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MASCHKE-TYPE THEOREM FOR PARTIAL SMASH PRODUCTS

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Abstract. In this paper, we mainly study the trace function for partial Hopf actions and give a Maschke-type theorem for partial smash products.

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1. Introduction

In [10], Exel first considered partial group actions in the context of operator algebras, and studied C^*-algebras generated by partial isometries on a Hilbert space. In [6], Caenepeel and Janssen introduced partial Hopf actions regarded as a generalization of partial group actions, who was motivated by an attempt to generalize the notion of partial Galois extensions of commutative rings (see [8]), and also introduced the concept of partial smash products, which is an unital subalgebra of the usual smash products. In [12], Lomp developed the theory of partial Hopf actions, and extended the well-known results of Hopf algebras concerning smash products, such as the Blattner-Montgomery and Cohen-Montgomery theorems in [13]. Recently, the authors in [3, 9] gave the Morita context between the invariant subalgebra and the partial smash product.

Let H be a finite-dimensional Hopf algebra over a field k and A a partial H-module algebra. Then, the partial smash product $\underline{A\#H}$ is a ring extension of A, which is familiar as the partial skew group ring $A * G$ for the partial group action. In [11], the authors proved the Maschke-type theorem for the partial skew group rings. So, we naturally have the following question.

Does the Maschke-type theorem for the partial smash product $\underline{A\#H}$ hold?

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In this note we give a positive answer to this question by using a new method which is not just a generalization of the proof of the classical result in [7].

We always work over a fixed field k. Unless otherwise specified, linearity, modules and $\otimes$ are all meant over k. And we freely use the Hopf algebras terminology introduced in [13]. For a coalgebra C, we write its comultiplication $\Delta(c) = c_1 \otimes c_2$, for any $c \in C$, in which we omit the summation symbols for convenience.

A partial action of the Hopf algebra H on the algebra A is a linear map $\alpha : H \otimes A \to A$, denoted by $\alpha(h \otimes a) = h \cdot a$, for any $a, b \in A,\quad h, g \in H$, such that

(P1) $h \cdot (ab) = (h_1 \cdot a)(h_2 \cdot b)$,

(P2) $1_H \cdot a = a$,

(P3) $h \cdot (g \cdot a) = (h_1 \cdot 1_A)((h_2 g) \cdot a)$.

We will also call A a *partial H-module algebra*. It is easy to see every action is also a partial action.

Given a Hopf algebra H and a partial H-module algebra A, one can form the *partial smash product* $\underline{A\#H}$ which is the unital subalgebra of $A \otimes H$ defined as follows: put an algebra structure in $A \otimes H$ with the product

$$(a \otimes h)(b \otimes k) = a(h_1 \cdot b) \otimes h_2 k.$$

The partial smash product is given by

$$\underline{A\#H} = (A \otimes H)(1_A \otimes 1_H)$$

that is, the subalgebra $\underline{A\#H}$ is spanned by the elements of the form $\{a(h_1 \cdot 1_A) \otimes h_2$, for any $a \in A,\quad h \in H\}$. One can easily verify that the multiplication of partial smash product satisfies

$$(\underline{a\#h})(\underline{b\#k}) = \underline{a(h_1 \cdot a)\#h_2 k}. \tag{1}$$

For a partial H-module algebra A and its enveloping action B given in [4], a special case which will be useful for further results is the case when $\theta(A)$ is an ideal of H-module algebra B, where the map $\theta : A \to B$ is a monomorphism of algebras. The authors in [4, Proposition 4] gave the sufficient and necessary condition, that is, for any $h, g \in H, a \in A$,

$$h \cdot (g \cdot a) = ((h_1 g) \cdot a)(h_2 \cdot 1_A), \tag{2}$$

for the element $\theta(1_A)$ to be a central idempotent in B. In our note we always assume that A is an ideal of B, since the map $\theta : A \to B$ is a monomorphism of algebras. So, 1_A becomes a central idempotent in B.

Throughout this note we suppose that H is always a finite dimensional Hopf algebra.

2. Central trace functions and invariants

Similar to the partial group action in [11], we can define the invariants for a partial H-module algebra A as follows:

$$A^H = \{a \in A \mid h \cdot a = (h \cdot 1_A)a, \text{ for any } h \in H\}.$$

Note that A^H is a subalgebra of A with identity 1_A. Define the trace map

$$\hat{t}_A : A \to A^H, \quad \hat{t}_A(a) = t \cdot a,$$

where $0 \neq t \in \int_H^l$ (the space of left integrals in H).

It is clear that $\hat{t}_A$ is a right A^H-linear map. But we hope that it is an A^H-bimodule map.

According to the references [1, 5], we know that lazy 1-cocycles are related with (co)homology and extensions.

A lazy 1-cocycle is a map $\ell \in \mathrm{Hom}(H, A)$ which is convolution invertible and satisfies

$$\ell(h_1) \otimes h_2 = \ell(h_2) \otimes h_1,$$

for any $h \in H$, where A is a left H-module algebra. In particular, the unit of $\mathrm{Hom}(H, A)$, the map $h \mapsto \varepsilon(h)1_A$, is a lazy 1- cocycle. If H acts globally on A then the unit is equal to the map $\ell(h) = h \cdot 1_A$.

For a partial H-module algebra A, if for any $h \in H$, the condition of lazy 1-cocycles (forgetting about the condition of being convolution invertible) holds:

$$h_1 \cdot 1_A \otimes h_2 = h_2 \cdot 1_A \otimes h_1, \tag{3}$$

then, it is easy to check that $H \cdot 1_A$ is in $C(A)$ (the center of the algebra A), that is, for any $h \in H, a \in A$,

$$\begin{aligned}(h \cdot 1_A)a &= (h_1 \cdot 1_A)\varepsilon(h_2)a = (h_1 \cdot 1_A)(h_2 S(h_3) \cdot a) \\ &= (h_1 \cdot 1_A)(h_2 \cdot (S(h_3) \cdot a)) = h_1 \cdot (S(h_2) \cdot a) \\ &\overset{(2)}{=} (h_1 S(h_3) \cdot a)(h_2 \cdot 1_A) \overset{(3)}{=} (h_1 S(h_2) \cdot a)(h_3 \cdot 1_A) \\ &= a(h \cdot 1_A).\end{aligned}$$

In what follows, we call the partial H-module algebra A satisfying the equality (3) *a strong partial H-module algebra.*

Remark.

(1) *The invariant subalgebra A^H as above in this case becomes $A^H = \{a \in A \mid h \cdot a = (h \cdot 1_A)a = a(h \cdot 1_A)$, for any $h \in H\}$, see [3, Definition 5].*

(2) *If H is cocommutative as coalgebra, then A is a strong partial H-module algebra automatically.*

In particular, for the partial group action, we know that it is a strong partial H-module algebra obviously.

(3) *Let B be an H-module algebra. Then B is a trivial strong partial H-module algebra.*

Before the next lemma we recall the definition of trace map for H-module algebras: let H be a finite-dimensional Hopf algebra acting on an algebra B with action "$\triangleright$" and choose $0 \neq t \in \int_H^l$. Then the map $\hat{t}_B : B \to B^H$ given by $\hat{t}_B(b) = t \triangleright b$ is a B^H-bimodule map. We call $\hat{t}_B$ a *(left) trace function for H on B*. From [2] we know that if B is an H-module algebra, the surjectivity of $\hat{t}_B$ onto B^H is equivalent to the existence of an element $b \in B$ with $\hat{t}_B(b) = 1_B$.

In what follows, we discuss the surjectivity of trace map for a partial H-module algebra A, and throughout the rest of this section we always assume that for a partial H-module algebra A,

$$A^H = \{a \in A \mid h \cdot a = (h \cdot 1_A)a = a(h \cdot 1_A), \text{ for any } h \in H\}.$$

Lemma 2.1. (1) *$\hat{t}_A : A \to A^H$ is an A^H-bimodule map with values in A^H.*

Let (B, θ) be an enveloping action of a partial H-module algebra A. Then

(2) *$\hat{t}_A(a) = \hat{t}_B(a)1_A$, for any $a \in A$;*

(3) *$\hat{t}_B(B) = \hat{t}_B(A)$.*

Proof. (1) For any $a \in A, c \in A^H$, we have

$$\begin{aligned} c\hat{t}_A(a) &= c(t \cdot a) = c(t_1 \cdot 1_A)(t_2 \cdot a) \\ &= (t_1 \cdot c)(t_2 \cdot a) = t \cdot (ca) = \hat{t}_A(ca), \\ \hat{t}_A(a)c &= (t \cdot a)c = (t_1 \cdot a)(t_2 \cdot 1_A)c \\ &= (t_1 \cdot a)(t_2 \cdot c) = t \cdot (ac) = \hat{t}_A(ac). \end{aligned}$$

(2) It is obvious from [4, Proposition 1].

(3) We only show that $\hat{t}_B(B) \subseteq \hat{t}_B(A)$, the opposite is obvious. Assume that there exists an element $x \in B$ such that $\hat{t}_B(x) = b \in \hat{t}_B(B)$, where the element x is of the form $\Sigma_i h_i \triangleright a_i$, for a finite number of elements $h_i \in H, a_i \in A$. Then

$$b = \hat{t}_B(x) = t \triangleright (\Sigma_i h_i \triangleright a_i) = \Sigma(th_i) \triangleright a_i \in \hat{t}_B(A),$$

so $\hat{t}_B(B) = \hat{t}_B(A)$. □

Proposition 2.2. (1) *$\hat{t}_A$ is onto A^H if and only if there exists an element $a \in A$ such that $\hat{t}_A(a) = 1_A$.*

(2) *Assume that (B, θ) is an enveloping action of a partial H-module algebra A. If $\hat{t}_B$ is onto B^H, then $\hat{t}_A$ is onto A^H.*

Proof. (1) Let there exist an element $a \in A$ such that $\hat{t}_A(a) = 1_A$. Then, for any $c \in A^H$, $c = c1_A = c\hat{t}_A(a) = \hat{t}_A(ca)$, that is, $\hat{t}_A$ is onto A^H. Conversely, it is straightforward.

(2) If there is an element $b \in B$ with $\hat{t}_B(b) = 1_B$, then, by Lemma 2.1, there exists an element $a \in A$ such that $\hat{t}_B(a) = 1_B$. So, the fact that $h \cdot a = 1_A(h \triangleright a) = (h \triangleright a)1_A$ implies $\hat{t}_A(a) = \hat{t}_B(a)1_A = 1_A$. According to the above conclusion, we know that $\hat{t}_A$ is onto A^H. □

3. Maschke-type theorem for partial smash products

In this section, we assume that A is a strong partial H-module algebra, and give the Maschke-type theorem for partial smash product by using a kind of new method.

Lemma 3.1. *In partial smash product $\underline{A\#H}$: for any $a \in A$, $h \in H$,*

$$\underline{a\#h} = (\underline{1_A\#h_2})(S^{-1}(h_1) \cdot a\#1_H). \tag{4}$$

Proof. For any $a \in A$, $h \in H$, we have

$$\begin{aligned}
\underline{a\#h} &= a(h_1 \cdot 1_A)\#h_2 = ((h_2 S^{-1}(h_1)) \cdot a)(h_3 \cdot 1_A)\#h_4 \\
&\overset{(2)}{=} h_2 \cdot (S^{-1}(h_1) \cdot a)\#h_3 = (h_2 \cdot 1_A)(h_3 \cdot (S^{-1}(h_1) \cdot a))\#h_4 \\
&= (h_2 \cdot 1_A\#h_3)(S^{-1}(h_1) \cdot a\#1_H) \\
&= (\underline{1_A\#h_2})(S^{-1}(h_1) \cdot a\#1_H).
\end{aligned}$$

□

Lemma 3.2. *Let V be a left $\underline{A\#H}$-module, W a submodule of V and $\hat{t}_A(1_A)$ be invertible in A. Assume that $\lambda : V \to W$ is a projection as A-modules. Then, there is also a projection from V to W as $\underline{A\#H}$-modules.*

Proof. Assume that $\lambda : V \to W$ be the projection as A-modules. Define the map

$$\widetilde{\lambda} : V \to W \text{ by } \widetilde{\lambda}(v) = u(\underline{1_A\#S(x_1)})\lambda((\underline{1_A\#x_2})v),$$

where $u = (\hat{t}_A(1_A))^{-1}$, $S(x) = t$, as in Section 2, $\hat{t}_A(1_A) = t \cdot 1_A$, $0 \neq t \in \int_H^l$, $x \in \int_H^r$.

We show that $\widetilde{\lambda}$ is a projection as $\underline{A\#H}$-module. First we check that $\widetilde{\lambda}$ is $\underline{A\#H}$-linear. Since S is bijective, we can choose $\underline{a\#S(h)}\in\underline{A\#H}$:

$$
\begin{aligned}
&u^{-1}(\underline{a\#S(h)})\widetilde{\lambda}(v)\\
&=u^{-1}(\underline{a\#S(h)})u(\underline{1_A\#S(x_1)})\lambda((\underline{1_A\#x_2})v)\\
&=(t\cdot 1_A\#1_H)(a(S(h_2)\cdot 1_A)\#S(h_1))u(\underline{1_A\#S(x_1)})\lambda((\underline{1_A\#x_2})v)\\
&=((t\cdot 1_A)a(S(h_2)\cdot 1_A)\#S(h_1))u(\underline{1_A\#S(x_1)})\lambda((\underline{1_A\#x_2})v)\\
&=(a(S(h_2)\cdot 1_A)(t\cdot 1_A)\#S(h_1))u(\underline{1_A\#S(x_1)})\lambda((\underline{1_A\#x_2})v)\\
&=(a(S(h_3)\cdot 1_A)(S(h_2)t\cdot 1_A)\#S(h_1))u(\underline{1_A\#S(x_1)})\lambda((\underline{1_A\#x_2})v)\\
&=(a(S(h_4)\cdot 1_A)(S(h_3)\cdot 1_A)(S(h_2)t\cdot 1_A)\#S(h_1))u(\underline{1_A\#S(x_1)})\lambda((\underline{1_A\#x_2})v)\\
&\overset{(2)}{=}(a(S(h_3)\cdot 1_A)(S(h_2)\cdot(t\cdot 1_A))\#S(h_1))u(\underline{1_A\#S(x_1)})\lambda((\underline{1_A\#x_2})v)\\
&=(a(S(h_2)\cdot 1_A)\#S(h_1))(t\cdot 1_A\#1_H)u(\underline{1_A\#S(x_1)})\lambda((\underline{1_A\#x_2})v)\\
&=(\underline{a\#S(h)})(\underline{1_A\#S(x_1)})\lambda((\underline{1_A\#x_2})v)\\
&\overset{(1)}{=}(\underline{a(S(h_2)\cdot 1_A)\#S(x_1h_1)})\lambda((\underline{1_A\#x_2})v)\\
&\overset{(4)}{=}(\underline{1_A\#S(x_1h_1)})(S^{-1}(S(x_2h_2))\cdot(a(S(h_3)\cdot 1_A))\#1_H)\lambda((\underline{1_A\#x_3})v)\\
&=(\underline{1_A\#S(x_1h_1)})(x_2h_2\cdot(a(S(h_3)\cdot 1_A))\#1_H)\lambda((\underline{1_A\#x_3})v)\\
&=(\underline{1_A\#S(x_1h_1)})\lambda((\underline{x_2h_2\cdot(a(S(h_3)\cdot 1_A))\#x_3})v)\\
&=(\underline{1_A\#S(x_1h_1)})\lambda((\underline{x_2h_2\cdot a)(x_3h_3\cdot(S(h_4)\cdot 1_A))\#x_4})v)\\
&=(\underline{1_A\#S(x_1h_1)})\lambda(((\underline{x_2h_2\cdot a)(x_3h_3S(h_4)\cdot 1_A)\#x_4})v)\\
&=(\underline{1_A\#S(x_1h_1)})\lambda(((\underline{x_2h_2\cdot a)(x_3\cdot 1_A)\#x_4})v)\\
&=(\underline{1_A\#S(x_1h_1)})\lambda((\underline{x_2h_2\cdot a\#x_3})v). \qquad (5)
\end{aligned}
$$

Since x is a right integral in H, we have

$$
\begin{aligned}
x_1h_1\otimes x_2h_2\otimes x_3 &=((\Delta\otimes id)\Delta(xh_1))(1_H\otimes 1_H\otimes S(h_2))\\
&=((\Delta\otimes id)\Delta(x))(1_H\otimes 1_H\otimes S(h))\\
&=x_1\otimes x_2\otimes x_3S(h).
\end{aligned}
$$

Now we use above equation to compute:

$$
\begin{aligned}
(5)\quad &=(\underline{1_A\#S(x_1)})\lambda((\underline{x_2\cdot a\#x_3S(h)})v)\\
&\overset{(1)}{=}(\underline{1_A\#S(x_1)})\lambda(((\underline{1_A\#x_2})(\underline{a\#S(h)}))v)\\
&=u^{-1}u(\underline{1_A\#S(x_1)})\lambda(((\underline{1_A\#x_2})(\underline{a\#S(h)}))v)\\
&=u^{-1}\widetilde{\lambda}((\underline{a\#S(h)})v),
\end{aligned}
$$

so $\widetilde{\lambda}$ is $\underline{A\#H}$-linear. From the above computation, we conclude that

$$(\underline{a\#S(h)})(\underline{1_A\#S(x_1)})\otimes_A(\underline{1_A\#x_2})=(\underline{1_A\#S(x_1)})\otimes_A(\underline{1_A\#x_2})(\underline{a\#S(h)}). \qquad (6)$$

It remains to check that $\widetilde{\lambda}$ is a projection. If $w \in W$, then we have

$$\begin{aligned}\widetilde{\lambda}(w) &= u(\underline{1_A\#S(x_1))(1_A\#x_2})w = u(\underline{S(x_2)\cdot 1_A\#S(x_1)x_3})w \\ &\overset{(3)}{=} u(\underline{S(x_1)\cdot 1_A\#S(x_2)x_3})w = u(S(x)\cdot 1_A\#1_H)w \\ &= u(S(x)\cdot 1_A)w = u(t\cdot 1_A)w = w.\end{aligned}$$

□

According to Lemma 3.2, we get the following main result.

Theorem 3.3. *Under the same assumptions as above. If A is semisimple Artinian, then $\underline{A\#H}$ is semisimple Artinian.*

Remark. *Since H is not a subalgebra of the partial smash product $\underline{A\#H}$, from the proof of Lemma 3.2, we can see that we use a new method which is not just a generalization of the proof of the classical result in [7] to prove the Maschke-type theorem.*

Note that an H-module algebra B is a trivial strong partial H-module algebra, H is semisimple iff $\varepsilon(t) \neq 0$, where $0 \neq t \in \int_H^l$, and $\hat{t}_B(1_B) = t \triangleright 1_B = \varepsilon(t)1_B$ is invertible in B iff $\varepsilon(t) \neq 0$. So, in this case the semisimplity of H is equivalent to the invertibility of $\hat{t}_B(1_B)$ in B. What's more, the partial smash product $\underline{A\#H}$ become a partial skew group ring $A \star_\alpha G$ in case of replacing H by kG. Therefore, we have the following results.

Corollary 3.4. *Let H be a finite-dimensional semisimple Hopf algebra, and B an H-module algebra. If B is semisimple, then $B\#H$ is semisimple.*

The above corollary is a generalization of Theorem 6 in [7].

Corollary 3.5. *Let α be a partial action of a finite group G on a unital algebra R. If R is semisimple and $\hat{t}_R(1_R)$ is invertible in R, then the partial skew group ring $R \star_\alpha G$ is semisimple.*

The above corollary is a generalization of Corollary 3.3 in [11].

In what follows, we consider the separability of $\underline{A\#H}$ under the condition that $\hat{t}_A(1_A)$ is invertible in A.

Proposition 3.6. *Assume that $\hat{t}_A(1_A)$ is invertible in A. Then $\underline{A\#H}$ is separable over A.*

Proof. As in Section 2, let $\hat{t}_A(1_A) = t\cdot 1_A$ be invertible in A with the inverse u. It is easy to prove $u \in C(A)$. Moreover, for any $h \in H$,

$$\begin{aligned} h\cdot u - (h\cdot 1_A)u &= (h\cdot u - (h\cdot 1_A)u)(t\cdot 1_A)u = (h\cdot u)(t\cdot 1_A)u - (h\cdot 1_A)u \\ &= (h_1\cdot u)(h_2t\cdot 1_A)u - (h\cdot 1_A)u = (h\cdot(u(t\cdot 1_A)))u - (h\cdot 1_A)u \\ &= (h\cdot 1_A)u - (h\cdot 1_A)u = 0\end{aligned}$$

shows $u \in A^H$. Hence $u \in C(A) \cap A^H$. Consider the element

$$w = (\underline{1_A \# t_2}) \otimes_A (\underline{u \# S^{-1}(t_1)}) \in \underline{A \# H} \otimes_A \underline{A \# H}.$$

In the following, we will show that w is a separability idempotent for $\underline{A\#H}$.
Let $\mu : \underline{A\#H} \otimes_A \underline{A\#H} \to \underline{A\#H}$ denote the multiplication map. Then

$$\begin{aligned}
\mu(w) &= (\underline{1_A\#t_2})(\underline{u\#S^{-1}(t_1)}) = \underline{t_2 \cdot u\#t_3S^{-1}(t_1)} \\
&= \underline{(t_2 \cdot 1_A)u\#t_3S^{-1}(t_1)} = \underline{(t_1 \cdot 1_A)u\#t_3S^{-1}(t_2)} \\
&= (t \cdot 1_A)u\#1_H = 1_A\#1_H.
\end{aligned}$$

As in Lemma 3.2, we choose $S(x) = t$, where $0 \neq t \in \int_H^l, x \in \int_H^r$, and choose $\underline{a\#S(h)} \in \underline{A\#H}$,

$$\begin{aligned}
(\underline{a\#S(h)})w = &\ (\underline{a\#S(h)})(\underline{1_A\#t_2}) \otimes_A (\underline{u\#S^{-1}(t_1)}) \\
= &\ (\underline{a\#S(h)})(\underline{1_A\#S(x_1)}) \otimes_A (\underline{u\#x_2}) \\
= &\ (\underline{a\#S(h)})(\underline{1_A\#S(x_1)}) \otimes_A u(\underline{1_A\#x_2}) \\
\overset{(6)}{=} &\ (\underline{1_A\#S(x_1)}) \otimes_A u(\underline{1_A\#x_2})(\underline{a\#S(h)}) \\
= &\ (\underline{1_A\#S(x_1)}) \otimes_A (\underline{u\#x_2})(\underline{a\#S(h)}) \\
= &\ (\underline{1_A\#t_2}) \otimes_A (\underline{u\#S^{-1}(t_1)})(\underline{a\#S(h)}) \\
= &\ w(\underline{a\#S(h)}),
\end{aligned}$$

which shows that w is a separability idempotent. Hence $\underline{A\#H}$ is separable over A. □

Question. In [3], the authors defined the partial invariants $A^H = \{a \in A \mid h \cdot a = (h \cdot 1_A)a = a(h \cdot 1_A), \text{ for any } h \in H\}$, and gave the Morita context between the invariant subalgebra A^H and the partial smash product $\underline{A\#H}$. In our note, we introduce the condition (3) of lazy 1-cocycles related with cohomology and extensions in order to prove the Maschke-type theorem. We hope that this condition in the future can be improved.

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References

[1] A. L. Agore and G. Militaru, *Extending structures II: the quantum version*, J. Algebra, 336 (2011), 321–341.

[2] E. Aljadeff, *On the surjectivity of some trace maps*, Israel J. Math., 86 (1994), 221–232.

[3] M. M. S. Alves and E. Batista, *Partial Hopf actions, partial invariants and a Morita context*, Algebra Discrete Math., 3 (2009), 1–19.

[4] M. M. S. Alves and E. Batista, *Enveloping actions for partial Hopf actions*, Comm. Algebra, 38(8) (2010), 2872–2902.

[5] J. Bichon and C. Kassel, *The lazy homology of a Hopf algebra*, J. Algebra, 323(9) (2010), 2556–2590.

[6] S. Caenepeel and K. Janssen, *Partial (co)actions of Hopf algebras and partial Hopf-Galois theory*, Comm. Algebra, 36(8) (2008), 2923–2946.

[7] M. Cohen and D. Fishman, *Hopf algebra actions*, J. Algebra, 100(2) (1986), 363–379.

[8] M. Dokuchaev, M. Ferrero and A. Paques, *Partial actions and Galois theory*, J. Pure Appl. Algebra, 208(1) (2007), 77–87.

[9] C. Z. Du and J. F. Lin, *Morita context, partial Hopf Galois extensions and partial entwining structure*, Math. Notes, 95 (2014), 43–52.

[10] R. Exel, *Circle actions on C^*-algebras, partial automorphisms and generalized Pimsner-Voiculescu exact sequences*, J. Funct. Anal., 122(2) (1994), 361–401.

[11] M. Ferrero and J. Lazzarin, *Partial actions and partial skew group rings*, J. Algebra, 319(12) (2008), 5247–5264.

[12] C. Lomp, *Duality for partial group actions*, Int. Electron. J. Algebra, 4 (2008), 53–62.

[13] S. Montgomery, Hopf Algebras and Their Actions on Rings, CBMS Regional Conference Series in Mathematics, 82, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the Amer. Math. Soc., Providence, RI, 1993.

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