac{1}{2}{}^T(R(X,Y)u).\end{aligned}$$

2. We have $\widehat{\nabla}_{{}^HX}{}^TY = \widetilde{\nabla}_{{}^HX}{}^TY - \widetilde{G}(\widetilde{\nabla}_{{}^HX}{}^TY,\mathcal{N})\mathcal{N}$, by direct calculation, we get

$$\widetilde{\nabla}_{{}^HX}{}^TY = \frac{1}{2}{}^H(R(u,Y)X) + {}^T(\nabla_XY) \ \ and \ \ \widetilde{G}(\widetilde{\nabla}_{{}^HX}{}^TY,\mathcal{N})\mathcal{N} = 0.$$

Hence

$$\widehat{\nabla}_{{}^HX}{}^TY = \frac{1}{2}{}^H(R(u,Y)X) + {}^T(\nabla_XY).$$

3. Also, we have $\widehat{\nabla}_{{}^TX}{}^HY = \widetilde{\nabla}_{{}^TX}{}^HY - \widetilde{G}(\widetilde{\nabla}_{{}^TX}{}^HY,\mathcal{N})\mathcal{N}$, by direct calculation, we get

$$\widetilde{\nabla}_{{}^TX}{}^HY = \frac{1}{2}{}^H(R(u,Y)X) \ \ and \ \ \widetilde{G}(\widetilde{\nabla}_{{}^HX}{}^TY,\mathcal{N})\mathcal{N} = 0.$$

Hence

$$\widehat{\nabla}_{{}^HX}{}^TY = \frac{1}{2}{}^H(R(u,Y)X).$$

4. In the same way above, we have $\widehat{\nabla}_{{}^TX}{}^TY = \widetilde{\nabla}_{{}^TX}{}^TY - \widetilde{G}(\widetilde{\nabla}_{{}^TX}{}^TY,\mathcal{N})\mathcal{N}$,

$$\begin{aligned}\widetilde{\nabla}_{{}^TX}{}^TY &= -\frac{f}{\lambda-1}g(Y,\varphi u)^V(\varphi X) + \frac{f^2}{(\lambda-1)^2}g(X,\varphi u)g(Y,\varphi u)^TU\\ &+\Big(\frac{(\lambda+1)f^2}{\lambda(\lambda-1)^2}g(X,u)g(Y,\varphi u) + \frac{f^2}{\lambda(\lambda-1)^2}g(X,\varphi u)g(Y,u)\\ &-\frac{(\lambda+1)f^3}{\lambda(\lambda-1)^3}g(X,\varphi u)g(Y,\varphi u) - \frac{f}{\lambda(\lambda-1)}g(X,\varphi Y)\Big)^V(\varphi U)\end{aligned}$$

and

$$\begin{aligned}\widetilde{G}(\widetilde{\nabla}_{{}^TX}{}^TY,\mathcal{N})\mathcal{N} &= -\frac{f}{\lambda-1}g(Y,\varphi u)\widetilde{G}({}^V(\varphi X),\mathcal{N})\mathcal{N}\\ &+\frac{f^2}{(\lambda-1)^2}g(X,\varphi u)g(Y,\varphi u)\widetilde{G}({}^VU,\mathcal{N})\mathcal{N}\\ &+\Big(\frac{(\lambda+1)f^2}{\lambda(\lambda-1)^2}g(X,u)g(Y,\varphi u) + \frac{f^2}{\lambda(\lambda-1)^2}g(X,\varphi u)g(Y,u)\\ &-\frac{(\lambda+1)f^3}{\lambda(\lambda-1)^3}g(X,\varphi u)g(Y,\varphi u) - \frac{f}{\lambda(\lambda-1)}g(X,\varphi Y)\Big)^V(\varphi U).\end{aligned}$$

Hence

$$\widehat{\nabla}_{{}^TX}{}^TY = \frac{f^2}{(\lambda-1)^2}g(X,\varphi u)g(Y,\varphi u)^TU - \frac{f}{\lambda-1}g(Y,\varphi u)^T(\varphi X).$$

□

Now, we shall calculate the Riemannian curvature tensor of $T_1^{\varphi}M$ with the vertical generalized Berger type deformed Sasaki metric $\widehat{G}$. Denoting by $\widehat{R}$ the Riemannian curvature tensors of $(T_1^{\varphi}M,\widehat{G})$, from the Gauss equation for hypersurfaces we deduce that $\widehat{R}(\widehat{X},\widehat{Y})\widehat{Z}$ satisfies

$$\widehat{R}(\widehat{X},\widehat{Y})\widehat{Z} = {}^t(\widetilde{R}(\widehat{X},\widehat{Y})\widehat{Z}) - B(\widehat{X},\widehat{Z}).A_{\mathcal{N}}\widehat{Y} + B(\widehat{Y},\widehat{Z}).A_{\mathcal{N}}\widehat{X} \tag{5.5}$$

for all vector fields $\widehat{X}, \widehat{Y}$ and $\widehat{Z}$ on $T_1^{\varphi}M$. Here ${}^t(\widetilde{R}(\widehat{X},\widehat{Y})\widehat{Z})$ is the tangential component of $\widetilde{R}(\widehat{X},\widehat{Y})\widehat{Z}$ with respect to the direct sum decomposition (5.2), $A_{\mathcal{N}}$ is the shape operator of $T_1^{\varphi}M$ in $(TM,\widetilde{G})$ derived from $\mathcal{N}$, and B is the second fundamental form of $T_1^{\varphi}M$ (as a hypersurface immersed in TM), associated to $\mathcal{N}$ on $T_1^{\varphi}M$.

$A_{\mathcal{N}}\widehat{X}$ is the tangential component of $(-\widetilde{\nabla}_{\widehat{X}}\mathcal{N})$, i.e.,

$$A_{\mathcal{N}}\widehat{X} = -{}^t(\widetilde{\nabla}_{\widehat{X}}\mathcal{N}).$$

$B(\widehat{X},\widehat{Y})$ is given by Gauss's formula, $\widetilde{\nabla}_{\widehat{X}}\widehat{Y} = \widehat{\nabla}_{\widehat{X}}\widehat{Y} + B(\widehat{X},\widehat{Y}).\mathcal{N}$, so

$$B(\widehat{X},\widehat{Y}) = \widetilde{G}(\widetilde{\nabla}_{\widehat{X}}\widehat{Y},\mathcal{N}).$$

Theorem 5.3. *Let (M^{2m},φ,g) be an anti-paraKähler manifold and $(T_1^{\varphi}M,\widehat{G})$ its φ-unit tangent bundle equipped with the vertical generalized Berger type deformed Sasaki metric. Then we have the following formulas*

$$\begin{aligned}
\widehat{R}({}^HX,{}^HY){}^HZ &= {}^H(R(X,Y)Z) + \frac{1}{2}{}^H(R(u,R(X,Y)u)Z)\\
&\quad +\frac{1}{4}{}^H(R(u,R(X,Z)u)Y) - \frac{1}{4}{}^H(R(u,R(Y,Z)u)X)\\
&\quad +\frac{1}{2}{}^T((\nabla_Z R)(X,Y)u),\\
\widehat{R}({}^HX,{}^TY){}^HZ &= \frac{1}{2}{}^H((\nabla_X R)(u,Y)Z) - \frac{1}{4}{}^T(R(X,R(u,Y)Z)u)\\
&\quad +\frac{1}{2}{}^T(R(X,Z)\bar{Y}),\\
\widehat{R}({}^HX,{}^HY){}^TZ &= \frac{1}{2}{}^H((\nabla_X R)(u,Z)Y)) - \frac{1}{2}{}^H((\nabla_Y R)(u,Z)X))\\
&\quad -\frac{1}{4}{}^T(R(X,R(u,Z)Y)u) + \frac{1}{4}{}^T(R(Y,R(u,Z)X)u)\\
&\quad +{}^T(R(X,Y)\bar{Z}),\\
\widehat{R}({}^HX,{}^TY){}^TZ &= -\frac{1}{4}{}^H(R(u,Y)R(u,Z)X) - \frac{1}{2}{}^H(R(\bar{Y},\bar{Z})X),\\
\widehat{R}({}^TX,{}^TY){}^HZ &= \frac{1}{4}{}^H(R(u,X)R(u,Y)Z) - \frac{1}{4}{}^H(R(u,Y)R(u,X)Z)\\
&\quad +{}^H(R(\bar{X},\bar{Y})Z),\\
\widehat{R}({}^TX,{}^TY){}^TZ &= \frac{f}{\lambda-1}\Big(g(\bar{Y},\varphi\bar{Z})^T(\varphi X) - g(\bar{X},\varphi\bar{Z})^T(\varphi Y)\Big)\\
&\quad +\frac{f^2}{(\lambda-1)^2}(g(Y,\varphi u)g(\bar{X},\varphi\bar{Z}) - g(X,\varphi u)g(\bar{Y},\varphi\bar{Z}))^T U
\end{aligned}$$

for all vector fields X, Y and Z on M^{2m}, where $\bar{X} = X - \dfrac{f}{\lambda-1}g(X,\varphi u)\varphi U$.

Proof. Using Theorem 3.6 and Lemma 3.7, we obtain

$$A_{\mathcal{N}}{}^HX = 0\,,\quad A_{\mathcal{N}}{}^TX = -\sqrt{\frac{f}{\lambda(\lambda-1)}}\Big({}^T(\varphi X) - \frac{f}{\lambda-1}g(X,\varphi u)^T U\Big),$$

$$B({}^HX,{}^HY) = B({}^HX,{}^TY) = B({}^TX,{}^HY) = 0,$$

and

$$B({}^TX,{}^TY) = -\sqrt{\frac{f}{\lambda(\lambda-1)}}g(\bar{X},\varphi\bar{Y}). \tag{5.6}$$

It suffices to use Theorem 4.1 and the formulas (5.5)-(5.6). Thus, we obtain the required formulas for the curvature tensor (see [2]). □

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