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A GENERAL THEORY OF ZERO-DIVISOR GRAPHS OVER A COMMUTATIVE RING

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Abstract. Let R be a commutative ring with $1 \neq 0$, I a proper ideal of R, and $\sim$ a multiplicative congruence relation on R. Let $R/\!\sim\, = \{\, [x]_\sim \mid x \in R\,\}$ be the commutative monoid of $\sim$-congruence classes under the induced multiplication $[x]_\sim[y]_\sim = [xy]_\sim$, and let $Z(R/\!\sim)$ be the set of zero-divisors of $R/\!\sim$. The *$\sim$-zero-divisor graph* of R is the (simple) graph $\Gamma_\sim(R)$ with vertices $Z(R/\!\sim) \setminus \{[0]_\sim\}$ and with distinct vertices $[x]_\sim$ and $[y]_\sim$ adjacent if and only if $[x]_\sim[y]_\sim = [0]_\sim$. Special cases include the usual zero-divisor graphs $\Gamma(R)$ and $\Gamma(R/I)$, the ideal-based zero-divisor graph $\Gamma_I(R)$, and the compressed zero-divisor graphs $\Gamma_E(R)$ and $\Gamma_E(R/I)$. In this paper, we investigate the structure and relationship between the various $\sim$-zero-divisor graphs.

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1. Introduction and definitions

Let R be a commutative ring with $1 \neq 0$, and let $Z(R)$ be its set of zero-divisors. The *zero-divisor graph* of R is the (simple) graph $\Gamma(R)$ with vertices $Z(R)^* = Z(R)\setminus\{0\}$, the set of nonzero zero-divisors of R, and with distinct vertices x and y adjacent if and only if $xy = 0$. There have been several other related "zero-divisor" graphs associated to R. The *ideal-based zero-divisor graph* of R with respect to an ideal I of R is the (simple) graph $\Gamma_I(R)$ with vertices $\{\, x \in R \setminus I \mid xy \in I$ for some $y \in R \setminus I\,\}$ and with distinct vertices x and y adjacent if and only if $xy \in I$. For example, $\Gamma_{\{0\}}(R) = \Gamma(R)$. Define an (congruence) equivalence relation $\sim$ on R by $x \sim y \Leftrightarrow \mathrm{ann}_R(x) = \mathrm{ann}_R(y)$, and let $R_E = \{\, [x] \mid x \in R\,\}$ be the commutative monoid of (congruence) equivalence classes under the induced multiplication $[x][y] = [xy]$. Note that $[0] = \{0\}$ and $[1] = R\setminus Z(R)$; so $[x] \subseteq Z(R)^*$ for every $x \in R\setminus([0]\cup[1])$. The *compressed zero-divisor graph* of R is the (simple) graph $\Gamma_E(R)$ with vertices $R_E \setminus \{[0],[1]\}$ and with distinct vertices $[x]$ and $[y]$ adjacent if and only if $[x][y] = [0]$, if and only if $xy = 0$.

More generally, for any (multiplicative) commutative semigroup S with 0, let $Z(S) = \{\, x \in S \mid xy = 0$ for some $0 \neq y \in S \,\}$ be the set of zero-divisors of S. The *zero-divisor graph* of S is the (simple) graph $\Gamma(S)$ with vertices $Z(S)^* = Z(S) \setminus \{0\}$ and with distinct vertices x and y adjacent if and only if $xy = 0$. Thus $\Gamma(R) = \Gamma(S)$, where $S = R$ considered as a monoid under the given ring multiplication; $\Gamma_I(R) = \Gamma(S)$, where $S = R_I$ is the Rees semigroup of R with respect to the ideal I; and $\Gamma_E(R) = \Gamma(S)$, where $S = R_E$ as defined above.

The concept of a zero-divisor graph was introduced by I. Beck [11], and then further studied by D. D. Anderson and M. Naseer [1]. However, they let all the elements of R be vertices of the graph, and they were mainly interested in colorings. Our definition of $\Gamma(R)$ and the emphasis on studying the interplay between the graph-theoretic properties of $\Gamma(R)$ and the ring-theoretic properties of R are from [8]. In [22], S. P. Redmond introduced the ideal-based zero-divisor graph $\Gamma_I(R)$. The compressed zero-divisor graph $\Gamma_E(R)$ (using different notation) was first defined by S. B. Mulay [21, p. 3551]. The semigroup zero-divisor graph $\Gamma(S)$ was given by F. R. DeMeyer, T. McKenzie, and K. Schneider in [16]. For additional information and references about zero-divisor graphs, see [4], [6], [7], [9], [23], and the two survey articles [2] and [14].

In this paper, we introduce a unifying concept of zero-divisor graph over a commutative ring R with $1 \neq 0$ based on a multiplicative congruence relation $\sim$ on R (i.e., $\sim$ is an equivalence relation on R and $x \sim y$ implies $xz \sim yz$ for $x, y, z \in R$). Then $R/{\sim} = \{\, [x]_\sim \mid x \in R \,\}$, the set of $\sim$-congruence classes of R, is a commutative monoid under the induced multiplication $[x]_\sim [y]_\sim = [xy]_\sim$ with identity element $[1]_\sim$ and zero element $[0]_\sim$. Thus $\Gamma_\sim(R) = \Gamma(R/{\sim})$, called the *congruence-based zero-divisor graph* of R with respect to $\sim$ (or the $\sim$*-zero-divisor graph* of R for short), is the (simple) graph with vertices $Z(R/{\sim})^* = Z(R/{\sim}) \setminus \{[0]_\sim\}$ and with distinct vertices $[x]_\sim$ and $[y]_\sim$ adjacent if and only if $[x]_\sim [y]_\sim = [xy]_\sim = [0]_\sim$, if and only if $xy \sim 0$. Special cases of $\Gamma_\sim(R)$ include the usual zero-divisor graphs $\Gamma(R)$ and $\Gamma(R/I)$, the ideal-based zero-divisor graph $\Gamma_I(R)$, and the condensed zero-divisor graphs $\Gamma_E(R)$ and $\Gamma_E(R/I)$ (see Example 2.1). This approach clarifies the many isolated results concerning the various zero-divisor graphs spread throughout the literature. While $\Gamma_\sim(R)$ is a special case of a semigroup zero-divisor graph, that concept is much too general for our purposes as our $\sim$-zero-divisor graphs are all based on the multiplication in R.

There are other "multiplicative" zero-divisor graphs of R that are not $\sim$-zero-divisor graphs. For example, A. Badawi [10] defined the *annihilator graph* of R

to be the (simple) graph $\mathrm{AG}(R)$ with vertices $Z(R)^*$ and with distinct vertices x and y adjacent if and only if $\mathrm{ann}_R(x) \cup \mathrm{ann}_R(y) \neq \mathrm{ann}_R(xy)$. Then $\Gamma(R)$ is a subgraph of $\mathrm{AG}(R)$, but $\mathrm{AG}(R)$ need not be a $\Gamma_\sim(R)$. (Let $R = \mathbb{Z}_8$. Then $\mathrm{AG}(R) = K^3$. However, $\Gamma(R) = K^{1,2}$, and thus any $\Gamma_\sim(R)$ with 3 vertices is also a $K^{1,2}$.) For another example, let S be the monoid of ideals of R under the usual ideal multiplication. Then, following S. Behboodi and Z. Rakeei [12], $\mathbb{AG}(R) = \Gamma(S)$, the *annihilating ideal graph* of R, is a semigroup zero-divisor graph that need not be a $\Gamma_\sim(R)$. (Let $R = \mathbb{Z}_2[X,Y]/(X^2, XY, Y^2)$. Then $\mathbb{AG}(R) = K^4$. However, any $\Gamma_\sim(R)$ has at most 3 vertices since $|Z(R)^*| = 3$.)

Let $\mathcal{C}(R)$ be the set of all multiplicative congruence relations on R. Then $\mathcal{C}(R)$ is partially ordered by inclusion, i.e., for $\sim_1, \sim_2 \in \mathcal{C}(R)$, $\sim_1 \leq \sim_2$ if and only if $\sim_1 \subseteq \sim_2$ as subsets of $R \times R$, if and only if $x \sim_1 y$ implies $x \sim_2 y$ for $x, y \in R$, if and only if $[x]_{\sim_1} \subseteq [x]_{\sim_2}$ for every $x \in R$. Moreover, $\mathcal{C}(R)$ has a least element $=_R = \{\, (x,x) \mid x \in R \,\}$ and a greatest element $R \times R$. For $\sim \in \mathcal{C}(R)$, $I = [0]_\sim$ is a semigroup ideal of R. (Recall that a $\emptyset \neq I \subseteq S$ of a (multiplicative) commutative semigroup S is an *(semigroup) ideal* of S if $xy \in I$ for all $x \in S$ and $y \in I$. A proper ideal I of S is a *prime* (resp., *radical*) *(semigroup) ideal* of S if $xy \in I$ implies $x \in I$ or $y \in I$ (resp., $x^n \in I$ for some integer $n \geq 1$ implies $x \in I$).) In this paper, R will always be considered a monoid under the given ring multiplication and "ideal of R" will always mean a ring ideal of R. Note that an ideal of R is always a semigroup ideal of R, but a semigroup ideal of R need not be an ideal of R. For example, $Z(R)$ and $R \setminus U(R)$ are always prime semigroup ideals of R, but need not be ideals of R. In fact, it is easily shown that a $\emptyset \neq I \subseteq R$ is a semigroup ideal of R if and only if I is a union of ideals of R, if and only if I is a union of principal ideals of R. Also, a prime (resp., radical) ideal of R is always a prime (resp., radical) semigroup ideal of R, but the converse may fail since a union of prime (resp., radical) ideals of R is a prime (resp., radical) semigroup ideal of R. However, $\{0\}$ is an (prime, radical) ideal of R if and only if it is a (prime, radical) semigroup ideal of R. For I an (semigroup) ideal of R and $x \in R$, $(I : x) = \{\, y \in R \mid xy \in I \,\}$ is an (semigroup) ideal of R containing I. Moreover, $(I : x)/I = \mathrm{ann}_{R/I}(x + I)$ when I is an ideal of R.

For I a semigroup ideal of R, let $\mathcal{C}_I(R) = \{\, \sim \in \mathcal{C}(R) \mid [0]_\sim = I \,\}$. Then $\mathcal{C}(R) = \bigsqcup \{\, \mathcal{C}_I(R) \mid I \text{ is a semigroup ideal of } R \,\}$, and each $\mathcal{C}_I(R)$ is nonempty (see Example 2.1(c)) and partially ordered by inclusion. Note that $[0]_\sim = R$ if and only if $\sim = R \times R$; so $\mathcal{C}_R(R) = \{R \times R\}$. Moreover, for $\sim, \sim_1, \sim_2 \in \mathcal{C}(R)$, $[0]_{\sim_1} \subseteq [0]_{\sim_2}$ if $\sim_1 \leq \sim_2$; and thus, if $\sim_1 \leq \sim \leq \sim_2$ with $\sim_1, \sim_2 \in \mathcal{C}_I(R)$, then $\sim \in \mathcal{C}_I(R)$.

We assume that all graphs are simple graphs, i.e., they are undirected graphs with no multiple edges or loops. By abuse of notation, we will let G, rather than $V(G)$, denote the vertices of a graph G. Recall that a graph G is *connected* if there is a path between any two distinct vertices of G. For vertices x and y of G, let $d(x,y)$ be the length of a shortest path from x to y ($d(x,x)=0$ and $d(x,y)=\infty$ if there is no such path). The *diameter* of G is $\mathrm{diam}(G)=\sup\{\, d(x,y) \mid x$ and y are vertices of $G\,\}$. The *girth* of G, denoted by $\mathrm{gr}(G)$, is the length of a shortest cycle in G ($\mathrm{gr}(G)=\infty$ if G contains no cycles). As usual, K^n will denote the complete graph on n vertices and $K^{m,n}$ will denote the complete bipartite graph on m,n vertices (m and n may be infinite cardinals). A subgraph G' of a graph G is an *induced subgraph* of G if two vertices of G' are adjacent in G' if and only if they are adjacent in G. For graphs G and G', a function $f : G \longrightarrow G'$ is a *graph homomorphism* if vertices x and y are adjacent in G implies that $f(x)$ and $f(y)$ are adjacent in G'. The function f is a *graph isomorphism* if it is bijective and f and f^{-1} are both graph homomorphisms (i.e., vertices x and y are adjacent in G if and only if $f(x)$ and $f(y)$ are adjacent in G'); in this case, we write $G \cong G'$ (again, by abuse of notation, we will often just write $G = G'$ when f is a naturally induced graph isomorphism).

In Section 2, we give some basic properties of $\Gamma_\sim(R)$ and investigate the structure of $\mathcal{C}_I(R)$. For example, we show that each $\mathcal{C}_I(R)$ has a least element (given by $x \sim y \Leftrightarrow x = y$ or $x, y \in I$) and a greatest element (given by $x \sim y \Leftrightarrow (I : x) = (I : y)$). In Section 3, we study functions between $\sim$-zero-divisor graphs. For $\sim_1, \sim_2 \in \mathcal{C}_I(R)$ with $\sim_1 \le \sim_2$, there is a surjective function $F : \Gamma_{\sim_1}(R) \longrightarrow \Gamma_{\sim_2}(R)$ given by $F([x]_{\sim_1}) = [x]_{\sim_2}$ and an injective graph homomorphism $G : \Gamma_{\sim_2}(R) \longrightarrow \Gamma_{\sim_1}(R)$ such that $FG = 1_{\Gamma_{\sim_2}(R)}$. In particular, there is a largest (resp., smallest) $\sim$-zero-divisor graph with $I = [0]_\sim$, namely, $\Gamma_I(R)$ (resp., $\Gamma_E(R/I)$ when I is an ideal of R). In Section 4, for a subring R of a commutative ring T with $1 \neq 0$, and $\sim_R \in \mathcal{C}(R)$ and $\sim_T \in \mathcal{C}(T)$ with $\sim_R \subseteq \sim_T$ and $[0]_{\sim_R} = [0]_{\sim_T} \cap R$, we consider the induced function $F : \Gamma_{\sim_R}(R) \longrightarrow \Gamma_{\sim_T}(T)$ given by $F([x]_{\sim_R}) = [x]_{\sim_T}$. If $\sim_R = \sim_T \cap (R \times R)$, then F is an injective graph homomorphism; so $\Gamma_{\sim_R}(R)$ is isomorphic to an induced subgraph of $\Gamma_{\sim_T}(T)$. In Section 5, we investigate the more general question of when a homomorphism $f : R \longrightarrow T$ of rings induces a function $F : \Gamma_\sim(R) \longrightarrow \Gamma_{\sim'}(T)$ of graphs given by $F([x]_\sim) = [f(x)]_{\sim'}$ for some $\sim \in \mathcal{C}(R)$ and $\sim' \in \mathcal{C}(T)$.

Throughout, R will be a commutative ring with $1 \neq 0$, $Z(R)$ its set of zero-divisors and $Z(R)^* = Z(R) \setminus \{0\}$, $\mathrm{nil}(R)$ its set of nilpotent elements, $U(R)$ its group of units, $T(R) = R_S$, where $S = R \setminus Z(R)$, its total quotient ring, and $\dim(R)$ its Krull dimension. As usual, we assume that a subring has the same identity element as the ring, and all ring and monoid homomorphisms send the identity to the identity. We say that R is *reduced* if $\mathrm{nil}(R) = \{0\}$. Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Z}_n$, and $\mathbb{Q}$ denote the positive integers, integers, integers modulo n, and rational numbers, respectively, and $A^* = A \setminus \{0\}$. For any undefined ring-theoretic terminology, see [18] or [19]; for semigroups, see [17]. A general reference for graph theory is [13]. To avoid trivialities, we will implicitly assume when necessary that $\Gamma_\sim(R)$ is not the empty graph.

Most of the results in the first four sections of this paper are from the second-named author's PhD dissertation [20] at The University of Tennessee under the direction of the first-named author.

2. Basic results

In this section, we give some basic properties of $\Gamma_\sim(R)$ and investigate the structure of $\mathcal{C}_I(R)$. We start with some examples of multiplicative congruence realations and their corresponding congruence-based zero-divisor graphs. In each case, it is easily verified that $\sim \in \mathcal{C}(R)$.

Example 2.1. *Let R be a commutative ring with $1 \neq 0$.*

(a) *Let $\sim$ be $=_R$. Then $R/\sim = R$, $\Gamma_\sim(R) = \Gamma(R)$, the usual zero-divisor graph of R (we identify $[x]_\sim = \{x\}$ with x), and $[0]_\sim = \{0\}$. Thus $\sim \in \mathcal{C}_{\{0\}}(R)$.*

(b) *Let I be a proper ideal of R, and let $\sim_I \in \mathcal{C}(R)$ be defined by $x \sim_I y \Leftrightarrow x - y \in I$ for $x, y \in R$. Then $R/\sim_I = R/I$, $\Gamma_{\sim_I}(R) = \Gamma(R/I)$, and $[0]_{\sim_I} = I$. Thus $\sim_I \in \mathcal{C}_I(R)$, and $\sim_I \leq \sim_J$ if and only if $I \subseteq J$. (Note that for I a semigroup ideal of R, $\sim_I$ defines a multiplicative congruence relation on R if and only if I is an ideal of R.)*

(c) *Let I be a semigroup ideal of R, and let $\sim_I \in \mathcal{C}(R)$ be defined by $x \sim_I y \Leftrightarrow x = y$ or $x, y \in I$ for $x, y \in R$. Then $R/\sim_I = R_I$ is the Rees semigroup of R with respect to I, and $[0]_{\sim_I} = I$. (The Rees semigroup is usually denoted by R/I, where the semigroup ideal I collapses to 0.) Thus $\sim_I \in \mathcal{C}_I(R)$, and $\sim_I \leq \sim_J$ if and only if $I \subseteq J$. If I is an ideal of R, then $\Gamma_{\sim_I}(R) = \Gamma_I(R)$, the ideal-based zero-divisor graph of R with respect to I (again, we identify $[x]_{\sim_I} = \{x\}$ with x for $x \in R \setminus I$). Hence, for any semigroup ideal I of R, we will denote $\Gamma_{\sim_I}(R)$ by $\Gamma_I(R)$.*

(d) *Let I be a semigroup ideal of R, and let $\sim \in \mathcal{C}(R)$ be defined by $x \sim y \Leftrightarrow (I : x) = (I : y)$ for $x, y \in R$. Then $[0]_\sim = I$; so $\sim \in \mathcal{C}_I(R)$. If*

I is a proper ideal of R, then $\Gamma_\sim(R) = \Gamma_E(R/I)$, the compressed zero-divisor graph of R/I (we identify $[x]_\sim$ in $R/\sim$ with $[x+I]$ in $(R/I)_E$, see Corollary 5.2). Throughout this paper, we will identify this $\Gamma_\sim(R)$ with $\Gamma_E(R/I)$. If $I = \{0\}$, then $(I : x) = ann_R(x)$; so $\Gamma_\sim(R) = \Gamma_E(R)$, the compressed zero-divisor graph of R.

(e) *Let G be a (multiplicative) subgroup of $U(R)$, and let $\sim_G \in \mathcal{C}(R)$ be defined by $x \sim_G y \Leftrightarrow y = ux$ for $x, y \in R$ and $u \in G$. Then $[0]_{\sim_G} = \{0\}$; so $\sim_G \in \mathcal{C}_{\{0\}}(R)$. For example, $\Gamma_{\sim_{\{1\}}}(R) = \Gamma(R)$. Moreover, for subgroups $G_1, G_2 \subseteq U(R)$, we have $\sim_{G_1} \leq \sim_{G_2}$ if and only if $G_1 \subseteq G_2$, and thus $\sim_{G_1} = \sim_{G_2}$ if and only if $G_1 = G_2$.*

The next example illustrates the added diversity associated with $\sim$-zero-divisor graphs. Recall that for $m, n \in \mathbb{N}$, there is a commutative ring R with $1 \neq 0$ such that $\Gamma(R) = K^n$ if and only if $n+1$ is a prime power [8, Theorem 2.10], and $\Gamma(R) = K^{m,n}$ if and only if $m+1$ and $n+1$ are both prime powers [8, p. 439].

Example 2.2. (a) *Let $R = \mathbb{Q}[X]/(X^2) = \mathbb{Q}[x]$ with $x^2 = 0$. Then $Z(R)^* = \{\, \alpha x \mid \alpha \in \mathbb{Q}^* \,\}$; so $\Gamma(R) = K^\omega$ and $\Gamma_E(R) = K^1$. For an integer $n \geq 1$, let $G_n = \{\, 2^{nk} i/j \mid k \in \mathbb{Z}, i, j \in \mathbb{Z} \setminus 2\mathbb{Z} \,\}$, and let $G_\omega = \{\, i/j \mid i, j \in \mathbb{Z} \setminus 2\mathbb{Z} \,\}$. Then G_n is a multiplicative subgroup of $\mathbb{Q}^* \subseteq U(R)$ for every $n \in \mathbb{N} \cup \{\omega\}$. Define $\sim_n = \sim_{G_n} \in \mathcal{C}_{\{0\}}(R)$ as in Example 2.1(e). It is easily verified that $Z(R/\sim_\omega)^* = \{\, [x]_{\sim_\omega}, [2x]_{\sim_\omega}, \ldots, [2^n x]_{\sim_\omega}, \ldots \}$ and $Z(R/\sim_n)^* = \{\, [x]_{\sim_n}, [2x]_{\sim_n}, \ldots, [2^{(n-1)} x]_{\sim_n} \}$; so $\Gamma_{\sim_\omega}(R) = K^\omega$ and $\Gamma_{\sim_n}(R) = K^n$ for every integer $n \geq 1$.*

(b) *Let $R = \mathbb{Q} \times \mathbb{Q}$. Then $\Gamma(R) = K^{\omega,\omega}$ and $\Gamma_E(R) = K^{1,1} = K^2$. For $n \in \mathbb{N} \cup \{\omega\}$, let G_n be as in part (a) above. Then $G_{m,n} = G_m \times G_n \subseteq \mathbb{Q}^* \times \mathbb{Q}^* = U(R)$. Define $\sim_{G_{m,n}} \in \mathcal{C}_{\{0\}}(R)$ as in Example 2.1(e). It is easily verified that $\Gamma_{\sim_{G_{m,n}}}(R) = K^{m,n}$ for every $m, n \in \mathbb{N} \cup \{\omega\}$.*

It is well known that $\Gamma(R)$ and $\Gamma_E(R)$ are the empty graph if and only if R is an integral domain and that $\Gamma_I(R)$ is the empty graph if and only if $I = R$ or I is a prime ideal of R. A similar result holds for $\Gamma_\sim(R)$.

Theorem 2.3. *Let R be a commutative ring with $1 \neq 0$, and let $\sim \in \mathcal{C}(R)$.*

(a) *$\Gamma_\sim(R)$ is the empty graph if and only if $[0]_\sim = R$ or $[0]_\sim$ is a prime semigroup ideal of R.*

(b) *$\Gamma_\sim(R)$ is the empty graph for every $\sim \in \mathcal{C}(R)$ if and only if R is a field.*

Proof. (a) This follows since $Z(R/{\sim}) = \emptyset$ if and only if $R/{\sim} = \{[0]_\sim\}$ (i.e., $[0]_\sim = R$), and $Z(R/{\sim}) = \{[0]_\sim\}$ if and only if $[0]_\sim$ is a prime semigroup ideal of R.

(b) If R is a field, then $\{0\}$ and R are the only semigroup ideals of R and $\{0\}$ is a prime semigroup ideal of R. Thus $\Gamma_\sim(R)$ is the empty graph when R is a field by part (a). Conversely, suppose that R is not a field. Hence R has a proper ideal I that is not a prime ideal. Let $\sim \in \mathcal{C}_I(R)$ be defined by $x \sim y \Leftrightarrow x = y$ or $x, y \in I$. Then $\Gamma_\sim(R) = \Gamma_I(R)$ is not the empty graph. □

The following fundamental lemma will be used in several places throughout this paper.

Lemma 2.4. *Let R be a commutative ring with $1 \neq 0$, $x \in R$, and $\sim_1, \sim_2 \in \mathcal{C}(R)$ with $[0]_{\sim_1} = [0]_{\sim_2}$. Then $[x]_{\sim_1} \in Z(R/{\sim_1})^*$ if and only if $[x]_{\sim_2} \in Z(R/{\sim_2})^*$.*

Proof. Suppose that $[0]_{\sim_1} = [0]_{\sim_2}$, and let $x \in R$. Then $[x]_{\sim_1} = [0]_{\sim_1}$ if and only if $[x]_{\sim_2} = [0]_{\sim_2}$, and $[x]_{\sim_1}[y]_{\sim_1} = [0]_{\sim_1}$ for $[y]_{\sim_1} \in (R/{\sim_1})^*$ if and only if $[x]_{\sim_2}[y]_{\sim_2} = [0]_{\sim_2}$ for $[y]_{\sim_2} \in (R/{\sim_2})^*$. Thus $[x]_{\sim_1} \in Z(R/{\sim_1})^*$ if and only if $[x]_{\sim_2} \in Z(R/{\sim_2})^*$. □

Now, we investigate the relative size of congruence-based zero-divisor graphs compared to the usual zero-divisor graph. The next result demonstrates that, in certain cases, we have $|\Gamma_\sim(R)| \leq |\Gamma(R)|$ (also, see Corollary 3.3). However, the remark that follows provides an example for which $|\Gamma_\sim(R)| > |\Gamma(R)|$.

Theorem 2.5. *Let R be a commutative ring with $1 \neq 0$, and let $\sim \in \mathcal{C}(R)$. If $R = Z(R) \cup U(R)$ or $\sim \in \mathcal{C}_{\{0\}}(R)$, then $|\Gamma_\sim(R)| = |Z(R/{\sim})^*| \leq |Z(R)^*| = |\Gamma(R)|$. In particular, the inequality holds if dim$(R) = 0$ (e.g., R is finite).*

Proof. Let $x \in R$. If $\sim \in \mathcal{C}_{\{0\}}(R)$, then $[0]_{=_R} = \{0\} = [0]_\sim$. Thus $x \in Z(R)^*$ if and only if $[x]_\sim \in Z(R/{\sim})^*$ by Lemma 2.4. If $R = Z(R) \cup U(R)$, then clearly $x \in Z(R)^*$ if $[x]_\sim \in Z(R/{\sim})^*$ (the converse may fail, see Remark 2.6(b)). Hence $|\Gamma_\sim(R)| = |Z(R/{\sim})^*| \leq |Z(R)^*| = |\Gamma(R)|$. □

Remark 2.6. (a) *We may have $|\Gamma_\sim(R)| = |Z(R/{\sim})^*| > |Z(R)^*| = |\Gamma(R)|$ for $\sim \in \mathcal{C}(R)$. For example, let $R = \mathbb{Z}$ and $I = 4\mathbb{Z}$. Then the graphs $\Gamma_I(R)$ and $\Gamma(R/I)$ are both nonempty although $Z(R) = \{0\}$, and thus $\Gamma(R)$ is the empty graph.*

(b) *Also, $x \in Z(R)^*$ need not imply that $[x]_\sim \in Z(R/{\sim})$ for $\sim \in \mathcal{C}(R)$. For example, let $R = \mathbb{Z}_2 \times \mathbb{Z}_2$, $I = \{0\} \times \mathbb{Z}_2$, and $\sim \in \mathcal{C}_I(R)$ be defined by $x \sim y \Leftrightarrow x - y \in I$. Then $(1, 0) \in Z(R)^*$, but $[(1, 0)]_\sim = [(1, 1)]_\sim \notin Z(R/{\sim})$.*

(c) *We may have* $\Gamma_{\sim_1}(R) = \Gamma_{\sim_2}(R) \neq \emptyset$ *for distinct* $\sim_1, \sim_2 \in \mathcal{C}_I(R)$*. This is because distinct multiplicative congruence relations* $\sim$ *on* R *may yield the same* $Z(R/\sim)$*. Specifically, this would happen if they restrict to the same congruence relation on* $Z(R)$*. For example, let* $R = \mathbb{Z}_4$*. Then* $\mathcal{C}_{\{0\}}(R) = \{=_R, \sim\}$*, where* $x \sim y \Leftrightarrow ann_R(x) = ann_R(y)$*, and* $=_R < \sim$*. However,* $=_R$ *and* $\sim$ *agree on* $Z(R)$*; so* $\Gamma(R) = \Gamma_E(R)(= K^1)$*. In fact, for* R *a commutative ring with* $1 \neq 0$*,* $\Gamma(R) = \Gamma_E(R) \neq \emptyset$ *if and only if* R *is a Boolean ring (not* $\mathbb{Z}_2$*),* $\mathbb{Z}_4$*, or* $\mathbb{Z}_2[X]/(X^2)$ [4, Corollary 2.7]*, also cf. Corollary 2.12.*

Probably the two best known results for $\Gamma(R)$ are that $\Gamma(R)$ is connected with $\mathrm{diam}(\Gamma(R)) \in \{0,1,2,3\}$ [8, Theorem 2.3] and $\mathrm{gr}(\Gamma(R)) \in \{3,4,\infty\}$ ([8, Theorem 2.4], [15, Theorem 1.6], [21, (1.4)]). These two results also hold for $\Gamma_I(R)$ and $\Gamma_E(R)$; in fact, both are special cases for the semigroup zero-divisor graph $\Gamma(S)$ [16, Theorems 1.2 and 1.5]. These two results also hold for every $\Gamma_\sim(R)$. Moreover, $\mathrm{gr}(\Gamma_E(R)) \in \{3,\infty\}$ [6, Theorem 3.1], and thus $\mathrm{gr}(\Gamma_\sim(R)) \in \{3,\infty\}$ when I is a proper ideal of R and $\sim \in \mathcal{C}_I(R)$ is defined by $x \sim y \Leftrightarrow (I : x) = (I : y)$ since $\Gamma_\sim(R) = \Gamma_E(R/I)$ by Example 2.1(d).

Theorem 2.7. *Let* R *be a commutative ring with* $1 \neq 0$*, and let* $\sim \in \mathcal{C}(R)$*.*

(a) $\Gamma_\sim(R)$ *is connected with diam*$(\Gamma_\sim(R)) \in \{0,1,2,3\}$*.*
(b) *gr*$(\Gamma_\sim(R)) \in \{3,4,\infty\}$*.*

Proof. Since $\Gamma_\sim(R) = \Gamma(R/\sim)$ is a semigroup zero-divisor graph, part (a) follows from [16, Theorem 1.2] and part (b) follows from [16, Theorem 1.5]. □

Let $\sim \in \mathcal{C}(R)$. Many properties of $\Gamma_\sim(R)$ are determined by the semigroup ideal $I = [0]_\sim$ and happen in "levels" given by $\mathcal{C}_I(R)$ (see Section 3). We have already observed in Lemma 2.4 that for $\sim_1, \sim_2 \in \mathcal{C}_I(R)$, $[x]_{\sim_1} \in Z(R/\sim_1)^*$ if and only if $[x]_{\sim_2} \in Z(R/\sim_2)^*$. Of special interest are the cases when either $I = \{0\}$ or I is an ideal of R. For example, distinct vertices $[x]_\sim$ and $[y]_\sim$ are adjacent in $\Gamma_\sim(R)$ if and only if $[xy]_\sim = [0]_\sim$, and hence if and only if $xy \in I$. Thus, if $[0]_\sim = \{0\}$ (i.e., $\sim \in \mathcal{C}_{\{0\}}(R)$), then $[x]_\sim$ and $[y]_\sim$ are adjacent in $\Gamma_\sim(R)$ if and only if $xy = 0$. In fact, $[0]_\sim = \{0\}$ if and only if $xy = 0$ whenever $xy \sim 0$ for $x, y \in R$. Also, let $\sim \in \mathcal{C}_{\{0\}}(R)$; then $\Gamma_\sim(R) = \emptyset$ if and only if R is an integral domain by Theorem 2.3(a).

We next show that every $\mathcal{C}_I(R)$ has a least element $\sim_1$, a greatest element $\sim_3$, and $\mathcal{C}_I(R) = \{\, \sim \in \mathcal{C}(R) \mid \sim_1 \leq \sim \leq \sim_3 \,\}$. Of course, we may have $\sim \leq \sim'$ for $\sim$, $\sim'$ in distinct $\mathcal{C}_I(R)$ (cf. Example 2.1(b),(c)).

Theorem 2.8. *Let R be a commutative ring with $1 \neq 0$, and let I be a semigroup ideal of R. Define $\sim_1$, $\sim_3 \in \mathcal{C}_I(R)$ by $x \sim_1 y \Leftrightarrow x = y$ or $x, y \in I$ and $x \sim_3 y \Leftrightarrow (I : x) = (I : y)$. Then $\sim_1 \leq \sim \leq \sim_3$ for every $\sim \in \mathcal{C}_I(R)$. Moreover, if $\sim_1 \leq \sim \leq \sim_3$ for $\sim \in \mathcal{C}(R)$, then $\sim \in \mathcal{C}_I(R)$. Thus $\mathcal{C}_I(R) = \{ \sim \in \mathcal{C}(R) \mid \sim_1 \leq \sim \leq \sim_3 \}$.*

Proof. Let $\sim \in \mathcal{C}_I(R)$; so $[0]_\sim = I = [0]_{\sim_1} = [0]_{\sim_3}$. First, we show that $\sim_1 \leq \sim$. Suppose that $x \sim_1 y$; then $x = y$ or $x, y \in I$. If $x = y$, then $x \sim y$ since $\sim$ is reflexive. Otherwise, $x, y \in I = [0]_\sim$; so $x \sim y$. Thus $\sim_1 \leq \sim$.

Next, we show that $\sim \leq \sim_3$ for $\sim \in \mathcal{C}_I(R)$. Suppose that $x \sim y$, and let $z \in (I : x)$. Then $zx \in I = [0]_\sim$. Thus $zx \sim zy$ gives $zy \in [0]_\sim = I$, and hence $z \in (I : y)$. Thus $(I : x) \subseteq (I : y)$. The proof of the reverse inclusion is similar; so $(I : x) = (I : y)$. Hence $x \sim_3 y$; so $\sim \leq \sim_3$.

For the "moreover" statement, note that $\sim_1 \leq \sim \leq \sim_3$ gives $I = [0]_{\sim_1} \subseteq [0]_\sim \subseteq [0]_{\sim_3} = I$. Thus $[0]_\sim = I$; so $\sim \in \mathcal{C}_I(R)$. $\square$

Corollary 2.9. *Let R be a commutative ring with $1 \neq 0$, and let $\sim \in \mathcal{C}(R)$. Define $\sim_3$ by $x \sim_3 y \Leftrightarrow ann_R(x) = ann_R(y)$. Then $\sim \in \mathcal{C}_{\{0\}}(R)$ if and only if $=_R \leq \sim \leq \sim_3$. Thus $\mathcal{C}_{\{0\}}(R) = \{ \sim \in \mathcal{C}(R) \mid x \sim y \Rightarrow ann_R(x) = ann_R(y) \}$.*

Our next goal is to determine when $|\mathcal{C}_I(R)| = 1$ for I a proper semigroup ideal of R (recall that $|\mathcal{C}_R(R)| = 1$). By Theorem 2.8, $|\mathcal{C}_I(R)| = 1$ if and only if $\sim_1 = \sim_3$. First, we need the following lemma.

Lemma 2.10. *Let R be a commutative ring with $1 \neq 0$, and let I be a proper ideal of R. Define $\sim_2, \sim_3 \in \mathcal{C}_I(R)$ by $x \sim_2 y \Leftrightarrow x - y \in I$ and $x \sim_3 y \Leftrightarrow (I : x) = (I : y)$. If $\sim_2 = \sim_3$, then I is a radical ideal of R.*

Proof. Suppose that $\sim_2 = \sim_3$, and let $x^2 \in I$. Then $(1 + x + I)(1 - x + I) = 1 + I$; so $1 + x + I \in U(R/I)$. Thus $(I : 1 + x)/I = \mathrm{ann}_{R/I}(1 + x + I) = \{0 + I\}$; so $(I : 1 + x) = I = (I : 1)$. Hence $1 + x \sim_3 1$, and thus $1 + x \sim_2 1$. Hence $x = (1 + x) - 1 \in I$, and thus I is a radical ideal of R. $\square$

Recall that a ring R is a *Boolean ring* if $x^2 = x$ for every $x \in R$. A Boolean ring R is necessarily commutative and reduced with $\mathrm{char}(R) = 2$, $\dim(R) = 0$, $U(R) = \{1\}$, and $Z(R) = R \setminus \{1\}$. Also, a finite ring R is a Boolean ring if and only if $R \cong \mathbb{Z}_2^n$ for some integer $n \geq 1$.

Theorem 2.11. *Let R be a commutative ring with $1 \neq 0$, and let I be a proper ideal of R. Define $\sim_1, \sim_2, \sim_3 \in \mathcal{C}_I(R)$ by $x \sim_1 y \Leftrightarrow x = y$ or $x, y \in I$, $x \sim_2 y \Leftrightarrow x - y \in I$, and $x \sim_3 y \Leftrightarrow (I : x) = (I : y)$.*

(a) $\sim_1 = \sim_2$ *if and only if* $I = \{0\}$. *Moreover, in this case,* $\sim_1 = \sim_2 = =_R$.

(b) $\sim_2 = \sim_3$ *if and only if* R/I *is a Boolean ring.*

(c) $\sim_1 = \sim_3$ *if and only if* $I = \{0\}$ *and* R *is a Boolean ring. Moreover, in this case,* $\sim_1 = \sim_2 = \sim_3 = =_R$.

Proof. By Theorem 2.8, $\sim_1 \leq \sim_2 \leq \sim_3$.

(a) Clearly, $\sim_1 = \sim_2 = =_R$ when $I = \{0\}$. Conversely, suppose that $\sim_1 = \sim_2$. Let $x \in I$. Then $x + 1 \sim_2 1$ since $(x + 1) - 1 = x \in I$; so also $x + 1 \sim_1 1$. Thus $x + 1 = 1$ or $x + 1, 1 \in I$. Since I is a proper ideal of R, necessarily $x + 1 = 1$, and hence $x = 0$. Thus $I = \{0\}$.

(b) Suppose that $\sim_2 = \sim_3$. We show that R/I is a Boolean ring. Let $x \in R$. Then it is easily shown that $(I : x) = (I : x^2)$ since I is a radical ideal of R by Lemma 2.10. Thus $x^2 \sim_3 x$, and hence $x^2 \sim_2 x$. Thus $x^2 - x \in I$; so $(x + I)^2 = x^2 + I = x + I$. Hence R/I is a Boolean ring.

Conversely, suppose that R/I is a Boolean ring. Since $\sim_2 \leq \sim_3$ by Theorem 2.8, we need only show that $\sim_3 \leq \sim_2$. Let $x \sim_3 y$ for $x, y \in R$. Then $(I : x) = (I : y)$. Now $x^2 + I = x + I$ since R/I is a Boolean ring; so $(x - 1)x = x^2 - x \in I$. Thus $x - 1 \in (I : x) = (I : y)$; so $xy - y \in I$. Similarly, $yx - x \in I$; so $x - y = (xy - y) - (yx - x) \in I$. Hence $x \sim_2 y$. Thus $\sim_3 \leq \sim_2$; so $\sim_2 = \sim_3$.

(c) This follows directly from parts (a) and (b). $\square$

Corollary 2.12. *Let* R *be a commutative ring with* $1 \neq 0$, I *an ideal of* R, *and* $\sim \in \mathcal{C}_I(R)$. *Then* $|\mathcal{C}_I(R)| = 1$ *if and only if either* $I = R$, *or* $I = \{0\}$ *and* R *is a Boolean ring. Moreover, if* $I = R$, *then* $\Gamma_\sim(R) = \emptyset$; *and if* $I = \{0\}$ *and* R *is a Boolean ring, then* $\Gamma_\sim(R) = \Gamma(R)$, *and* $\Gamma(R) = \emptyset$ *if and only if* $R = \mathbb{Z}_2$. *Thus* $|\mathcal{C}_I(R)| \geq 2$ *when* I *is a nonzero, proper ideal of* R *(and* $\Gamma_\sim(R) = \emptyset$ *for* $\sim \in \mathcal{C}_I(R)$ *if and only if* I *is a prime ideal of* R*).*

Corollary 2.13. *Let* R *be a commutative ring with* $1 \neq 0$.

(a) *If* R *is a field, then* $|\mathcal{C}(R)| \geq 2$. *Moreover,* $|\mathcal{C}(R)| = 2$ *if and only if* $R = \mathbb{Z}_2$.

(b) *If* R *is a Boolean ring and not a field, then* $|\mathcal{C}(R)| \geq 7$.

(c) *If* R *is not a Boolean ring and not a field, then* $|\mathcal{C}(R)| \geq 5$.

Proof. (a) Suppose that R is a field. Then $\{0\}$ and R are the only semigroup ideals of R; so $\mathcal{C}(R) = \mathcal{C}_{\{0\}}(R) \sqcup \mathcal{C}_R(R)$. Thus $|\mathcal{C}(R)| = |\mathcal{C}_{\{0\}}(R)| + |\mathcal{C}_R(R)| \geq |\mathcal{C}_{\{0\}}(R)| + 1 \geq 2$. Moreover, $|\mathcal{C}(R)| = 2$ if and only if $|\mathcal{C}_{\{0\}}(R)| = 1$, if and only if R is a Boolean ring by Corollary 2.12, if and only if $R = \mathbb{Z}_2$ since R is also a field.

(b) and (c) Suppose that R is not a field. Then R has a nonzero, proper ideal I, and $|\mathcal{C}_I(R)| \geq 2$ by Corollary 2.12. If R is not a Boolean ring, then $|\mathcal{C}_{\{0\}}(R)| \geq 2$ by

Corollary 2.12 again; so $|\mathcal{C}(R)| \geq |\mathcal{C}_{\{0\}}(R)|+|\mathcal{C}_I(R)|+|\mathcal{C}_R(R)| \geq 2+2+1 = 5$. If R is a Boolean ring, then $|\mathcal{C}_{\{0\}}(R)| = 1$ by Corollary 2.12, R has another nonzero, proper ideal J with $|\mathcal{C}_J(R)| \geq 2$, and $Z(R) = R \setminus \{1\}$ is a semigroup ideal of R that is not an ideal of R. Thus $|\mathcal{C}(R)| \geq |\mathcal{C}_{\{0\}}(R)| + |\mathcal{C}_I(R)| + |\mathcal{C}_J(R)| + |\mathcal{C}_{Z(R)}(R)| + |\mathcal{C}_R(R)| \geq 1 + 2 + 2 + 1 + 1 = 7$. $\square$

Remark 2.14. (a) *One can easily show that* $|\mathcal{C}(\mathbb{Z}_4)| = |\mathcal{C}(\mathbb{Z}_2[X]/(X^2))| = 5$ *and* $|\mathcal{C}(\mathbb{Z}_2 \times \mathbb{Z}_2)| = 7$*; so the lower bounds in Corollary 2.13 may be realized. Moreover, for a Boolean ring* R, $|\mathcal{C}(R)| = 7$ *if and only if* $R = \mathbb{Z}_2 \times \mathbb{Z}_2$. *Also,* $|\mathcal{C}(\mathbb{Z}_3)| = 3$, *and* $|\mathcal{C}(\mathbb{Z}_5)| = 4$ *(see* [20, Example 3.16] *for details). However,* $\Gamma_\sim(R) = \emptyset$ *for every* $\sim \in \mathcal{C}(R)$ *when* R *is a field by Theorem 2.3(b).*

(b) *If* $U(R)$ *has incomparable subgroups, then* $\mathcal{C}_{\{0\}}(R)$ *has incomparable elements by Example 2.1(e). For example, if* R *contains* $\mathbb{Q}$, *then* $\mathcal{C}_{\{0\}}(R)$ *is uncountable and has incomparable elements (see* [20, Example 3.17] *for details; also see Example 2.2(b)).*

By Theorem 2.11(c), if I is a proper ideal of R, then $\sim_1 = \sim_3$ (i.e., $|\mathcal{C}_I(R)| = 1$) if and only if $I = \{0\}$ and R is a Boolean ring. This may fail, however, if I is only assumed to be a semigroup ideal of R. For example, let $R = \mathbb{Z}_2 \times \mathbb{Z}_2$ and $I = Z(R) = R \setminus \{(1,1)\}$. Then I is a nonzero, proper semigroup ideal of R, but not an ideal of R, and it is easily verified that $\sim_1 = \sim_3$; so $|\mathcal{C}_I(R)| = 1$. In fact, let R be any Boolean ring. Then $R = Z(R) \cup \{1\}$, $I = Z(R) = R \setminus \{1\}$ is a prime semigroup ideal of R, and $|\mathcal{C}_I(R)| = 1$. Note that I is an (prime) ideal of R if and only if it is the unique maximal ideal of R, i.e., $R = \mathbb{Z}_2$.

Next, we consider the case when $|\mathcal{C}_I(R)| = 1$ for I a semigroup ideal of R.

Theorem 2.15. *Let* R *be a commutative ring with* $1 \neq 0$, *and let* I *be a proper semigroup ideal of* R. *Define* $\sim_1, \sim_3 \in \mathcal{C}_I(R)$ *by* $x \sim_1 y \Leftrightarrow x = y$ *or* $x, y \in I$ *and* $x \sim_3 y \Leftrightarrow (I : x) = (I : y)$.

(a) *If* $\sim_1 = \sim_3$, *then* $U(R) = \{1\}$ *and* R *is reduced.*

(b) *If* $U(R) = \{1\}$, *then* $R \setminus \{1\}$ *is a prime semigroup ideal of* R *and* $\sim_1 = \sim_3$ *for* $I = R \setminus \{1\}$.

(c) *If* $U(R) = \{1\}$, *then* R *is a Boolean ring if and only if* $\dim(R) = 0$.

(d) *If* $\dim(R) = 0$ *(e.g.,* R *is finite), then* $U(R) = \{1\}$ *if and only if* R *is a Boolean ring.*

Proof. (a) Suppose that $\sim_1 = \sim_3$. Let $x \in U(R)$. Then $(I : x) = I = (I : 1)$; so $x \sim_3 1$. Hence $x \sim_1 1$; so $x = 1$ since I is a proper semigroup ideal of R. Thus

$U(R) = \{1\}$. If $x^2 = 0$ for $x \in R$, then $1 + x \in U(R) = \{1\}$; so $x = 0$. Hence R is reduced.

(b) Suppose that $U(R) = \{1\}$. Then it is easily verified that $R \setminus \{1\}$ is a prime semigroup ideal of R and $\sim_1 = \sim_3$ for $I = R \setminus \{1\}$.

(c) Suppose that $U(R) = \{1\}$. Any Boolean ring R has $\dim(R) = 0$. Conversely, suppose that $\dim(R) = 0$. Then R is reduced by parts (a) and (b) and zero-dimensional by hypothesis; so R is von Neumann regular [18, Theorem 3.1]. Thus every $x \in R$ has the form ue, where $u \in U(R)$ and $e \in R$ is idempotent [18, Theorem 3.2]. Hence every element of R is idempotent since $U(R) = \{1\}$; so R is a Boolean ring.

(d) Suppose that $\dim(R) = 0$. If R is a Boolean ring, then $U(R) = \{1\}$. Conversely, suppose that $U(R) = \{1\}$. Then R is a Boolean ring by part (c). $\square$

Corollary 2.16. *Let $R \neq \mathbb{Z}_2$ be a commutative ring with $1 \neq 0$.*

(a) *R has a nonzero, proper semigroup ideal I with $|\mathcal{C}_I(R)| = 1$ if and only if $U(R) = \{1\}$.*

(b) *If $\dim(R) = 0$ (e.g., R is finite), then R has a nonzero, proper semigroup ideal I with $|\mathcal{C}_I(R)| = 1$ if and only if R is a Boolean ring.*

Remark 2.17. (a) *The $\dim(R) = 0$ hypothesis is needed in Theorem 2.15(d) and Corollary 2.16(b). For example, let $R = \mathbb{Z}_2[X_1, \ldots, X_n]$ for $n \geq 1$. Then $\dim(R) = n \geq 1$ and $U(R) = \{1\}$; so $|\mathcal{C}_I(R)| = 1$ for $I = R \setminus \{1\}$ by Theorem 2.15(b), but R is not a Boolean ring.*

(b) *Let R be a commutative ring with $1 \neq 0$. Then $I = Z(R)$ and $J = R \setminus U(R)$ are always prime semigroup ideals of R; so $\Gamma_\sim(R) = \emptyset$ for every $\sim \in \mathcal{C}_I(R) \cup \mathcal{C}_J(R)$ by Theorem 2.3(a). Note that J is an ideal of R if and only if R is quasilocal (with maximal ideal J), and $|\mathcal{C}_J(R)| = 1$ if and only if $U(R) = \{1\}$ by Theorem 2.15(a),(b).*

(c) *Let R be a commutative ring with $1 \neq 0$. If $\dim(R) = 0$, then R is a Boolean ring if and only if $R = Z(R) \cup \{1\}$ by Theorem 2.15(d). We ask if the $\dim(R) = 0$ hypothesis is needed. See* [3, Section 2] *for other conditions that force R to be a Boolean ring.*

3. Zero-divisor graph maps

In this section, we study functions between $\sim$-zero-divisor graphs over a commutative ring R with $1 \neq 0$. Specifically, for $\sim_1, \sim_2 \in \mathcal{C}_I(R)$ with $\sim_1 \leq \sim_2$, we define a surjective function $F : \Gamma_{\sim_1}(R) \longrightarrow \Gamma_{\sim_2}(R)$ given by $F([x]_{\sim_1}) = [x]_{\sim_2}$ and an injective graph homomorphism $G : \Gamma_{\sim_2}(R) \longrightarrow \Gamma_{\sim_1}(R)$ such that $FG = 1_{\Gamma_{\sim_2}(R)}$.

Let I be a semigroup ideal of R and $\sim_1, \sim_2 \in \mathcal{C}(R)$ with $\sim_1 \leq \sim_2$. Then it is easily verified that $1_R : R \longrightarrow R$ induces a surjective monoid homomorphism $f : R/\sim_1 \longrightarrow R/\sim_2$ given by $f([x]_{\sim_1}) = [x]_{\sim_2}$ with $f([0]_{\sim_1}) = [0]_{\sim_2}$ and $f([1]_{\sim_1}) = [1]_{\sim_2}$. In fact, f is well-defined if and only if $\sim_1 \leq \sim_2$, and f is injective if and only if $\sim_1 = \sim_2$. Now suppose, in addition, that $\sim_1, \sim_2 \in \mathcal{C}_I(R)$ (i.e., $I = [0]_{\sim_1} = [0]_{\sim_2}$). Then $[x]_{\sim_1} \in Z(R/\sim_1)^*$ if and only if $[x]_{\sim_2} \in Z(R/\sim_2)^*$ by Lemma 2.4. Thus f induces a surjective function $F : \Gamma_{\sim_1}(R) \longrightarrow \Gamma_{\sim_2}(R)$ given by $F([x]_{\sim_1}) = [x]_{\sim_2}$, i.e., $F = f|_{Z(R/\sim_1)^*}$. Moreover, for distinct adjacent vertices $[x]_{\sim_1}$ and $[y]_{\sim_1}$ in $\Gamma_{\sim_1}(R)$, either $F([x]_{\sim_1}) = F([y]_{\sim_1})$ or $F([x]_{\sim_1})$ and $F([y]_{\sim_1})$ are adjacent in $\Gamma_{\sim_2}(R)$. Note that F may be well-defined or injective (and hence a graph isomorphism) without f being well-defined or injective (cf. Remark 2.6(c)). Since F is surjective, there is an (not necessarily unique) injective function $G : \Gamma_{\sim_2}(R) \longrightarrow \Gamma_{\sim_1}(R)$ such that $FG = 1_{\Gamma_{\sim_2}(R)}$ (i.e., for each $z \in Z(R/\sim_2)^*$, choose an $\alpha(z) \in Z(R/\sim_1)^*$ such that $F(\alpha(z)) = z$, and then define $G(z) = \alpha(z)$). Moreover, $G([x]_{\sim_2})$ and $G([y]_{\sim_2})$ are adjacent in $\Gamma_{\sim_1}(R)$ if and only if $[x]_{\sim_2}$ and $[y]_{\sim_2}$ are adjacent in $\Gamma_{\sim_2}(R)$ since $[0]_{\sim_1} = [0]_{\sim_2} = I$. Thus G is an injective graph homomorphism and embeds $\Gamma_{\sim_2}(R)$ as an induced subgraph of $\Gamma_{\sim_1}(R)$. We record these observations in the following theorem.

Theorem 3.1. *Let R be a commutative ring with $1 \neq 0$, and let $\sim_1, \sim_2 \in \mathcal{C}(R)$ with $\sim_1 \leq \sim_2$. Then there is a surjective monoid homomorphism $f : R/\sim_1 \longrightarrow R/\sim_2$ given by $f([x]_{\sim_1}) = [x]_{\sim_2}$. If $[0]_{\sim_1} = [0]_{\sim_2}$, then f induces a surjective function $F : \Gamma_{\sim_1}(R) \longrightarrow \Gamma_{\sim_2}(R)$ given by $F([x]_{\sim_1}) = [x]_{\sim_2}$ and an injective graph homomorphism $G : \Gamma_{\sim_2}(R) \longrightarrow \Gamma_{\sim_1}(R)$ such that $FG = 1_{\Gamma_{\sim_2}(R)}$. Moreover, for distinct adjacent vertices $[x]_{\sim_1}$ and $[y]_{\sim_1}$ in $\Gamma_{\sim_1}(R)$, either $F([x]_{\sim_1}) = F([y]_{\sim_1})$ or $F([x]_{\sim_1})$ and $F([y]_{\sim_1})$ are adjacent in $\Gamma_{\sim_2}(R)$; and $\Gamma_{\sim_2}(R)$ is isomorphic to an induced subgraph of $\Gamma_{\sim_1}(R)$.*

Thus, by Theorems 2.8 and 3.1, for a fixed semigroup ideal I of R, there is a largest and a smallest $\sim$-zero-divisor graph for $\sim \in \mathcal{C}_I(R)$. The largest is $\Gamma_I(R)$, and the smallest is $\Gamma_\sim(R)$, where $x \sim y \Leftrightarrow (I : x) = (I : y)$. Moreover, $\Gamma_\sim(R) = \Gamma_E(R/I)$ when I is a proper ideal of R. Hence, for $I = \{0\}$, $\Gamma(R)$ is the largest and $\Gamma_E(R)$ is the smallest zero-divisor graph. In [5, p. 1450070-4], it was observed that $\Gamma_E(R)$ is isomorphic to a subgraph of $\Gamma(R)$, and in [22, Corollary 2.7] that $\Gamma_I(R)$ contains $|I|$ disjoint subgraphs isomorphic to $\Gamma(R/I)$.

However, for $\sim_1, \sim_2 \in \mathcal{C}_I(R)$ with $\sim_1 \leq \sim_2$, the function $F : \Gamma_{\sim_1}(R) \longrightarrow \Gamma_{\sim_2}(R)$ given by $F([x]_{\sim_1}) = [x]_{\sim_2}$ need not be a graph homomorphism since distinct adjacent vertices in $\Gamma_{\sim_1}(R)$ may collapse to the same vertex in $\Gamma_{\sim_2}(R)$. For example, let $R = \mathbb{Z}_8$, $\sim_1 \, = \, =_R$, and define $\sim_2$ by $x \sim_2 y \Leftrightarrow \mathrm{ann}_R(x) = \mathrm{ann}_R(y)$. Then $\sim_1$, $\sim_2 \in \mathcal{C}_{\{0\}}(R)$ with $\sim_1 \leq \sim_2$, $\Gamma_{\sim_1}(R) = \Gamma(R) = K^2$, and $\Gamma_{\sim_2}(R) = \Gamma_E(R) = K^1$; so $F : \Gamma(R) \longrightarrow \Gamma_E(R)$ is not a graph homomorphism. The next theorem gives a sufficient condition for F to be a graph homomorphism.

Theorem 3.2. *Let R be a commutative ring with $1 \neq 0$, I a radical semigroup ideal of R, and $\sim_1$, $\sim_2 \in \mathcal{C}_I(R)$ with $\sim_1 \leq \sim_2$. Then $F : \Gamma_{\sim_1}(R) \longrightarrow \Gamma_{\sim_2}(R)$ given by $F([x]_{\sim_1}) = [x]_{\sim_2}$ is a surjective graph homomorphism.*

Proof. Suppose that distinct vertices $[x]_{\sim_1}$ and $[y]_{\sim_1}$ are adjacent in $\Gamma_{\sim_1}(R)$. Thus $[xy]_{\sim_1} = [x]_{\sim_1}[y]_{\sim_1} = [0]_{\sim_1}$; so $xy \sim_1 0$. Hence $xy \sim_2 0$ since $\sim_1 \leq \sim_2$. If $[x]_{\sim_2} = [y]_{\sim_2}$, then $x \sim_2 y$; so $x^2 \sim_2 xy \sim_2 0$. Thus $x^2 \in [0]_{\sim_2} = I$, and hence $x \in I$ since I is a radical semigroup ideal of R. But then $[x]_{\sim_1} = [0]_{\sim_1}$, a contradiction. Thus $[x]_{\sim_2}$ and $[y]_{\sim_2}$ are distinct adjacent vertices in $\Gamma_{\sim_2}(R)$, and hence F is a surjective graph homomorphism. $\square$

Corollary 3.3. *Let R be a commutative ring with $1 \neq 0$.*

(a) *Let $\sim \in \mathcal{C}_{\{0\}}(R)$. Then there are surjective functions $\Gamma(R) \longrightarrow \Gamma_\sim(R) \longrightarrow \Gamma_E(R)$. Thus $|\Gamma_E(R)| \leq |\Gamma_\sim(R)| \leq |\Gamma(R)|$. Moreover, if R is reduced, then these functions are graph homomorphisms.*

(b) *Let I be a proper ideal of R. Then there are surjective functions $\Gamma_I(R) \longrightarrow \Gamma(R/I) \longrightarrow \Gamma_E(R/I)$. Thus $|\Gamma_E(R/I)| \leq |\Gamma(R/I)| \leq |\Gamma_I(R)|$. Moreover, if R/I is reduced (i.e., I is a radical ideal of R), then these functions are graph homomorphisms.*

(c) *Let I be a proper semigroup ideal of R and $\sim \in \mathcal{C}_I(R)$. Then there are surjective functions $\Gamma_I(R) \longrightarrow \Gamma_\sim(R) \longrightarrow \Gamma_{\sim'}(R)$, where $\sim' \in \mathcal{C}_I(R)$ is defined by $x \sim' y \Leftrightarrow (I : x) = (I : y)$. Thus $|\Gamma_{\sim'}(R)| \leq |\Gamma_\sim(R)| \leq |\Gamma_I(R)|$. Moreover, if I is a radical semigroup ideal of R, then these functions are graph homomorphisms.*

In the following corollary, let the surjective and injective functions be those given in Theorem 3.1, and note that the injective functions may be chosen to make the diagrams commute.

Corollary 3.4. *Let R be a commutative ring with $1 \neq 0$, I a proper ideal of R, and $\sim \in \mathcal{C}_I(R)$. Then there are surjective functions*

$$\begin{tikzcd}
 & \Gamma_{\sim}(R) \arrow[dr, "F'"] & \\
\Gamma_I(R) \arrow[ur, "F"] \arrow[dr, "\overline{F}"'] & & \Gamma_E(R/I) \\
 & \Gamma(R/I) \arrow[ur, "\overline{F}'"'] &
\end{tikzcd}$$

and injective graph homomorphisms

$$\begin{tikzcd}
 & \Gamma_{\sim}(R) \arrow[dr, "G"] & \\
\Gamma_E(R/I) \arrow[ur, "G'"] \arrow[dr, "\overline{G}'"'] & & \Gamma_I(R) \\
 & \Gamma(R/I) \arrow[ur, "\overline{G}"'] &
\end{tikzcd}$$

such that the above diagrams commute and $F\,G = 1_{\Gamma_{\sim}(R)}, \overline{F}\,\overline{G} = 1_{\Gamma(R/I)}, F'\,G' = 1_{\Gamma_E(R/I)}$*, and* $\overline{F}'\,\overline{G}' = 1_{\Gamma_E(R/I)}$*. Moreover,* $\Gamma_{\sim}(R)$ *and* $\Gamma(R/I)$ *are each isomorphic to an induced subgraph of* $\Gamma_I(R)$*, and* $\Gamma_E(R/I)$ *is isomorphic to an induced subgraph of* $\Gamma_{\sim}(R)$*,* $\Gamma(R/I)$*, and* $\Gamma_I(R)$*.*

4. More zero-divisor graph maps

In this section, we extend the results from the previous section using the inclusion function $i : R \longrightarrow T$, where R is a subring of T, as a natural generalization of the identity function $1_R : R \longrightarrow R$. More precisely, let R be a subring of a commutative ring T with $1 \neq 0$, and let $\sim_R \in \mathcal{C}(R)$ and $\sim_T \in \mathcal{C}(T)$. We say that $\sim_R$ and $\sim_T$ are *compatible*, denoted by $\sim_R \leq_i \sim_T$, if $x \sim_R y$ implies $x \sim_T y$ for $x, y \in R$, equivalently, if $\sim_R \subseteq \sim_T$ (and thus $\sim_R \subseteq \sim_T \cap (R \times R)$). Note that if $T = R$, then $\sim_R \leq_i \sim_T$ if and only if $\sim_R \leq \sim_T$. Also, $\sim_R \leq_i \sim_T$ if and only if $[x]_{\sim_R} \subseteq [x]_{\sim_T} \cap R$ for every $x \in R$. In particular, $[0]_{\sim_R} \subseteq [0]_{\sim_T} \cap R$ when $\sim_R \leq_i \sim_T$. Note that the inclusion map $i : R \longrightarrow T$ induces a well-defined monoid homomorphism $f : R/{\sim_R} \longrightarrow T/{\sim_T}$ given by $f([x]_{\sim_R}) = [x]_{\sim_T}$ if and only if $\sim_R \leq_i \sim_T$.

Let $\sim_R \in \mathcal{C}(R)$ and $\sim_T \in \mathcal{C}(T)$ with $\sim_R \leq_i \sim_T$. Then the monoid homomorphism $f : R/{\sim_R} \longrightarrow T/{\sim_T}$ given by $f([x]_{\sim_R}) = [x]_{\sim_T}$ is injective if and only if $[x]_{\sim_T} = [y]_{\sim_T}$ (i.e., $x \sim_T y$) for $x, y \in R$ implies $[x]_{\sim_R} = [y]_{\sim_R}$ (i.e., $x \sim_R y$). Thus f is injective if and only if $\sim_T \cap (R \times R) \subseteq \sim_R$, if and only if $[x]_{\sim_T} \cap R \subseteq [x]_{\sim_R}$

for every $x \in R$. Note that if $T = R$, then f is injective if and only if $\sim_R = \sim_T$. In particular, if f is injective, then $[0]_{\sim_T} \cap R \subseteq [0]_{\sim_R}$, and hence $[0]_{\sim_T} \cap R = [0]_{\sim_R}$. Thus f is well-defined and injective if and only if $\sim_T \cap (R \times R) = \sim_R$, if and only if $[x]_{\sim_T} \cap R = [x]_{\sim_R}$ for every $x \in R$.

Now suppose that $\sim_R \in \mathcal{C}(R)$ and $\sim_T \in \mathcal{C}(T)$ with $\sim_R \leq_i \sim_T$ and $[0]_{\sim_T} \cap R = [0]_{\sim_R}$. Then $f(Z(R/\sim_R)^*) \subseteq Z(T/\sim_T)^*$ since for $x \in R$, $[x]_{\sim_R} = [0]_{\sim_R} \Leftrightarrow [x]_{\sim_T} = [0]_{\sim_T}$ because $[0]_{\sim_T} \cap R = [0]_{\sim_R}$ (cf. Lemma 2.4). Thus f induces a function $F : \Gamma_{\sim_R}(R) \longrightarrow \Gamma_{\sim_T}(T)$ given by $F([x]_{\sim_R}) = [x]_{\sim_T}$, i.e., $F = f|_{Z(R/\sim)^*}$. If $[x]_{\sim_R}$ and $[y]_{\sim_R}$ are distinct adjacent vertices in $\Gamma_{\sim_R}(R)$, then either $F([x]_{\sim_R}) = F([y]_{\sim_R})$ or $F([x]_{\sim_R})$ and $F([y]_{\sim_R})$ are adjacent in $\Gamma_{\sim_T}(T)$. These observations are recorded in the following theorem.

Theorem 4.1. *Let R be a subring of a commutative ring T with $1 \neq 0$, and let $\sim_R \in \mathcal{C}(R)$ and $\sim_T \in \mathcal{C}(T)$ with $\sim_R \leq_i \sim_T$. Then there is a monoid homomorphism $f : R/\sim_R \longrightarrow T/\sim_T$ given by $f([x]_{\sim_R}) = [x]_{\sim_T}$. If $[0]_{\sim_T} \cap R = [0]_{\sim_R}$, then f induces a function $F : \Gamma_{\sim_R}(R) \longrightarrow \Gamma_{\sim_T}(T)$ given by $F([x]_{\sim_R}) = [x]_{\sim_T}$. Moreover, if $[x]_{\sim_R}$ and $[y]_{\sim_R}$ are distinct adjacent vertices in $\Gamma_{\sim_R}(R)$, then either $F([x]_{\sim_R}) = F([y]_{\sim_R})$ or $F([x]_{\sim_R})$ and $F([y]_{\sim_R})$ are adjacent in $\Gamma_{\sim_T}(T)$.*

Just as in the previous section, F need not be a graph homomorphism. However, the next theorem gives several cases where F is a graph homomorphism (cf. Theorem 3.2).

Theorem 4.2. *Let R be a subring of a commutative ring T with $1 \neq 0$, and let $\sim_R \in \mathcal{C}(R)$ and $\sim_T \in \mathcal{C}(T)$ with $\sim_R \leq_i \sim_T$ and $[0]_{\sim_T} \cap R = [0]_{\sim_R}$. Then $F : \Gamma_{\sim_R}(R) \longrightarrow \Gamma_{\sim_T}(T)$ given by $F([x]_{\sim_R}) = [x]_{\sim_T}$ is a graph homomorphism if $[0]_{\sim_R}$ is a radical semigroup ideal of R or $[0]_{\sim_T}$ is a radical semigroup ideal of T. Moreover, F is an injective graph homomorphism if $\sim_T \cap (R \times R) = \sim_R$; so $\Gamma_{\sim_R}(R)$ is isomorphic to an induced subgraph of $\Gamma_{\sim_T}(T)$.*

Proof. If $[0]_{\sim_T}$ is a radical semigroup ideal of T, then $[0]_{\sim_R} = [0]_{\sim_T} \cap R$ is a radical semigroup ideal of R. So we need only show that F is a graph homomorphism when $[0]_{\sim_R}$ is a radical semigroup ideal of R. Let $[x]_{\sim_R}$ and $[y]_{\sim_R}$ be distinct adjacent vertices in $\Gamma_{\sim_R}(R)$. Then $[xy]_{\sim_R} = [x]_{\sim_R}[y]_{\sim_R} = [0]_{\sim_R}$; so $xy \sim_R 0$. If $[x]_{\sim_T} = [y]_{\sim_T}$, then $x \sim_T y$, and thus $x^2 \sim_T xy \sim_T 0$; so $x^2 \in [0]_{\sim_T} \cap R = [0]_{\sim_R}$. Hence $x \in [0]_{\sim_R}$; so $[x]_{\sim_R} = [0]_{\sim_R}$, a contradiction. The "moreover" statement follows from the discussion before Theorem 4.1. $\square$

Given $\sim_T \in \mathcal{C}(T)$, define $\sim_R = \sim_T \cap (R \times R)$. Clearly, $\sim_R \in \mathcal{C}(R)$, $\sim_R \leq_i \sim_T$, and $[0]_{\sim_R} = [0]_{\sim_T} \cap R$. Thus $\sim_T \in \mathcal{C}_I(T)$ implies $\sim_R \in \mathcal{C}_{I \cap R}(R)$. Moreover, $\sim$

$\in \mathcal{C}(R)$ is compatible with $\sim_T$ if and only if $\sim \leq \sim_R$ since $\sim \leq_i \sim_T$ if and only if $\sim \subseteq \sim_T \cap (R \times R) = \sim_R$. Hence $\sim_R$ is the greatest element of $\mathcal{C}(R)$ with $\sim_R \leq_i \sim_T$. With these observations, the next theorem follows directly from Theorems 4.1 and 4.2.

Theorem 4.3. *Let R be a subring of a commutative ring T with $1 \neq 0$, and let $\sim_T \in \mathcal{C}_I(T)$. Then $\sim_R \in \mathcal{C}_{I\cap R}(R)$ defined by $\sim_R = \sim_T \cap (R \times R)$ is the greatest element of $\mathcal{C}(R)$ with $\sim_R \leq_i \sim_T$. The function $F : \Gamma_{\sim_R}(R) \longrightarrow \Gamma_{\sim_T}(T)$ given by $F([x]_{\sim_R}) = [x]_{\sim_T}$ is an injective graph homomorphism; so $\Gamma_{\sim_R}(R)$ is isomorphic to an induced subgraph of $\Gamma_{\sim_T}(T)$.*

The following examples illustrate the preceding theorem.

Example 4.4. (a) *Let $\sim_T$ be $=_T$. Then $\sim_R = \sim_T \cap (R \times R)$ is $=_R$. Thus $\Gamma(R)$ is isomorphic to an induced subgraph of $\Gamma(T)$.*

(b) *Let I be a proper ideal of T. Then $I \cap R$ is a proper ideal of R. Thus $\sim_T \in \mathcal{C}_I(T)$ given by $x \sim y \Leftrightarrow x - y \in I$ for $x, y \in T$ induces $\sim_R \in \mathcal{C}_{I\cap R}(R)$ given by $x \sim_R y \Leftrightarrow x - y \in I \cap R$ for $x, y \in R$. Hence $\Gamma(R/(I \cap R))$ is isomorphic to an induced subgraph of $\Gamma(T/I)$.*

(c) *Let I be a proper semigroup ideal of T. Then $I \cap R$ is a proper semigroup ideal of R. Thus $\sim_T \in \mathcal{C}_I(T)$ given by $x \sim_T y \Leftrightarrow x = y$ or $x, y \in I$ for $x, y \in T$ induces $\sim_R \in \mathcal{C}_{I\cap R}(R)$ given by $x \sim_R y \Leftrightarrow x = y$ or $x, y \in I\cap R$ for $x, y \in R$. Hence $\Gamma_{I\cap R}(R)$ is isomorphic to an induced subgraph of $\Gamma_I(T)$. Of course, part (a) is just part (b) or (c) when $I = \{0\}$.*

However, not all congruence relations and corresponding congruence-based zero-divisor graphs behave so nicely. Let R be a subring of a commutative ring T with $1 \neq 0$. Define $\sim_T \in \mathcal{C}_{\{0\}}(T)$ by $x \sim_T y \Leftrightarrow \mathrm{ann}_T(x) = \mathrm{ann}_T(y)$ and $\sim \in \mathcal{C}_{\{0\}}(R)$ by $x \sim y \Leftrightarrow \mathrm{ann}_R(x) = \mathrm{ann}_R(y)$, and let $\sim_R = \sim_T \cap (R \times R)$. Note that $\Gamma_\sim(R) = \Gamma_E(R)$ and $\Gamma_{\sim_T}(T) = \Gamma_E(T)$. For $x, y \in R$, it is always true that $\mathrm{ann}_T(x) = \mathrm{ann}_T(y) \Rightarrow \mathrm{ann}_R(x) = \mathrm{ann}_R(y)$, but it need not be true that $\mathrm{ann}_R(x) = \mathrm{ann}_R(y) \Rightarrow \mathrm{ann}_T(x) = \mathrm{ann}_T(y)$ (see [4, p. 1630] for a specific example). Thus the "function" $F : \Gamma_E(R) \longrightarrow \Gamma_E(T)$ given by $F([x]_\sim) = [x]_{\sim_T}$ need not be well-defined. The function F is well-defined if and only if $\sim \leq_i \sim_T$, if and only if $\sim \leq \sim_T \cap (R \times R) = \sim_R$. But, by the above comments, $\sim_R \leq \sim$; so F is well-defined (and injective) if and only if $\sim_R = \sim$.

For some subrings R of T, things do behave nicely. For example, if T is a subring of $Q(R)$, the complete ring of quotients of R, then $F : \Gamma_E(R) \longrightarrow \Gamma_E(T)$ is an injective graph homomorphism. Furthermore, if T is a subring of $T(R)$, the total quotient ring of R, then $F : \Gamma_E(R) \longrightarrow \Gamma_E(T)$ is a graph isomorphism [4, Theorem 3.2]. More generally, if T is a flat R-module (e.g., $T = R_S$ for $S \subseteq R$ a multiplicative subset with $S \cap Z(R) = \emptyset$), then $\mathrm{ann}_R(x) = \mathrm{ann}_R(y) \Rightarrow \mathrm{ann}_T(x) = \mathrm{ann}_R(x)T = \mathrm{ann}_R(y)T = \mathrm{ann}_T(y)$ for $x, y \in R$. So, in this case, $\sim_R = \sim$ and $F : \Gamma_E(R) \longrightarrow \Gamma_E(T)$ is an injective graph homomorphism, and thus $\Gamma_E(R)$ is isomorphic to an induced subgraph of $\Gamma_E(T)$ [14, Proposition 3.1].

Let R be a subring of a commutative ring T with $1 \neq 0$, and let $\sim_T, \sim'_T \in \mathcal{C}(T)$ with $\sim_T \leq \sim'_T$. Define $\sim_R, \sim'_R \in \mathcal{C}(R)$ by $\sim_R = \sim_T \cap (R \times R)$ and $\sim'_R = \sim'_T \cap (R \times R)$. It follows that $\sim_R \leq \sim'_R$; so there are surjective monoid homomorphisms $f_R : R/{\sim_R} \longrightarrow R/{\sim'_R}$ and $f_T : T/{\sim_T} \longrightarrow T/{\sim'_T}$ given by $f_R([x]_{\sim_R}) = [x]_{\sim'_R}$ and $f_T([x]_{\sim_T}) = [x]_{\sim'_T}$. Since $\sim_R \leq_i \sim_T$ and $\sim'_R \leq_i \sim'_T$, there are injective monoid homomorphisms $g : R/{\sim_R} \longrightarrow T/{\sim_T}$ and $g' : R/{\sim'_R} \longrightarrow T/{\sim'_T}$ given by $g([x]_{\sim_R}) = [x]_{\sim_T}$ and $g'([x]_{\sim'_R}) = [x]_{\sim'_T}$. If $[0]_{\sim_T} = [0]_{\sim'_T}$, then $[0]_{\sim_R} = [0]_{\sim_T} \cap R = [0]_{\sim'_T} \cap R = [0]_{\sim'_R}$. So by Theorem 3.1, the monoid homomorphisms f_R and f_T induce surjective functions $F_R : \Gamma_{\sim_R}(R) \longrightarrow \Gamma_{\sim'_R}(R)$ and $F_T : \Gamma_{\sim_T}(T) \longrightarrow \Gamma_{\sim'_T}(T)$, respectively. By Theorem 4.3, the monoid homomorphisms g and g' induce injective graph homomorphisms $G : \Gamma_{\sim_R}(R) \longrightarrow \Gamma_{\sim_T}(T)$ and $G' : \Gamma_{\sim'_R}(R) \longrightarrow \Gamma_{\sim'_T}(T)$, respectively. It is easily checked that these functions give commutative diagrams as recorded in the following theorem.

Theorem 4.5. *Let R be a subring of a commutative ring T with $1 \neq 0$, and let $\sim_T, \sim'_T \in \mathcal{C}(T)$ with $\sim_T \leq \sim'_T$. Define $\sim_R, \sim'_R \in \mathcal{C}(R)$ by $\sim_R = \sim_T \cap (R \times R)$ and $\sim'_R = \sim'_T \cap (R \times R)$. Then the following diagram of monoid homomorphisms commutes.*

$$\begin{tikzcd} R/{\sim_R} \arrow[r, "g"] \arrow[d, "f_R"] & T/{\sim_T} \arrow[d, "f_T"] \\ R/{\sim'_R} \arrow[r, "g'"] & T/{\sim'_T} \end{tikzcd}$$

If $[0]_{\sim_T} = [0]_{\sim'_T}$, then the following diagram of induced maps of congruence-based zero-divisor graphs commutes. Moreover, G and G' are graph homomorphisms.

$$\begin{tikzcd} \Gamma_{\sim_R}(R) \arrow[r, "G"] \arrow[d, "F_R"] & \Gamma_{\sim_T}(T) \arrow[d, "F_T"] \\ \Gamma_{\sim'_R}(R) \arrow[r, "G'"] & \Gamma_{\sim'_T}(T) \end{tikzcd}$$

5. Induced zero-divisor graph maps

In this final section, we investigate when a homomorphism $f : R \longrightarrow T$ of commutative rings with $1 \neq 0$ induces a function $F : \Gamma_{\sim_R}(R) \longrightarrow \Gamma_{\sim_T}(T)$ of graphs given by $F([x]_{\sim_R}) = [f(x)]_{\sim_T}$. There are two cases, depending on whether we start with $\sim_R \in \mathcal{C}(R)$ or $\sim_T \in \mathcal{C}(T)$. These results generalize Section 3, where $R = T$ and f is the identity function on R, and Section 4, where f is the inclusion function from R into T.

We start this section with an earlier mentioned result that will be a special case of Theorem 5.9. Let I be a proper ideal of a commutative ring R with $1 \neq 0$. In Example 2.1(b),(d), we observed that in some cases, a $\sim \in \mathcal{C}_I(R)$ induces a $\sim' \in \mathcal{C}_{\{0\}}(R/I)$ such that $[x]_\sim \mapsto [x+I]_{\sim'}$ gives a graph isomorphism $\Gamma_\sim(R) \longrightarrow \Gamma_{\sim'}(R/I)$. Thus a $\sim$-zero-divisor graph may come from different base rings. We next formalize this “change of rings” result.

Theorem 5.1. *Let R be a commutative ring with $1 \neq 0$, and let I be a proper ideal of R. Define $\sim_1 \in \mathcal{C}_I(R)$ by $x \sim_1 y \Leftrightarrow x - y \in I$. Given $\sim \in \mathcal{C}_I(R)$ with $\sim_1 \leq \sim$, define $\sim' \in \mathcal{C}_{\{0\}}(R/I)$ by $x + I \sim' y + I \Leftrightarrow x \sim y$. Then $F : \Gamma_\sim(R) \longrightarrow \Gamma_{\sim'}(R/I)$ given by $F([x]_\sim) = [x+I]_{\sim'}$ is a graph isomorphism.*

Proof. It is easy to verify that $\sim'$ is well-defined and $\sim' \in \mathcal{C}_{\{0\}}(R/I)$ since $\sim_1 \leq \sim$. Define $f : R/\sim \longrightarrow (R/I)/\sim'$ by $f([x]_\sim) = [x+I]_{\sim'}$. It is also easy to verify that f is a monoid isomorphism and induces a graph isomorphism $F : \Gamma_\sim(R) \longrightarrow \Gamma_{\sim'}(R/I)$ given by $F([x]_\sim) = [x+I]_{\sim'}$, i.e., $F = f|_{Z(R/\sim)^*}$. $\square$

Corollary 5.2. *Let R be a commutative ring with $1 \neq 0$, and let I be a proper ideal of R. Define $\sim_1, \sim_2 \in \mathcal{C}_I(R)$ by $x \sim_1 y \Leftrightarrow x - y \in I$ and $x \sim_2 y \Leftrightarrow (I : x) = (I : y)$. Then $\Gamma_{\sim_1}(R) = \Gamma(R/I)$ and $\Gamma_{\sim_2}(R) = \Gamma_E(R/I)$.*

Remark 5.3. *The $\sim_1 \leq \sim$ hypothesis is needed in Theorem 5.1 since $\Gamma_I(R)$ is usually not a $\Gamma_{\sim'}(R/I)$ for any $\sim' \in \mathcal{C}(R/I)$. For example, let $R = \mathbb{Z}$ and $I = 4\mathbb{Z}$. Then $\Gamma_I(R)$ is infinite, but every $\Gamma_{\sim'}(R/I)$ is finite since $R/I = \mathbb{Z}_4$ is finite (actually, each $\Gamma_{\sim'}(R/I)$ has at most one point). In fact, $\sim'$ is well-defined if and only if $\sim_1 \leq \sim$ by Remark 5.10(a).*

Let R and T be commutative rings with $1 \neq 0$, $f : R \longrightarrow T$ a homomorphism, $\sim_R \in \mathcal{C}(R)$, and $\sim_T \in \mathcal{C}(T)$. If $x \sim_R y$ implies $f(x) \sim_T f(y)$ for $x, y \in R$, then we say that $\sim_R$ and $\sim_T$ are *f-compatible* and write $\sim_R \leq_f \sim_T$. (This notation agrees with the earlier notation of $\leq$ and $\leq_i$). For example, $=_R \leq_f \sim_T$ for every $\sim_T \in \mathcal{C}(T)$, and $\sim \leq_f =_T$ for $\sim \in \mathcal{C}(R)$ if and only if $\sim \leq \sim'$, where $x \sim' y \Leftrightarrow x - y \in$

ker$f$. Clearly, $\sim_R \leq_f \sim_T$ if and only if $[x]_{\sim_R} \subseteq f^{-1}([f(x)]_{\sim_T})$ for every $x \in R$. In particular, if $\sim_R \leq_f \sim_T$, then $[0]_{\sim_R} \subseteq f^{-1}([0]_{\sim_T})$. Also, note that if $\sim \leq \sim_R$ in $\mathcal{C}(R)$, $\sim_R \leq_f \sim_T$, and $\sim_T \leq \sim'$ in $\mathcal{C}(T)$, then $\sim \leq_f \sim'$.

If $\sim_R \leq_f \sim_T$, then f induces a monoid homomorphism $\overline{f} : R/{\sim_R} \longrightarrow T/{\sim_T}$ given by $\overline{f}([x]_{\sim_R}) = [f(x)]_{\sim_T}$. If, in addition, $\overline{f}(Z(R/{\sim_R})^*) \subseteq Z(T/{\sim_T})^*$, then $\overline{f}$ induces a function $F : \Gamma_{\sim_R}(R) \longrightarrow \Gamma_{\sim_T}(T)$ given by $F([x]_{\sim_R}) = [f(x)]_{\sim_T}$, i.e., $F = \overline{f}|_{Z(R/\sim_R)^*}$. Note that $\overline{f}(Z(R/{\sim_R})^*) \subseteq Z(T/{\sim_T})^*$ if $[0]_{\sim_R} = f^{-1}([0]_{\sim_T})$ (cf. Lemma 2.4). Moreover, if $[x]_{\sim_R}$ and $[y]_{\sim_R}$ are distinct adjacent vertices in $\Gamma_{\sim_R}(R)$, then either $F([x]_{\sim_R}) = F([y]_{\sim_R})$ or $F([x]_{\sim_R})$ and $F([y]_{\sim_R})$ are adjacent in $\Gamma_{\sim_T}(T)$. Also, as in Theorem 4.2, it is easily verified that F is a graph homomorphism if $f^{-1}([0]_{\sim_T})$ is a radical semigroup ideal of R (e.g., if $[0]_{\sim_T}$ is a radical semigroup ideal of T). This was the case for $f = 1_R$ in Section 3 and f the inclusion map in Section 4. We summarize this discussion in the following theorem.

Theorem 5.4. *Let R and T be commutative rings with $1 \neq 0$, $\sim_R \in \mathcal{C}(R)$, and $\sim_T \in \mathcal{C}(T)$. Suppose that $f : R \longrightarrow T$ is a homomorphism such that $\sim_R \leq_f \sim_T$ and $\overline{f}(Z(R/{\sim_R})^*) \subseteq Z(T/{\sim_T})^*$ (e.g., if $[0]_{\sim_R} = f^{-1}([0]_{\sim_T})$), where $\overline{f} : R/{\sim_R} \longrightarrow T/{\sim_T}$ is the monoid homomorphism given by $\overline{f}([x]_{\sim_R}) = [f(x)]_{\sim_T}$. Then $\overline{f}$ induces a function $F : \Gamma_{\sim_R}(R) \longrightarrow \Gamma_{\sim_T}(T)$ given by $F([x]_{\sim_R}) = [f(x)]_{\sim_T}$. Moreover, if $[x]_{\sim_R}$ and $[y]_{\sim_R}$ are distinct adjacent vertices in $\Gamma_{\sim_R}(R)$, then either $F([x]_{\sim_R}) = F([y]_{\sim_R})$ or $F([x]_{\sim_R})$ and $F([y]_{\sim_R})$ are adjacent in $\Gamma_{\sim_T}(T)$; and F is a graph homomorphism if $f^{-1}([0]_{\sim_T})$ is a radical semigroup ideal of R.*

Example 5.5. *Let R and T be commutative rings with $1 \neq 0$, $f : R \longrightarrow T$ a homomorphism, and $I = kerf$. Then one can easily verify directly that f induces a function $F : \Gamma_I(R) \longrightarrow \Gamma(T)$ given by $F(x) = f(x)$. Let $\sim \in \mathcal{C}(R)$ be defined by $x \sim y \Leftrightarrow x = y$ or $x, y \in I$. Then $\sim \leq_f =_T$ and $[0]_\sim = I = f^{-1}([0]_{=_T})$; so this is a special case of Theorem 5.4. Moreover, F is a graph homomorphism if I is a radical ideal of R, or more specifically, if T is reduced.*

In some cases, the condition $\overline{f}(Z(R/{\sim_R})^*) \subseteq Z(T/{\sim_T})^*$ forces f to be injective; so in these cases, we can assume that R is a subring of T.

Theorem 5.6. *Let R and T be commutative rings with $1 \neq 0$, $\sim_R \in \mathcal{C}_{\{0\}}(R)$, and $\sim_T \in \mathcal{C}(T)$. Suppose that $f : R \longrightarrow T$ is a homomorphism such that $\sim_R \leq_f \sim_T$ and $\overline{f}(Z(R/{\sim_R})^*) \subseteq Z(T/{\sim_T})^*$, where $\overline{f} : R/{\sim_R} \longrightarrow T/{\sim_T}$ is the monoid homomorphism given by $\overline{f}([x]_{\sim_R}) = [f(x)]_{\sim_T}$. If $Z(R) \neq \{0\}$, then f is injective.*

Proof. Suppose that $f(x) = 0$ for $0 \neq x \in R$, and let $0 \neq y \in Z(R)$. Then $xy \in Z(R)$ and $f(xy) = f(x)f(y) = 0$. If $xy \neq 0$, then $[xy]_{\sim_R} \in Z(R/{\sim_R})^*$

by Lemma 2.4. Thus $[0]_{\sim_T} = [f(xy)]_{\sim_T} = \overline{f}([xy]_{\sim_R}) \neq [0]_{\sim_T}$ by hypothesis, a contradiction. Hence $xy = 0$; so $x \in Z(R)^*$, and thus $[x]_{\sim_R} \in Z(R/\sim_R)^*$ by Lemma 2.4 again. But then $\overline{f}([x]_{\sim_R}) = [f(x)]_{\sim_T} = [0]_{\sim_T}$, a contradiction. Hence $\ker f = \{0\}$; so f is injective. □

Corollary 5.7. *Let R and T be commutative rings with $1 \neq 0$ and $Z(R) \neq \{0\}$, $\sim_T \in \mathcal{C}_{\{0\}}(T)$, and $f : R \longrightarrow T$ a homomorphism. Then f induces a function $F : \Gamma(R) \longrightarrow \Gamma_{\sim_T}(T)$ given by $F(x) = [f(x)]_{\sim_T}$ if and only if f is injective. In particular, f induces a function $F : \Gamma(R) \longrightarrow \Gamma(T)$ given by $F(x) = f(x)$ if and only if f is injective.*

Given commutative rings R and T with $1 \neq 0$ and a homomorphism $f : R \longrightarrow T$, we now consider the two problems of when $\sim_R \in \mathcal{C}(R)$ induces a compatible $\sim_T \in \mathcal{C}(T)$ and when $\sim_T \in \mathcal{C}(T)$ induces a compatible $\sim_R \in \mathcal{C}(R)$. In both cases, we will have $x \sim_R y \Leftrightarrow f(x) \sim_T f(y)$.

First, let $\sim_T \in \mathcal{C}_I(T)$. For $x, y \in R$, define $\sim_R$ by $x \sim_R y \Leftrightarrow f(x) \sim_T f(y)$. It is easily verified that $\sim_R \in \mathcal{C}_{f^{-1}(I)}(R)$ and $\sim_R \leq_f \sim_T$. We include these and more in the following theorem.

Theorem 5.8. *Let R and T be commutative rings with $1 \neq 0$, I a semigroup ideal of T, $\sim_T \in \mathcal{C}_I(T)$, and $f : R \longrightarrow T$ a homomorphism. For $x, y \in R$, define $\sim_R$ by $x \sim_R y \Leftrightarrow f(x) \sim_T f(y)$.*

(a) $\sim_R \in \mathcal{C}_{f^{-1}(I)}(R)$ *and* $\sim_R \leq_f \sim_T$.

(b) *Let $\sim \in \mathcal{C}(R)$. Then $\sim \leq_f \sim_T$ if and only if $\sim \leq \sim_R$.*

(c) *The function $F : \Gamma_{\sim_R}(R) \longrightarrow \Gamma_{\sim_T}(T)$ given by $F([x]_{\sim_R}) = [f(x)]_{\sim_T}$ is an injective graph homomorphism. If f is surjective, then F is a graph isomorphism.*

Proof. (a) It is easily verified that $\sim_R \in \mathcal{C}(R)$, and $\sim_R \leq_f \sim_T$ by definition. Also, $x \in [0]_{\sim_R} \Leftrightarrow x \sim_R 0 \Leftrightarrow f(x) \sim_T 0 \Leftrightarrow f(x) \in [0]_{\sim_T} = I$. Thus $[0]_{\sim_R} = f^{-1}(I)$; so $\sim_R \in \mathcal{C}_{f^{-1}(I)}(R)$.

(b) Let $\sim \in \mathcal{C}(R)$. We have already observed that $\sim \leq_f \sim_T$ when $\sim \leq \sim_R$. Conversely, suppose that $\sim \leq_f \sim_T$. Then $x \sim y \Rightarrow f(x) \sim_T f(y) \Rightarrow x \sim_R y$; so $\sim \leq \sim_R$.

(c) By part (a), $\sim_R \leq_f \sim_T$ and $[0]_{\sim_R} = f^{-1}([0]_{\sim_T})$. Thus by Theorem 5.4, f induces the function $F : \Gamma_{\sim_R}(R) \longrightarrow \Gamma_{\sim_T}(T)$ given by $F([x]_{\sim_R}) = [f(x)]_{\sim_T}$. If $F([x]_{\sim_R}) = F([y]_{\sim_R})$, then $[f(x)]_{\sim_T} = [f(y)]_{\sim_T}$. Hence $f(x) \sim_T f(y)$; so $x \sim_R y$ by definition. Thus $[x]_{\sim_R} = [y]_{\sim_R}$; so F is injective. Hence F is a graph homomorphism by Theorem 5.4 again.

Suppose that f is surjective, and let $[z]_{\sim_T} \in Z(T/{\sim_T})^*$. Since f is surjective, $z = f(x)$ for some $x \in R$. It is easily verified that $[x]_{\sim_R} \in Z(R/{\sim_R})^*$. Thus $F([x]_{\sim_R}) = [f(x)]_{\sim_T} = [z]_{\sim_T}$; so F is surjective, and hence a graph isomorphism. $\square$

Next, let $\sim_R \in \mathcal{C}_I(R)$. In this case, we also need to assume that f is surjective, $\ker f \subseteq I$, and $\sim' \leq \sim_R$, where $x \sim' y \Leftrightarrow x - y \in \ker f$, to guarantee that $\sim_T$ is well-defined (see Remark 5.10(a)). For $w, z \in T$, define $\sim_T$ by $w \sim_T z \Leftrightarrow w = f(x)$ and $z = f(y)$ for some $x, y \in R$ with $x \sim_R y$. Then $\sim_T \in \mathcal{C}_{f(I)}(R)$, and as above, we have $f(x) \sim_T f(y) \Leftrightarrow x \sim_R y$ and $\sim_R \leq_f \sim_T$. The following theorem generalizes Theorem 5.1.

Theorem 5.9. *Let R and T be commutative rings with $1 \neq 0$, $f : R \longrightarrow T$ a surjective homomorphism, I a ideal of R with $\ker f \subseteq I$, and $\sim_R \in \mathcal{C}_I(R)$ with $\sim' \leq \sim_R$, where $x \sim' y \Leftrightarrow x - y \in \ker f$. For $w, z \in T$, define $\sim_T$ by $w \sim_T z \Leftrightarrow w = f(x)$ and $z = f(y)$ for some $x, y \in R$ with $x \sim_R y$.*

(a) *Let $x, y \in R$. Then $f(x) \sim_T f(y) \Leftrightarrow x \sim_R y$.*
(b) *$\sim_T \in \mathcal{C}_{f(I)}(T)$ and $\sim_R \leq_f \sim_T$.*
(c) *Let $\sim \in \mathcal{C}(T)$. Then $\sim_R \leq_f \sim$ if and only if $\sim_T \leq \sim$.*
(d) *The function $F : \Gamma_{\sim_R}(R) \longrightarrow \Gamma_{\sim_T}(T)$ given by $F([x]_{\sim_R}) = [f(x)]_{\sim_T}$ is a graph isomorphism.*

Proof. (a) We show that $\sim_T$ is independent of the choices of $x, y \in R$. Suppose that $w = f(x) = f(x')$ and $z = f(y) = f(y')$ for $x, x', y, y' \in R$. Then $x - x', y - y' \in \ker f$; so $x \sim' x'$ and $y \sim' y'$. Thus $x \sim_R x'$ and $y \sim_R y'$ since $\sim' \leq \sim_R$. Hence $x \sim_R y \Leftrightarrow x' \sim_R y'$. Thus $f(x) \sim_T f(y) \Leftrightarrow x \sim_R y$.

(b) Using part (a), it is now easily verified that $\sim_R \in \mathcal{C}(R)$ and $\sim_R \leq_f \sim_T$. Next, let $w = f(x) \in [0]_{\sim_T}$. Then $f(x) \sim_T f(0)$, and thus $x \sim_R 0$ by definition; so $x \in [0]_{\sim_R} = I$. Hence $[0]_{\sim_T} = f([0]_{\sim_R}) = f(I)$; so $\sim_T \in \mathcal{C}_{f(I)}(R)$.

(c) Let $\sim \in \mathcal{C}(T)$. We have already observed that $\sim_R \leq_f \sim$ if $\sim_T \leq \sim$. Conversely, suppose that $\sim_R \leq_f \sim$; we show that $\sim_T \leq \sim$. Suppose that $f(x) \sim_T f(y)$. Then $x \sim_R y$ by definition, and hence $f(x) \sim f(y)$ since $\sim_R \leq_f \sim$. Thus $\sim_T \leq \sim$ as desired.

(d) Since f is surjective and I is an ideal of R with $\ker f \subseteq I = [0]_{\sim_R}$, we have $f^{-1}([0]_{\sim_T}) = f^{-1}(f([0]_{\sim_R})) = [0]_{\sim_R}$. By Theorem 5.4, f induces the function $F : \Gamma_{\sim_R}(R) \longrightarrow \Gamma_{\sim_T}(T)$ given by $F([x]_{\sim_R}) = [f(x)]_{\sim_T}$. If $[f(x)]_{\sim_T} = [f(y)]_{\sim_T}$, then $f(x) \sim_T f(y)$ and hence $x \sim_R y$. Thus F is an injective graph homomorphism by Theorem 5.4 again. The proof that F is surjective is similar to that in Theorem 5.8(c) above. Hence F is a graph isomorphism. $\square$

Remark 5.10. (a) *Note that* $\sim_T$ *in Theorem 5.9 is well-defined if and only if* $\sim' \leq \sim_R$, *where* $x \sim' y \Leftrightarrow x - y \in kerf$. *Suppose that* $\sim_T$ *is well-defined and* $x \sim' y$ *for* $x, y \in R$. *Then* $x - y \in kerf$; *so* $f(x) = f(y)$, *and thus* $f(x) \sim_T f(y)$. *Hence* $x \sim_R y$ *by definition; so* $\sim' \leq \sim_R$. *Also, necessarily* $kerf \subseteq I$. *Let* $x \in kerf$. *Then* $f(x) = 0 = f(0) \Rightarrow f(x) \sim_T f(0) \Rightarrow x \sim_R 0 \Rightarrow x \in [0]_{\sim_R}$. *Thus* $kerf \subseteq [0]_{\sim_R} = I$.

(b) *With the hypotheses of Theorem 5.8, let* $\sim \in \mathcal{C}_{f^{-1}(I)}(R)$ *with* $\sim \leq \sim_R$. *Then the function* $F : \Gamma_\sim(R) \longrightarrow \Gamma_{\sim_T}(T)$ *given by* $F([x]_\sim) = [f(x)]_{\sim_T}$ *is the composition of the two functions* $\Gamma_\sim(R) \longrightarrow \Gamma_{\sim_R}(R) \longrightarrow \Gamma_{\sim_T}(T)$. *The first function is surjective, and the second function is injective.*

(c) *With the hypotheses of Theorem 5.9, let* $\sim \in \mathcal{C}_{f(I)}(T)$ *with* $\sim_T \leq \sim$. *Then the function* $F : \Gamma_{\sim_R}(R) \longrightarrow \Gamma_\sim(T)$ *given by* $F([x]_{\sim_R}) = [f(x)]_\sim$ *is the composition of the two functions* $\Gamma_{\sim_R}(R) \longrightarrow \Gamma_{\sim_T}(T) \longrightarrow \Gamma_\sim(T)$. *The first function is a graph isomorphism, and the second function is surjective.*

(d) *With the hypotheses of Theorem 5.8 (resp., Theorem 5.9), given* $\sim_T, \sim'_T \in \mathcal{C}(T)$ *(resp.,* $\sim_R, \sim'_R \in \mathcal{C}(R)$*) with* $\sim_T \leq \sim'_T$ *(resp.,* $\sim_R \leq \sim'_R$*), then it is easily shown that the induced* $\sim_R, \sim'_R \in \mathcal{C}(R)$ *(resp.,* $\sim_T, \sim'_T \in \mathcal{C}(T)$*) satisfy* $\sim_R \leq \sim'_R$ *(resp.,* $\sim_T \leq \sim'_T$*). Moreover, if* $[0]_{\sim_T} = [0]_{\sim'_T}$ *and* $[0]_{\sim_R} = f^{-1}([0]_{\sim_T}) = f^{-1}([0]_{\sim'_T}) = [0]_{\sim'_R}$, *then we have the associated commutative diagrams given in Theorem 5.11 below.*

We have the following commutative diagrams; details are left to the reader.

Theorem 5.11. *Let* R *and* T *be commutative rings with* $1 \neq 0$, $\sim_R, \sim'_R \in \mathcal{C}(R)$ *and* $\sim_T, \sim'_T \in \mathcal{C}(T)$ *with* $\sim_R \leq \sim'_R$ *and* $\sim_T \leq \sim'_T$, *and* $f : R \longrightarrow T$ *a homomorphism such that* $\sim_R \leq_f \sim_T$ *and* $\sim'_R \leq_f \sim'_T$. *Let* $\overline{f}([x]_{\sim_R}) = [f(x)]_{\sim_T}$, $\overline{f'}([x]_{\sim_R}) = [f'(x)]_{\sim_T}$*),* $g_R([x]_{\sim_R}) = [x]_{\sim'_R}$, *and* $g_T([x]_{\sim_T}) = [x]_{\sim'_T}$. *Then the following diagram of monoid homomorphisms commutes.*

$$\begin{tikzcd}
R/{\sim_R} \arrow[r, "\overline{f}"] \arrow[d, "g_R"] & T/{\sim_T} \arrow[d, "g_T"] \\
R/{\sim'_R} \arrow[r, "\overline{f'}"] & T/{\sim'_T}
\end{tikzcd}$$

If $[0]_{\sim_T} = [0]_{\sim'_T}$ *and* $[0]_{\sim_R} = f^{-1}([0]_{\sim_T}) = f^{-1}([0]_{\sim'_T}) = [0]_{\sim'_R}$, *then the following diagram of induced maps of congruence-based zero-divisor graphs commutes. Moreover,* F, F', G_R, *and* G_T *are graph homomorphisms if* $[0]_{\sim_T}$ *is a radical semigroup ideal of* T.

$$\begin{CD}
\Gamma_{\sim_R}(R) @>{F}>> \Gamma_{\sim_T}(T) \\
@V{G_R}VV @VV{G_T}V \\
\Gamma_{\sim'_R}(R) @>{F'}>> \Gamma_{\sim'_T}(T)
\end{CD}$$

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