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THE GROUP OF SELF-HOMOTOPY EQUIVALENCES OF A SIMPLY CONNECTED AND 4-DIMENSIONAL CW-COMPLEX

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Abstract. Let X be a CW complex, $\mathcal{E}(X)$ the group of homotopy classes of self-homotopy equivalences of X and $\mathcal{E}_*(X)$ its subgroup of the elements that induce the identity on homology. This paper deals with the problem 19 in [Contemp. Math., 519 (2010), 217-230]. Given a group G, find a space X such that $\frac{\mathcal{E}(X)}{\mathcal{E}_*(X)} = G$. For a simply connected and 4-dimensional CW-complex X we define a group $\mathcal{B}^4 \subset \mathrm{aut}(H_*(X,\mathbb{Z}))$ in term of the Whitehead exact sequence of X and we show that this problem has a solution if $G \cong \mathcal{B}^4$ for some space X.

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1. Introduction

Let X be a CW complex, and let $\mathcal{E}(X)$ denote the group of homotopy classes of self-homotopy equivalences of X. The determination of the group $\mathcal{E}(X)$ presents a challenging problem of computation with a long history of progress on special cases (cf. [1,3,5,6,8,9,10,13,15,16]). Several problems related to the group $\mathcal{E}(X)$ are given in the literature especially the realizability of $\mathcal{E}(X)$ as a given group G [13] and the (in)finiteness of the nilpotent group $\mathcal{E}_*(X)$ [2,6,9].

A variant of the realizability problem is the following:

Problem [12, Problem 19]: Given a group G, find X such that $\widehat{\mathcal{E}}(X) = G$.

Here $\widehat{\mathcal{E}}(X)$ is a distinguished subgroup or quotient of $\mathcal{E}(X)$. It may be the subgroup $\mathcal{E}_\sharp(X)$ of self-equivalences that induce the identity on the homotopy groups, the subgroup $\mathcal{E}_*(X)$, or the derived subgroup, or $\widehat{\mathcal{E}}(X)$ may be the quotient $\dfrac{\mathcal{E}(X)}{\mathcal{E}_*(X)}$.

The aim of this paper is to investigate the problem quoted above for a simply connected and 4-dimensional CW-complex X. For this purpose we define the group

$\mathcal{B}^4$ in terms of the Whitehead exact sequence of X [17, page 72]:

$$H_4(X,\mathbb{Z}) \longrightarrow \Gamma(H_2(X,\mathbb{Z})) \longrightarrow \pi_3(X) \longrightarrow H_3(X,\mathbb{Z}) \to 0$$

and a "certain" notion of automorphisms, called the Γ-automorphisms, of this sequence given in Definition 2.6.

Our main result is the following:

Theorem 1. *If X is a simply connected and 4-dimensional CW-complex, then*

$$\frac{\mathcal{E}(X)}{\mathcal{E}_*(X)} \cong \mathcal{B}^4.$$

The idea of using rational homotopy methods to translate the problem of computing or at least getting some informations regarding the (in)finiteness of the groups $\mathcal{E}(X)$ and $\mathcal{E}_*(X)$ within the framework of minimal commutative differential graded algebras and algebraic homotopy of DGA maps traces back to the results of Arkowitz-Lupton [2] in which they exhibited conditions under which $\mathcal{E}_*(X)$ is finite or infinite where X is a rational space having a 2 -stage Postnikov-like decomposition (for example, rationalizations of homogeneous spaces). Using rational homotopy theory we show the following result:

Theorem 2. *Let X be a simply connected 4-dimensional CW-complex having 4-cells. Then $\mathcal{E}_*(X)$ is finite in the following two cases:*

(1) $H_2(X,\mathbb{Q}) = 0$.
(2) $\dim H_2(X,\mathbb{Q}) = 1$ *and* $H_3(X,\mathbb{Q}) = 0$

and infinite if $H_2(X,\mathbb{Q}) \neq 0$ and $H_3(X,\mathbb{Q}) \neq 0$.

In Section 2, we recall the basic definitions of Whitehead's certain exact sequence and his theorem about 4-dimensional simply-connected CW-complexes and in Section 3, we define the group $\mathcal{B}^4$ and give some of its important properties, moreover we formulate and prove the main theorem. In Section 4 we end this work by giving some applications.

2. The certain exact sequence of Whitehead

2.1. The definition of Whitehead's certain exact sequence. All the materials of this section which is essential and fundamental in this work can be found in details in [4,17].

Let X be a simply connected CW-complex defined by the collection of its skeleta $(X_n)_{n\geq 0}$, where we can suppose $X_0 = X_1 = \star$.

The long exact sequence of the pair (X_n, X_{n-1}) in homotopy and in homology are connected by the Hurewicz morphism h_* in order to give the following commutative diagram:

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{i_{m,n}} & \pi_m(X_n) & \xrightarrow{j_{m,n}} & \pi_m(X_n, X_{n-1}) & \xrightarrow{\beta_{m,n}} & \pi_{m-1}(X_{n-1}) & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle h_m} & & \downarrow{\scriptstyle h_m} & & \downarrow{\scriptstyle h_{m-1}} & & \\ \cdots & \xrightarrow{i^H_{m,n}} & H_m(X_n, \mathbb{Z}) & \xrightarrow{j^H_{m,n}} & H_m((X_n, X_{n-1}), \mathbb{Z}) & \xrightarrow{\beta^H_{m,n}} & H_{m-1}(X_{n-1}, \mathbb{Z}) & \longrightarrow & \cdots \end{array}$$

Remark 2.1. *The group $C_n X = \pi_n(X_n, X_{n-1})$ with the differential $d_n = j_n \circ \beta_n$, where $\beta_n = \beta_{n,n}$ and $j_n = j_{n,n}$, defines the cellular chain complex of X. Moreover $\beta_n : C_n X \to \pi_{n-1}(X^{n-1})$ represents by adjunction the attaching map for the n-cells $\vee S^n \to X^{n-1}$.*

Now Whitehead [17, page 72] inserted the Hurewicz homomorphism in a long exact sequence connecting homology and homotopy. First he defined the following abelian group

$$\Gamma_n^X = \text{Im } (i_n : \pi_n(X_{n-1}) \to \pi_n(X_n)) = \ker j_n, \forall n \geq 2. \tag{1}$$

We notice that $\beta_{n+1} \circ d_{n+1} = 0$ and so $\beta_{n+1} : \pi_{n+1}(X_{n+1}, X_n) \to \pi_n(X_n)$ factors through the quotient: $b_{n+1} : H_{n+1}(X) \to \Gamma_n^X$.
With this map, Whitehead [17] defined the following sequence:

$$\cdots \to H_{n+1}(X, \mathbb{Z}) \xrightarrow{b_{n+1}} \Gamma_n^X \longrightarrow \pi_n(X) \xrightarrow{h_n} H_n(X, \mathbb{Z}) \to \cdots \tag{2}$$

and proved the following.

Theorem 2.2. *The sequence (2), called the Whitehead exact sequence of X, is a natural exact sequence.*

2.2. **4-dimensional CW-complexes.** Let $\mathbb{A}b$ be the category of abelian groups. Using the notion of quadratic maps, Whitehead constructed a functor $\Gamma : \mathbb{A}b \to \mathbb{A}b$ called Whitehead's quadratic functor [17].

Proposition 2.3. *The Whitehead's quadratic functor has the following properties (see for example [4, page 448]) for more details):*

(1) $\Gamma_3^X = \Gamma(H_2(X, \mathbb{Z}))$;
(2) $\Gamma(\mathbb{Z}) = \mathbb{Z}$;
(3) $\Gamma(\mathbb{Z}_n) = \mathbb{Z}_{2n}$, *n even;*
(4) $\Gamma(\mathbb{Z}_n) = \mathbb{Z}_n$, *n odd;*

(5) *Let A an abelian group and let $n. : A \to A$ denote the multiplication by n, that means $a \longmapsto na$, If $n.$ is an isomorphism of abelian groups, then $\Gamma(n.) : \Gamma(A) \to \Gamma(A)$ is the multiplication by n^2 i.e., $\Gamma(n.) = n^2$ and it is also an isomorphism of abelian groups.*

Definition 2.4. Given four abelian groups H_4, H_3, H_2, π where H_4 is free. A Γ-sequence is an exact sequence of abelian groups:

$$H_4 \to \Gamma(H_2) \to \pi \to H_3 \to 0$$

where Γ is the Whitehead's quadratic functor.

Example 2.5. *According to Proposition 2.3, if X is a simply connected 4-dimensional CW-complex, then its Whitehead exact sequence can be written as follows:*

$$H_4(X,\mathbb{Z}) \xrightarrow{b_4} \Gamma(H_2(X,\mathbb{Z})) \longrightarrow \pi_3(X) \longrightarrow H_3(X,\mathbb{Z}) \to 0 \tag{3}$$

thus its a Γ-sequence.
Notice that in this case $H_4(X,\mathbb{Z})$ is a free abelian groups which admits the set of all the 4-cells as a basis.

Definition 2.6. Let X be a simply connected 4-dimensional CW-complex and let $(f_4, f_3, f_2) \in \text{aut}(H_4(X,\mathbb{Z})) \times \text{aut}(H_3(X,\mathbb{Z})) \times \text{aut}(H_2(X,\mathbb{Z}))$. We say that the triple (f_4, f_3, f_2) is a Γ-automorphism of the Whitehead exact sequence of X if there exists an automorphism $\Omega : \pi_3(X) \to \pi_3(X)$ making the following diagram commutes:

$$\begin{array}{ccccccc} H_4(X,\mathbb{Z}) & \xrightarrow{b_4} & \Gamma(H_2(X,\mathbb{Z})) & \xrightarrow{i_3} & \pi_3(X) & \overset{h_3}{\twoheadrightarrow} & H_3(X,\mathbb{Z}) \\ \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle \Gamma(f_2)} & & \downarrow{\scriptstyle \Omega} & & \downarrow{\scriptstyle f_3} \\ H_4(X,\mathbb{Z}) & \xrightarrow{b_4} & \Gamma(H_2(X,\mathbb{Z})) & \xrightarrow{i_3} & \pi_3(X) & \overset{h_3}{\twoheadrightarrow} & H_3(X,\mathbb{Z}) \end{array}$$

Example 2.7. *In [4, page 450] it is shown that if X is a simply connected 4-dimensional CW-complex and if $\alpha : X \to X$ is a homotopy equivalence, then $(H_4(\alpha), H_3(\alpha), H_2(\alpha))$ is a Γ-automorphism.*

In order to state Whitehead's theorems on 4-dimensional CW-complexes. We need the following two definitions.

Definition 2.8. A Γ-sequence $H_4 \to \Gamma(H_2) \to \pi \to H_3 \to 0$ is said to be realizable if there exists a simply connected 4-dimensional CW-complex X such that its Whitehead exact sequence coincides with the given Γ-sequence.

Definition 2.9. Let X be a simply connected 4-dimensional CW-complex and let (f_4, f_3, f_2) be a Γ-automorphism of the Whitehead exact sequence of X. We say that (f_4, f_3, f_2) is realizable if there exists a homotopy equivalence $\alpha : X \to X$ such that $(f_4, f_3, f_2) = (H_4(\alpha), H_3(\alpha), H_2(\alpha))$.

Now we are ready to formulate Whitehead's theorems which give a complete classification of homotopy types of simply connected 4-dimensional CW-complexes.

Theorem 2.10. [4, Theorem 4.9] *Every Γ-sequence is realizable. Every Γ-morphism of the Whitehead exact sequence of a simply connected 4-dimensional CW-complex X is realizable.*

Remark 2.11. *Theorem 2.10 is not valid for CW-complexes of higher dimensions. Nevertheless the author* [7] *generalize Whitehead's theorems for simply connected n-dimensional CW-complexes where $n \geq 5$ by introducing the notion of strong automorphisms of the Whitehead exact sequences of simply connected n-dimensional CW-complex extending the notion of the Γ-automorphisms.*

3. The main results

3.1. The group $\mathcal{B}^4$. In this paragraph we introduce the group $\mathcal{B}^4$ given in the introduction and which plays a crucial role in this paper.

Definition 3.1. Let X be a simply connected 4-dimensional CW-complex. We define $\mathcal{B}^4$ to be the set of all the Γ-automorphisms of the Whitehead exact sequence of X.

Remark 3.2. *Example 2.7 gives that* $(id_{H_4(X,\mathbb{Z}))}, id_{H_3(X,\mathbb{Z}))}, id_{H_2(X,\mathbb{Z}))}) \in \mathcal{B}^4$.

Proposition 3.3. $\mathcal{B}^4$ *is a subgroup of* $\text{aut}(H_4(X,\mathbb{Z}))\times\text{aut}(H_3(X,\mathbb{Z}))\times\text{aut}(H_2(X,\mathbb{Z}))$.

Proof. First let us prove that $(f_4, f_3, f_2), (f'_4, f'_3, f'_2) \in \mathcal{B}^4$, then the composition:

$$(f'_4, f'_3, f'_2) \circ (f_4, f_3, f_2) = (f'_4 \circ f_4, f'_3 \circ f_3, f'_2 \circ f_2) \in \mathcal{B}^4$$

Indeed, since $(f_4, f_3, f_2), (f'_4, f'_3, f'_2) \in \mathcal{B}^4$ from Definition 2.6 we deduce that there exist two automorphisms $\Omega, \Omega' : \pi_3(X) \to \pi_3(X)$ making the diagrams (1) and (2) commute:

$$\begin{array}{ccccccc}
H_4(X,\mathbb{Z}) & \xrightarrow{b_4} & \Gamma(H_2(X,\mathbb{Z})) & \xrightarrow{i_3} & \pi_3(X) & \overset{h_3}{\twoheadrightarrow} & H_3(X,\mathbb{Z}) \\
\Big\downarrow{\scriptstyle f_4} & & \Big\downarrow{\scriptstyle \Gamma(f_2)} & & \Big\downarrow{\scriptstyle \Omega} & & \Big\downarrow{\scriptstyle f_3} \\
H_4(X,\mathbb{Z}) & \xrightarrow{b_4} & \Gamma(H_2(X,\mathbb{Z})) & \xrightarrow{i_3} & \pi_3(X) & \overset{h_3}{\twoheadrightarrow} & H_3(X,\mathbb{Z}) \\
\Big\downarrow{\scriptstyle f'_4} & & \Big\downarrow{\scriptstyle \Gamma(f'_2)} & & \Big\downarrow{\scriptstyle \Omega'} & & \Big\downarrow{\scriptstyle f'_3} \\
H_4(X,\mathbb{Z}) & \xrightarrow{b_4} & \Gamma(H_2(X,\mathbb{Z})) & \xrightarrow{i_3} & \pi_3(X) & \overset{h_3}{\twoheadrightarrow} & H_3(X,\mathbb{Z})
\end{array}
\begin{array}{c} (1) \\ \\ (2) \end{array}$$

Next as Γ is a functor, so $\Gamma(f'_2)\circ\Gamma(f_2) = \Gamma(f'_2\circ f_2)$. Therefore the commutativity of the diagrams (1) and (2) implies that the following diagram commutes:

$$\begin{array}{ccccccc}
H_4(X,\mathbb{Z}) & \xrightarrow{b_4} & \Gamma(H_2(X,\mathbb{Z})) & \xrightarrow{i_3} & \pi_3(X) & \overset{h_3}{\twoheadrightarrow} & H_3(X,\mathbb{Z}) \\
\Big\downarrow{\scriptstyle f'_4\circ f_4} & & \Big\downarrow{\scriptstyle \Gamma(f'_2\circ f_2)} & & \Big\downarrow{\scriptstyle \Omega'\circ\Omega} & & \Big\downarrow{\scriptstyle f'_3\circ f_3} \\
H_4(X,\mathbb{Z}) & \xrightarrow{b_4} & \Gamma(H_2(X,\mathbb{Z})) & \xrightarrow{i_3} & \pi_3(X) & \overset{h_3}{\twoheadrightarrow} & H_3(X,\mathbb{Z})
\end{array} \quad (3)$$

It follows that $(f'_4\circ f_4, f'_3\circ f_3, f'_2\circ f_2)\in\mathcal{B}^4$.

Finally if $(f_4, f_3, f_2)\in\mathcal{B}^4$, then by definition f_4, f_3, f_2 are automorphisms so we get the triple $(f_4^{-1}, f_3^{-1}, f_2^{-1})$. As $(f_4, f_3, f_2)\in\mathcal{B}^4$ there is an automorphism $\Omega : \pi_3(X)\to\pi_3(X)$ making the diagram (1) commutes which implies that the following diagram is also commutative:

$$\begin{array}{ccccccc}
H_4(X,\mathbb{Z}) & \xrightarrow{b_4} & \Gamma(H_2(X,\mathbb{Z})) & \xrightarrow{i_3} & \pi_3(X) & \overset{h_3}{\twoheadrightarrow} & H_3(X,\mathbb{Z}) \\
\Big\downarrow{\scriptstyle f_4^{-1}} & & \Big\downarrow{\scriptstyle (\Gamma(f_2))^{-1}} & & \Big\downarrow{\scriptstyle \Omega^{-1}} & & \Big\downarrow{\scriptstyle f_3^{-1}} \\
H_4(X,\mathbb{Z}) & \xrightarrow{b_4} & \Gamma(H_2(X,\mathbb{Z})) & \xrightarrow{i_3} & \pi_3(X) & \overset{h_3}{\twoheadrightarrow} & H_3(X,\mathbb{Z})
\end{array} \quad (4)$$

Since $(\Gamma(f_2))^{-1} = \Gamma(f_2^{-1})$ it follows that the triple $(f_4^{-1}, f_3^{-1}, f_2^{-1})\in\mathcal{B}^4$. □

Let X be a simply connected 4-dimensional CW-complex. Example 2.7 allows us to define a map $\Psi : \mathcal{E}(X)\to\mathcal{B}^4$ by setting:

$$\Psi\big([\alpha]\big) = (H_4(\alpha), H_3(\alpha), H_2(\alpha)) \quad (4)$$

Proposition 3.4. *The map Ψ is a surjective homomorphism of groups whose kernel is $\mathcal{E}_*(X)$.*

Proof. First let $[\alpha], [\alpha'] \in \mathcal{E}(X)$. Using the formula (4), an easy computation shows that:

$$\begin{aligned} \Psi([\alpha].[\alpha']) &= \Psi([\alpha \circ \alpha']) \\ &= (H_4(\alpha \circ \alpha'), H_3(\alpha \circ \alpha'), H_2(\alpha \circ \alpha')) \\ &= (H_4(\alpha) \circ H_4(\alpha'), H_3(\alpha) \circ H_3(\alpha'), H_2(\alpha) \circ H_2(\alpha')) \\ &= (H_4(\alpha), H_3(\alpha), H_2(\alpha)) \circ (H_4(\alpha'), H_3(\alpha'), H_2(\alpha')) \\ &= \Psi([\alpha]).\Psi([\alpha']) \end{aligned} \quad (5)$$

it follows that Ψ is a homomorphism of groups. Clearly $\ker\Psi = \mathcal{E}_*(X)$ and finally the surjection of the homomorphism Ψ is given by Theorem 2.10. □

Accordingly we are now ready the announce our main result.

Theorem 3.5. *If X is a simply connected and 4-dimensional CW-complex, then*

$$\frac{\mathcal{E}(X)}{\mathcal{E}_*(X)} \cong \mathcal{B}^4.$$

Corollary 3.6. *Let G be a group. If $G \cong \mathcal{B}^4$, then the problem quoted in the introduction has a solution.*

4. Applications

Let X be a simply connected 4-dimensional CW-complex. Our first application deals with the question of the (in)finiteness of the groups $\mathcal{E}(X)$ and $\mathcal{E}_*(X)$. More precisely from Theorem 3.5 we derive the following corollary which is is straightforward.

Corollary 4.1. *Let X be a simply connected 4-dimensional CW-complex.*

(1) *$\mathcal{E}(X)$ is finite if and only if $\mathcal{E}_*(X)$ and $\mathcal{B}^4$ are finite;*
(2) *if $\mathcal{B}^4$ is an infinite group, then so is $\mathcal{E}(X)$.*

Next the following theorem concerns the finiteness of the group $\mathcal{E}_*(X)$.

Theorem 4.2. *Let X be a simply connected 4-dimensional CW-complex having 4-cells. Then $\mathcal{E}_*(X)$ is finite in the following two cases:*

(1) *$H_2(X, \mathbb{Q}) = 0$.*
(2) *$\dim H_2(X, \mathbb{Q}) = 1$ and $H_3(X, \mathbb{Q}) = 0$*

and infinite if $H_2(X, \mathbb{Q}) \neq 0$ and $H_3(X, \mathbb{Q}) \neq 0$.

Proof. First let us consider the space $X_{\mathbb{Q}}$ which is the rationalized of the space X. That is a simply connected CW-complex which satisfies:

$$H_*(X_{\mathbb{Q}}, \mathbb{Z}) = H_*(X, \mathbb{Q}), \qquad \pi_*(X_{\mathbb{Q}}) = \pi_*(X) \otimes \mathbb{Q}.$$

By rational homotopy theory $X_{\mathbb{Q}}$ admits a Quillen model. That means there exists a free differential graded Lie algebra $(L(V), \partial)$, where V is a graded vector space such that each V_{i-1} admits the set of the *i-cells* of X as a basis. In addition we have:

$$V_{*-1} = H_*(X_{\mathbb{Q}}, \mathbb{Z}), \qquad H_{*-1}((L(V), \partial)) \cong \pi_*(X) \otimes \mathbb{Q}.$$

As the Quillen model determines completely the rational homotopy type of a simply connected CW-complex X, we can derive that:

$$\mathcal{E}(X_{\mathbb{Q}}) \cong \mathcal{E}\big((L(V), \partial)\big), \qquad \mathcal{E}_*(X_{\mathbb{Q}}) \cong \mathcal{E}_*\big((L(V), \partial)\big) \tag{6}$$

where $\mathcal{E}\big((L(V), \partial)\big)$ denotes the group of DG Lie homotopy self-equivalences of $(L(V), \partial)$ and where $\mathcal{E}_*\big((L(V), \partial)\big)$ denotes the subgroup of $\mathcal{E}\big((L(V), \partial)\big)$ consists of maps inducing the identity automorphism of the indecomposables.

Next according to Dror-Zabrodsky [11], we know that $\mathcal{E}_*(X)$ is a nilpotent group and in [14] Maruyama proved that $\mathcal{E}_*(X)_{\mathbb{Q}} = \mathcal{E}_*(X_{\mathbb{Q}})$. Here $\mathcal{E}_*(X)_{\mathbb{Q}}$ is the localized of the nilpotent group $\mathcal{E}_*(X)$ at $\mathbb{Q}$. Using (6) we get:

$$\mathcal{E}_*(X)_{\mathbb{Q}} = \mathcal{E}_*\big((L(V), \partial)\big). \tag{7}$$

Then let $\{v_1, \dots, v_n\}$ be a basis of the vector space V_3 (here we assume that the CW-complex has n 4-cells). For every $r \in \mathbb{Q}$, we define $\alpha_r : (L(V), \partial) \to (L(V), \partial)$ as follows:

$$\begin{aligned} \alpha_r(v_i) &= v_i + rx_i + ry_i \ , \quad \text{on } V_3 \text{ where } x_i \in [V_2, V_1] \text{ and } y_i \in [V_1, [V_1, V_1]] \\ \alpha_r &= id \ , \quad \text{on } V_1 \text{ and } V_2. \end{aligned} \tag{8}$$

As the differential ∂ is quadratic, the following diagram is obviously commutative:

$$\begin{array}{ccc} V_3 & \xrightarrow{\alpha_r} & V_3 \oplus [V_2, V_1] \oplus [V_1, [V_1, V_1]] \\ \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} \\ [V_1, V_1] & \xrightarrow{id} & [V_1, V_1] \end{array}$$

so α_r is a DG Lie morphism which induces the identity on the indecomposables. It follows that $[\alpha_r] \in \mathcal{E}_*\big((L(V), \partial)\big)$.

Now if $H_2(X,\mathbb{Q}) = 0$ or ($\dim H_2(X,\mathbb{Q}) = 1$ and $H_3(X,\mathbb{Q}) = 0$), then the vector space $[V_2, V_1] \oplus [V_1, [V_1, V_1]]$ is nil. So the elements x_i and y_i, given in the formula (8), are also nil and obviously $\mathcal{E}_*\big((L(V), \partial)\big)$ is trivial. It follows by (7) that $\mathcal{E}_*(X)$ is finite. If $H_2(X,\mathbb{Q}) \neq 0$ and $H_3(X,\mathbb{Q}) \neq 0$, then $[V_2, V_1] \neq 0$. So the elements x_i can be chosen non-zero so that α_r and $\alpha_{r'}$ are not homotopic provides that $r \neq r'$. Consequently $\mathcal{E}_*\big((L(V), \partial)\big)$ contains an infinity of elements $[\alpha_r]$, $r \in \mathbb{Q}$. Hence $\mathcal{E}_*\big((L(V), \partial)\big)$ is infinite and by (7) $\mathcal{E}_*(X)$ is also infinite. □

Theorem 4.3. *Let X be a simply connected 4-dimensional CW-complex. If the groups $H_*(X,\mathbb{Z})$ are finite, then so is $\mathcal{E}(X)$.*

Proof. First since the groups $H_*(X,\mathbb{Z})$ are finite, the group $\mathcal{B}^4$ is also finite. Next the finiteness of $H_*(X,\mathbb{Z})$ implies that the Quillen model of X is trivial so the group $\mathcal{E}_*(X)_{\mathbb{Q}}$ is also trivial. Therefore by Maruyama Theorem we deduce that $\mathcal{E}_*(X)$ is finite. As a result $\mathcal{E}(X)$ is also finite. □

Theorem 4.2 implies the following corollary.

Corollary 4.4. *Let X be a simply connected 4-dimensional CW-complex having 4-cells. Then $\mathcal{E}(X)$ is infinite if $H_2(X,\mathbb{Q}) \neq 0$ and $H_2(X,\mathbb{Q}) \neq 0$.*

If $H_2(X,\mathbb{Q}) = 0$ (or $\dim H_2(X,\mathbb{Q}) = 1$ and $H_3(X,\mathbb{Q}) = 0$), then $\mathcal{E}(X)$ is finite if and only if the group $\mathcal{B}^4$ is finite.

The next result relates the finiteness of the $\mathcal{E}(X)$ to the Hurewicz homomorphism.

Theorem 4.5. *Let X be a simply connected 4-dimensional CW-complex. Assume that the Hurewicz homomorphism $h_4 : \pi_4(X) \to H_4(X,\mathbb{Z})$ is surjective. Then the group $\mathcal{B}^4$ contains a subgroup isomorphic to $\mathrm{aut}(H_4(X,\mathbb{Z}))$.*

Proof. First according to the exact sequence of Whitehead of X, the surjectivity of the Hurewicz homomorphism h_4 implies that the homomorphism b_4 is nil. It follows that every automorphism $f_4 \in \mathrm{aut}(H_4(X,\mathbb{Z}))$ makes the following diagram commutes:

$$\begin{array}{ccccccc}
H_4(X,\mathbb{Z}) & \xrightarrow{b_4} & \Gamma(H_2(X,\mathbb{Z})) & \xrightarrow{i_3} & \pi_3(X^4) & \overset{h_3}{\twoheadrightarrow} & H_3(X,\mathbb{Z}) \\
\downarrow f_4 & & \downarrow \Gamma(id)=id & & \downarrow id & & \downarrow id \\
H_4(X,\mathbb{Z}) & \xrightarrow{b_4} & \Gamma(H_2(X,\mathbb{Z})) & \xrightarrow{i_3} & \pi_3(X) & \overset{h_3}{\twoheadrightarrow} & H_3(X,\mathbb{Z})
\end{array}$$

therefore for every $f_4 \in \operatorname{aut}(H_4(X,\mathbb{Z}))$ the triple $(f_4, id_{H_3(X,\mathbb{Z})}, id_{H_2(X,\mathbb{Z})})$ belongs to the group $\mathcal{B}^4$. Thus $\operatorname{aut}(H_4(X,\mathbb{Z})) \times \{id_{H_3(X,\mathbb{Z})}\} \times \{id_{H_2(X,\mathbb{Z})}\}$ is a subgroup of $\mathcal{B}^4$. Finally we conclude Theorem 4.5 by observing that the two groups $\operatorname{aut}(H_4(X,\mathbb{Z}))$ and $\operatorname{aut}(H_4(X,\mathbb{Z})) \times \{id_{H_3(X,\mathbb{Z})}\} \times \{id_{H_2(X,\mathbb{Z})}\}$ are isomorphic. □

Now as X is a simply connected 4-dimensional CW-complex, then the group $H_4(X,\mathbb{Z})$ is free of rank n, where n is the number of the 4-cells of X.

Corollary 4.6. *Let X be a simply connected 4-dimensional CW-complex. Assume that h_4 is surjective.*

(1) *If $n \geq 2$, then the index $[\mathcal{E}(X) : \mathcal{E}_*(X)]$ is infinite and the quotient group $\frac{\mathcal{E}(X)}{\mathcal{E}_*(X)}$ contains a subgroup isomorphic to $GL(n,\mathbb{Z})$.*
(2) *If $n = 1$, then $\frac{\mathcal{E}(X)}{\mathcal{E}_*(X)}$ contains an element of order 2.*

Proof. If $n \geq 2$, then we have $\operatorname{aut}(H_4(X,\mathbb{Z})) = GL(n,\mathbb{Z})$ and when $n = 1$ we have $\operatorname{aut}(H_4(X,\mathbb{Z})) = \operatorname{aut}(\mathbb{Z}) \cong \mathbb{Z}_2$. □

4.1. Examples. In the following examples we give explicit computations of the group $\mathcal{B}^4$ showing that it may be finite or infinite.

Example 4.7. *Let X be a simply connected 4-dimensional CW-complex such that:*

$$H_2(X,\mathbb{Z}) = H_3(X,\mathbb{Z}) = H_4(X,\mathbb{Z}) = \mathbb{Z}. \tag{9}$$

First using the properties of Whitehead's quadratic functor given in Proposition 2.3 we obtain that:

$$\Gamma(H_2(X,\mathbb{Z})) = \Gamma(\mathbb{Z}) = \mathbb{Z}.$$

Therefore the Whitehead exact sequence of X which is an example of a Γ-sequence can be written as follows:

$$\mathbb{Z} \xrightarrow{b} \mathbb{Z} \longrightarrow \pi_3(X) \longrightarrow \mathbb{Z} \to 0. \tag{10}$$

It is important to notice that by virtue of Theorem 2.10 this Γ-sequence is realizable, so there exists a simply connected 4-dimensional CW-complex X having (10) as the Whitehead exact sequence.
Next let us compute the group $\mathcal{B}^4$ in this case. Since the group $\operatorname{aut}(\mathbb{Z}) \cong \mathbb{Z}_2$ we deduce that $\mathcal{B}^4$ is a subgroup of $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Consequently if $(f_4, f_3, f_2) \in \mathcal{B}^4$, then:

$$f_4 = \pm 1, \qquad f_3 = \pm 1, \qquad f_2 = \pm 1. \tag{11}$$

It follows, using the properties of Whitehead's quadratic functor given in Proposition 2.3, that $\Gamma(f_2) = 1$. Therefore we seek the automorphisms $f_4 = \pm 1$ and $f_3 = \pm 1$

for which there exists an automorphism $\Omega : \pi_3(X) \to \pi_3(X)$ *making the following diagram commutes;*

$$\begin{array}{ccccccccc} \mathbb{Z} & \xrightarrow{b} & \mathbb{Z} & \longrightarrow & \pi_3(X) & \longrightarrow & \mathbb{Z} & \to & 0 \\ \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \Omega} & & \downarrow{\scriptstyle f_3} & & \\ \mathbb{Z} & \xrightarrow{b} & \mathbb{Z} & \longrightarrow & \pi_3(X) & \longrightarrow & \mathbb{Z} & \to & 0 \end{array} \tag{5}$$

so we have to treat two cases.

Case 1: *The homomorphism* $\mathbb{Z} \xrightarrow{b} \mathbb{Z}$ *is not nil. In this case to have the diagram* (5) *commutes we must have* $f_4 = 1$*. Moreover we get the extension:*

$$0 \to \operatorname{coker} b \longrightarrow \pi_3(X) \longrightarrow \mathbb{Z} \to 0$$

which splits since $\mathbb{Z}$ *is free, i.e.,*

$$\pi_3(X) \cong \operatorname{coker} b \oplus \mathbb{Z}. \tag{12}$$

This implies that any automorphism $\Omega : \pi_3(X) \to \pi_3(X)$ *making the diagram* (5) *commutes splits also i.e.,* $\Omega = 1 \oplus f_3$*. As a result we deduce that:*

- *if* $f_3 = 1$*, then we take* $\Omega = 1 \oplus 1$*. That means if* $x \in \pi_3(X)$*, then using the splitting (12) we can decompose* x *into* $y \oplus z$ *where* $y \in \operatorname{coker} b$ *and* $z \in \mathbb{Z}$ *and we get* $\Omega(x) = (1 \oplus 1)(y \oplus z) = y \oplus z$
- *if* $f_3 = 1$*, then we take* $\Omega = 1 \oplus (-1)$*. That means* $\Omega(x) = (1 \oplus (-1))(x \oplus y) = y \oplus (-z)$*.*

Consequently we get only 4 triples which are:

$$(f_4 = 1, f_3 = 1, f_2 = 1), \qquad (f_4 = 1, f_3 = 1, f_2 = -1)$$
$$(f_4 = 1, f_3 = -1, f_2 = 1), \qquad (f_4 = 1, f_3 = -1, f_2 = -1).$$

As every triple is obviously of order 2 we conclude, in this case, that:

$$\mathcal{B}^4 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2. \tag{13}$$

Case 2: *The homomorphism* $\mathbb{Z} \xrightarrow{b} \mathbb{Z}$ *is nil. In this case any automorphism* f_4 *makes the diagram* (5) *commutes. Moreover we get the extension:*

$$0 \to \mathbb{Z} \longrightarrow \pi_3(X) \longrightarrow \mathbb{Z} \to 0$$

which also splits and we conclude as in the case 1. Consequently we get only 8 triples which are:

$$(f_4 = 1, f_3 = 1, f_2 = 1), \qquad (f_4 = 1, f_3 = 1, f_2 = -1)$$

$$(f_4 = 1, f_3 = -1, f_2 = 1), \qquad (f_4 = 1, f_3 = -1, f_2 = -1)$$

$$(f_4 = -1, f_3 = 1, f_2 = 1), \qquad (f_4 = -1, f_3 = 1, f_2 = -1)$$

$$(f_4 = -1, f_3 = -1, f_2 = 1), \qquad (f_4 = -1, f_3 = -1, f_2 = -1).$$

As every triple is obviously of order 2 we conclude, in this case, that:

$$\mathcal{B}^4 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2. \tag{14}$$

Notice that in this example, according to Theorem 4.2, the groups $\mathcal{E}_*(X)$ and $\mathcal{E}(X)$ are both infinite.

Example 4.8. *Let X be a simply connected 4-dimensional CW-complex such that:*

$$H_2(X,\mathbb{Z}) = \mathbb{Z}_2, \qquad H_3(X,\mathbb{Z}) = \mathbb{Z}_7, \qquad H_4(X,\mathbb{Z}) = \mathbb{Z}. \tag{15}$$

By the properties of Whitehead's quadratic functor given in Proposition 2.3 we obtain that $\Gamma(H_2(X,\mathbb{Z})) = \Gamma(\mathbb{Z}_2) = \mathbb{Z}_4$. Therefore the Whitehead exact sequence of X can be written as follows:

$$\mathbb{Z} \xrightarrow{b} \mathbb{Z}_4 \longrightarrow \pi_3(X) \longrightarrow \mathbb{Z}_7 \to 0. \tag{16}$$

As $\text{aut}(\mathbb{Z}) \cong \mathbb{Z}_2$, $\text{aut}(\mathbb{Z}_2) \cong \{id\}$, $\text{aut}(\mathbb{Z}_7) \cong \mathbb{Z}_6$, so if $(f_4, f_3, f_2) \in \mathcal{B}^4$, then:

$$f_4 = id, (-1), \qquad f_3 = id, 2., 3., 4., 5., 6., \qquad f_2 = id, \qquad \Gamma(f_2) = id. \tag{17}$$

Here the notation $n.$ means the multiplication by the number n. Thus we seek the automorphisms f_4, f_3 for which there exists an automorphism $\Omega : \pi_3(X) \to \pi_3(X)$ making the following diagram commutes;

$$\begin{array}{ccccccccc} \mathbb{Z} & \xrightarrow{b} & \mathbb{Z}_4 & \longrightarrow & \pi_3(X) & \longrightarrow & \mathbb{Z}_7 & \to & 0 \\ \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \Omega} & & \downarrow{\scriptstyle f_3} & & \\ \mathbb{Z} & \xrightarrow{b} & \mathbb{Z}_4 & \longrightarrow & \pi_3(X) & \longrightarrow & \mathbb{Z}_7 & \to & 0 \end{array} \tag{6}$$

so we have to treat two cases.
Case 1: *The homomorphism $\mathbb{Z} \xrightarrow{b} \mathbb{Z}_4$ is not nil. In this case in order that the diagram (6) commutes we must have $f_4 = 1$. Moreover we get the extension:*

$$0 \to \text{coker}\, b \longrightarrow \pi_3(X) \longrightarrow \mathbb{Z}_7 \to 0. \tag{18}$$

Since $\text{coker}\, b$ is either $\mathbb{Z}_2$ or $\mathbb{Z}_4$, in both cases the extension (18) splits, it follows that $\pi_3(X) \cong \text{coker}\, b \oplus \mathbb{Z}_7$. This implies that any automorphism $\Omega : \pi_3(X) \to \pi_3(X)$

making the diagram (6) *commutes splits also i.e.,* $\Omega = 1 \oplus f_3$. *As a result we get 6 triples:*

$$(f_4 = id, f_3 = n., f_2 = id) \quad , \quad \textit{where} \;\; n = 1,2,3,4,5,6$$

which form a group of order 6. Consequently we derive that $\mathcal{B}^4 \cong \mathbb{Z}_6$.
Case 2: *The homomorphism* $\mathbb{Z} \xrightarrow{b} \mathbb{Z}_4$ *is nil. In this case any automorphism* f_4 *makes the diagram* (6) *commutes. Moreover we get the extension:*

$$0 \to \mathbb{Z}_4 \longrightarrow \pi_3(X) \longrightarrow \mathbb{Z}_7 \to 0$$

which also splits and we conclude as in the case 1. Consequently we get only 12 triples:

$$(f_4 = id, f_3 = n., f_2 = id) \;,\; (f_4 = -1., f_3 = n., f_2 = id) \;,\; n = 1,2,3,4,5,6$$

which form a group of order 12. As $\mathcal{B}^4$ *is a subgroup of* $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_6$ *we get* $\mathcal{B}^4 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_6$.

Notice that in this example, according to Theorem 4.2, the group $\mathcal{E}_*(X)$ is finite and as $\mathcal{B}^4$ is finite it follows that $\mathcal{E}(X)$ is also finite.

Example 4.9. *Let* X *be a simply connected* 4*-dimensional CW-complex having:*

$$\mathbb{Z} \xrightarrow{b} \mathbb{Z}_4 \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{\cong} \mathbb{Z}_2 \to 0 \tag{19}$$

as the Whitehead exact sequence. Recall that in this case we have:

$$H_4(X,\mathbb{Z}) = \mathbb{Z}, \qquad H_3(X,\mathbb{Z}) = H_2(X,\mathbb{Z}) = \mathbb{Z}_2, \qquad \pi_3(X) = \mathbb{Z}_2. \tag{20}$$

Here we use the properties of the Whitehead's quadratic functor which assert that $\Gamma(\mathbb{Z}_2) = \mathbb{Z}_4$ *implying that the sequence (19) is a* Γ*-sequence, so its realizable by Theorem 2.10. The group* $\mathcal{B}^4$ *is a subgroup of* $\mathrm{aut}(H_4(X,\mathbb{Z})) \times \mathrm{aut}(H_3(X,\mathbb{Z})) \times \mathrm{aut}(H_2(X,\mathbb{Z})) = \mathrm{aut}(\mathbb{Z}) \times \mathrm{aut}(\mathbb{Z}_2) \times \mathrm{aut}(\mathbb{Z}_2) \cong \mathbb{Z}_2$.
Therefore the identity is the only element in $\mathcal{B}^4$ *making the following diagram commutes:*

$$\begin{array}{ccccccccc}
\mathbb{Z} & \xrightarrow{b} & \mathbb{Z}_4 & \xrightarrow{0} & \mathbb{Z}_2 & \xrightarrow{\cong} & \mathbb{Z}_2 & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathbb{Z} & \xrightarrow{b} & \mathbb{Z}_4 & \xrightarrow{0} & \mathbb{Z}_2 & \xrightarrow{\cong} & \mathbb{Z}_2 & \to & 0
\end{array}$$

so $\mathcal{B}^4$ *is trivial and* $\mathcal{E}_*(X) = \mathcal{E}(X)$. *Also in this example, according to Theorem 4.2,* $\mathcal{E}(X)$ *is finite.*

The following example gives a simply connected 4-dimensional complex with $\mathcal{E}_*(X)$ finite and $\mathcal{E}_(X)$ infinite.

Example 4.10. *Let* $\mathbb{Z}\oplus\mathbb{Z} \xrightarrow{b} \Gamma(\mathbb{Z}) = \mathbb{Z} \longrightarrow \pi_3 \longrightarrow \mathbb{Z}_2 \to 0$ *be a given* Γ*-sequence such that:*

$$b(x+y) = x \tag{21}$$

First we have $\operatorname{aut}(\mathbb{Z}) \cong \mathbb{Z}_2, \operatorname{aut}(\mathbb{Z}\oplus\mathbb{Z}) \cong GL(2,\mathbb{Z})$ *and* $\operatorname{aut}(\mathbb{Z}_2) \cong \{id_{\mathbb{Z}_2}\}$*. Next if* $(f_4, f_3, f_2) \in \mathcal{B}^4$*, then:*

$$f_4 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \ , \ a_{11}a_{22} - a_{12}a_{21} = \pm 1 \ \ , \ \ f_3 = id \ \ , \ \ f_2 = id. \tag{22}$$

As in the Example 2.7 we have $\Gamma(f_2) = id$*. Therefore we seek the invertible matrices* $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ *for which there exists an automorphism* $\Omega : \pi_3 \to \pi_3$ *making the following diagram commutes;*

$$\begin{array}{ccccccccc} \mathbb{Z}\oplus\mathbb{Z} & \xrightarrow{b} & \mathbb{Z} & \longrightarrow & \pi_3(X) & \xrightarrow{h} & \mathbb{Z}_2 & \to & 0 \\ \downarrow{\scriptsize\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}} & & \downarrow{id} & & \downarrow{\Omega} & & \downarrow{id} & & \\ \mathbb{Z}\oplus\mathbb{Z} & \xrightarrow{b} & \mathbb{Z} & \longrightarrow & \pi_3(X) & \xrightarrow{h} & \mathbb{Z}_2 & \to & 0 \end{array} \tag{7}$$

Now the commutativity of the diagram (7) *and the formula (21) imply the following equation:*

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix} \tag{23}$$

it follows that $a_{11} = 1, a_{12} = 0$ *and using (22) we get* $a_{22} = \pm 1$*. As a result the matrices of* $GL(2,\mathbb{Z})$ *which satisfy the relation (23) form the following subgroup:*

$$G = \left\{ \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} , \begin{pmatrix} 1 & 0 \\ c & -1 \end{pmatrix} \ , \ a, c \in \mathbb{Z} \right\} \tag{24}$$

notice that $\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$ *is of infinite order and* $\begin{pmatrix} 1 & 0 \\ c & -1 \end{pmatrix}$ *is of order 2.*
Next as the homomorphism b *given by the formula (21) is surjective,* $\operatorname{coker} b$ *is nil. So the homomorphism* h *in the diagram* (7) *is automorphism. This implies that if we choose* $\Omega = id$*, the diagram* (7) *commutes. Hence from our precedent arguments we derive that the group* $\mathcal{B}^4$ *is isomorphic to* G*. In this example as* $H_3(X,\mathbb{Z}) = \mathbb{Z}_2$ *and* $H_2(X,\mathbb{Z}) = \mathbb{Z}$*, from Theorem 4.2 we deduce that* $\mathcal{E}_*(X)$ *is finite.*

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