ON COMPLETELY PRIME SUBMODULES

David Ssevviiri

ABSTRACT. The formal study of completely prime modules was initiated by
N. J. Groenewald and the current author in the paper: *Completely prime
submodules*, Int. Electron. J. Algebra, 13(2013), 1–14. In this paper, the study
of completely prime modules is continued. Firstly, the advantage completely
prime modules have over prime modules is highlighted and different situations
that lead to completely prime modules given. Later, emphasis is put on fully
completely prime modules (i.e., modules whose all submodules are completely
prime). For a fully completely prime left R-module M, if $a, b \in R$ and $m \in M$,
then $abm = bam$, $am = a^k m$ for all positive integers k, and either $am = abm$
or $bm = abm$. In the last section, two different torsion theories induced by the
completely prime radical are given.

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prime radical, torsion theory

1. Introduction

Completely prime modules were first formally studied in [6] as a generalization
of prime modules. These modules had earlier appeared informally in most cases as
examples in the works of: Andrunakievich [1], de la Rosa and Veldsman [5, p. 466,
Section 5.6], Lomp and Peña [12, Proposition 3.1], and Tuganbaev [18, p. 1840]
which were published in the years 1962, 1994, 2000 and 2003, respectively. In [1]
and [5] these modules were called modules without zero-divisors, in [12] they were
not given any special name and [18] they were called completely prime modules. In
this paper just like in [6], we follow the nomenclature of Tuganbaev.

Definition 1.1. Let R be a ring. A left R-module M for which $RM \neq \{0\}$ is
called

    (1) *completely prime* if for all $a \in R$ and every $m \in M$, $am = 0$ implies $m = 0$
        or $aM = \{0\}$;

    (2) *completely semiprime* if for all $a \in R$ and every $m \in M$, $a^2 m = 0$ implies
        $aRm = \{0\}$;

(3) *prime if for all $a \in R$ and every $m \in M$, $aRm = \{0\}$ implies $m = 0$ or $aM = \{0\}$.*

A submodule P of M is a completely prime (resp. completely semiprime, prime) submodule if the factor module M/P is a completely prime (resp. completely semiprime, prime) module. A completely prime module is prime but not conversely in general. Over a commutative ring, completely prime modules are indistinguishable from prime modules.

Example 1.2. *We know that the ring $R = M_n(\mathbb{F})$ of all $n \times n$ matrices over a field $\mathbb{F}$ is prime but not completely prime, i.e., it is not a domain. Since for a unital ring R, R is prime (resp. completely prime) if and only if the module $_RR$ is prime (resp. completely prime), see [6, Proposition 2.4], we conclude that the R-module R (where $R = M_n(\mathbb{F})$) is prime but not completely prime.*

Example 1.3 below is motivated by Example 3.2 in [10].

Example 1.3. *Let $M = \left\{ \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}, \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{1} & \bar{1} \end{pmatrix}, \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix}, \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{1} & \bar{1} \end{pmatrix} \right\}$ where entries of matrices in M are from the ring $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ of integers modulo 2 and $R = M_2(\mathbb{Z})$ the ring of all 2×2 matrices defined over integers. M is a prime R-module which is not completely prime.*

Proof. First, we show that M is simple and hence prime since all simple modules are prime. Let $r = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R$, then

$$rM = \left\{ \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}, \begin{pmatrix} a & a \\ c & c \end{pmatrix}, \begin{pmatrix} b & b \\ d & d \end{pmatrix}, \begin{pmatrix} a+b & a+b \\ c+d & c+d \end{pmatrix} \right\} \subseteq M$$

for any $a, b, c, d \in \mathbb{Z}$. The would be non-trivial proper submodules, namely; $N_1 = \left\{ \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}, \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix} \right\}$, $N_2 = \left\{ \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}, \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{1} & \bar{1} \end{pmatrix} \right\}$, and $N_3 = \left\{ \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}, \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{1} & \bar{1} \end{pmatrix} \right\}$ are not closed under multiplication by R since for a and c odd, $rN_1 \nsubseteq N_1$, for b and d odd, $rN_2 \nsubseteq N_2$ and for a odd but b, c, d even, $rN_3 \nsubseteq N_3$. Now, take $a = \begin{pmatrix} 3 & 3 \\ 2 & 2 \end{pmatrix} \in R$ and $m = \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{1} & \bar{1} \end{pmatrix} \in M$. It follows that $am = 0$ but $aM \neq \{0\}$ since $a = \begin{pmatrix} 3 & 3 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix} = \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix} \neq 0$. Thus, M is not completely prime. $\qquad\square$

1.1. Notation. All modules considered are left unital modules defined over rings. The rings are unital and associative. Let M be an R-module. If S is a subset of M and $m \in M \setminus S$, by $(S : m)$ we denote the set $\{r \in R : rm \in S\}$. If N is a submodule of a module M, we write $N \leq M$. If $N \leq M$, $(N : M)$ is the ideal $\{r \in R : rM \subseteq N\}$ which is the annihilator of the R-module M/N. For an R-module M, $\mathrm{End}_R(M)$ denotes the ring of all R-endomorphisms of M.

1.2. A road map for the paper. This paper contains five sections. In Section 1, we give an introduction, define some of the notation used and describe how the paper is organized. In Section 2, we state the advantage completely prime modules have over prime modules. They behave as though they are defined over a commutative ring, a behavior prime modules do not have in general. The aim of Section 3 is two fold; we provide situations under which a module becomes completely prime and furnish concrete examples for completely prime modules. In Section 4, we define completely co-prime modules by drawing motivation from how prime, completely prime and co-prime modules are defined. A chart of implications is established between completely co-prime modules, co-prime modules, completely prime modules and prime modules, see Proposition 4.3. In Proposition 4.3 it is established that the notion of completely co-prime modules is the same as that for fully completely prime modules, i.e., modules whose all submodules are completely prime. Many other equivalent formulations for completely co-prime modules are given. It is shown that if M is a fully completely prime R-module, then for all $a, b \in R$ and every $m \in M$, $abm = bam$, $am = a^k m$ for all positive integers k, and either $am = abm$ or $bm = abm$. In Section 5, which is the last section, we give two torsion theories induced by the completely prime radical of a module. On the class of IFP modules (i.e., modules with the insertion-of-factor property), the faithful completely prime radical is hereditary and hence leads to a torsion theory, see Theorem 5.2. Lastly, we show in Theorem 5.3 that the completely prime radical is also hereditary on the class of semisimple R-modules and therefore it induces another torsion theory.

2. Advantage of completely prime modules over prime modules

Where as prime modules form a much bigger class than that of completely prime modules, completely prime modules possess nice properties which prime modules lack in general. Completely prime modules over noncommutative rings behave like modules over commutative rings. In particular, they lead to the following properties on an R-module M:

P1. for all $a, b \in R$ and $m \in M$, $abm = 0$ implies $bam = 0$;

P2. for all subsets S of M and $m \in M \setminus S$, $(S : m)$ is a two sided ideal of R;

P3. for all $a \in R$ and $m \in M$, $am = 0$ implies $arm = 0$ for all $r \in R$;

P4. the prime radical of M coincides with its completely prime radical, i.e., the intersection of all prime submodules of M coincides with the intersection of all its completely prime submodules.

A module which satisfies property P1, P3 and P4 is respectively called *symmetric, IFP (i.e., has insertion-of-factor property) and 2-primal.* Properties P2 and P3 are equivalent. To prove the claims made in this section, one only needs to prove the following implications for a module:

completely prime $\Rightarrow$ completely semiprime $\Rightarrow$ symmetric $\Rightarrow$ IFP $\Rightarrow$ 2-primal,

see [7, Theorems 2.2 and 2.3] and [10] for the proof. A submodule P of an R-module M is said to be *symmetric (resp. IFP)* if the module M/P is symmetric (resp. IFP). A comparison with what happens for rings indicates that these results on modules are what one would expect. Every domain (completely prime ring) is reduced (i.e., completely semiprime) so it is symmetric, IFP and 2-primal, see [13]. Note that the IFP condition is called SI in [13]. The notions of IFP and symmetry first existed for rings before they were extended to modules.

3. Properties and some examples

An R-module M is completely prime if and only if for all nonzero $m \in M$, $(0 : m) = (0 : M)$. This characterization is used in the proof of Proposition 3.1, in Example 3.3, in Proposition 3.7 and in Section 4.

Proposition 3.1. *Let M be an R-module. If every nonzero endomorphism $f \in End_R(M)$ is a monomorphism, then M is a completely prime module.*

Proof. Let $r \in R$ such that $r \notin (0 : M)$ and let $0 \neq m \in M$. Then there exists $n \in M$ such that $rn \neq 0$. The endomorphism $g : M \to M$ given by $g(x) = rx$ is nonzero since $g(n) = rn \neq 0$. By hypothesis, g is a monomorphism. Thus, $g(m) = rm \neq 0$ since by assumption $m \neq 0$. So, $r \notin (0 : m)$. Hence, $(0 : m) \subseteq (0 : M)$ which shows that $(0 : m) = (0 : M)$ for all $0 \neq m \in M$ since the reverse inclusion always holds. $\square$

According to Reyes [15, Definition 2.1], a left ideal P of a ring R is *completely prime* if for any $a, b \in R$ such that $Pa \subseteq P$, $ab \in P$ implies that either $a \in P$ or $b \in P$.

Example 3.2. *If P is a left ideal of a ring R which is completely prime in the sense of Reyes, then R/P is a completely prime module and $S = End_R(R/P)$ is a domain. This is because, according to [15, Proposition 2.5], P is a completely prime left ideal of R if and only if every nonzero $f \in S := End_R(R/P)$ is injective if and only if S is a domain and the right S-module R/P is torsion-free. Now apply Proposition 3.1.*

Example 3.3. *A torsion-free module is completely prime and faithful. If M is torsion-free, $(0 : m) = \{0\}$ for all $0 \neq m \in M$. So, $(0 : M) \subseteq (0 : m) = \{0\}$ and hence $(0 : M) = (0 : m) = \{0\}$ for all $0 \neq m \in M$.*

Let N be a submodule of an R-module M, the zero divisor set of the R-module M/N is the set

$$\mathrm{Zd}_R(M/N) := \{r \in R \ : \ \text{there exists } m \in M \setminus N \text{ with } rm \in N\}.$$

In Proposition 3.4, we characterize completely prime submodules in terms of zero divisor sets of their factor modules.

Proposition 3.4. *A submodule N of an R-module M is a completely prime submodule if and only if $(N : M) = Zd_R(M/N)$. In particular, $Zd_R(M/N)$ is a completely prime ideal of R whenever N is a completely prime submodule of M.*

Proof. Suppose $(N : M) = \mathrm{Zd}_R(M/N)$, i.e., $\displaystyle\bigcap_{m \in M \setminus N} (N : m) = \bigcup_{m \in M \setminus N} (N : m)$. Then this equality is possible if and only if the set $\{(N : m) \ : \ m \in M \setminus N\}$ is a singleton. Thus, N is a completely prime submodule by [6, Proposition 2.5]. For the converse, if N is a completely prime submodule, it follows by [6, Proposition 2.5] that the set $\{(N : m) \ : \ m \in M \setminus N\}$ is a singleton. So, $\displaystyle\bigcap_{m \in M \setminus N} (N : m) = \bigcup_{m \in M \setminus N} (N : m)$ and $(N : M) = \mathrm{Zd}_R(M/N)$. The last statement follows from the fact that, if N is a completely prime submodule of an R-module M, then $(N : M)$ is a completely prime ideal of R. $\qquad\square$

Corollary 3.5. *An R-module M is completely prime if and only if $(0 : M) = Zd_R(M)$. In particular, $Zd_R(M)$ is a completely prime ideal of R whenever M is a completely prime module.*

Corollary 3.6. *If M is a faithful completely prime module, then the set $Zd_R(M)$ is a domain.*

Proposition 3.7. *If M is a uniform module, then M is completely prime if and only if every cyclic submodule of M is a completely prime module.*

Proof. The if part is clear. For the converse, we prove by contradiction. Suppose there exists $0 \neq m \in M$ such that $(0 : M) \neq (0 : m)$, i.e., $(0 : M) \subsetneq (0 : m)$. Then, there exists $a \in R$ and $0 \neq x \in M$ such that $am = 0$ and $ax \neq 0$. Since M is uniform, there exists a nonzero element z such that $z \in Rm \cap Rx$. $z, m \in Rm$ and $z, x \in Rx$. Since by hypothesis, Rm and Rx are completely prime modules, it follows that $(0 : z) = (0 : m) = (0 : Rm)$ and $(0 : z) = (0 : x) = (0 : Rx)$. Hence, $(0 : m) = (0 : x)$ which contradicts the fact that $am = 0$ and $ax \neq 0$. □

Completely prime modules are generalizations of torsion-free modules. Torsion-free modules form the module analogue of domains. If M is a faithful completely prime R-module, then R is a domain. We show in Propositions 3.9 and 3.11 (resp. Proposition 3.8) that under "suitable conditions" the R-module M is completely prime whenever R (resp. $\text{End}_R(M)$) is a domain. We define a retractable module and a torsionless module first. A module M is *retractable* if for any nonzero submodule N of M, $\text{Hom}_R(M, N) \neq \{0\}$. An R-module M is *torsionless* if for each $0 \neq m \in M$ there exists $f \in \text{Hom}_R(M, R)$ such that $f(m) \neq 0$. Free modules, generators and semisimple modules are retractable. Torsionless modules over semiprime rings are also retractable, see [16, Sec. 2, p.685].

Proposition 3.8. *Let M be a retractable R-module and $S = End_R(M)$. If S is a domain, then M is a completely prime module.*

Proof. By [19, Proposition 1.7], S is a domain if and only if any nonzero endomorphism of M is a monomorphism. By Proposition 3.1, M is a completely prime module. □

Proposition 3.9. *Let M be a torsionless R-module, if R is a domain, then M is a completely prime module.*

Proof. Suppose $am = 0$ for some $a \in R$ and $m \in M$ but $m \neq 0$ and $aM \neq \{0\}$. M torsionless implies $f(m) \neq 0$ for some $f \in \text{Hom}_R(M, R)$. Now, $a \neq 0$ and $f(m) \neq 0$ imply $af(m) \neq 0$ since R is a domain. Thus, $f(am) \neq 0$ and $am \neq 0$ which is a contradiction. □

Example 3.10. *By [2, p. 477], a submodule of a projective module is a torsionless module. Thus, if R is a domain, a submodule of a projective module is a completely prime module by Proposition 3.9.*

Proposition 3.11. *A free module M over a domain R is completely prime.*

Proof. Suppose $am = 0$ for some $a \in R$ and $m \in M$. If $m = 0$, M is a completely prime module. Suppose $m \neq 0$. Then $am = a \sum_{i=1}^{n} r_i m_i = \sum_{i=1}^{n} (ar_i) m_i = 0$ for some $r_i \in R$ and $m_i \in M$ with $i \in \{1, 2, \cdots, n\}$. M being free implies $ar_i = 0$. $m \neq 0$ implies there exists $j \in \{1, 2, \cdots, n\}$ such that $r_j \neq 0$. $ar_j = 0$ implies $a = 0$ since R is a domain and $r_j \neq 0$. Hence, $aM = \{0\}$ and M is completely prime. $\square$

4. Completely co-prime modules

Recall that an R-module M for which $RM \neq \{0\}$ is called

(1) *prime* if for all nonzero submodules N of M, $(0 : N) = (0 : M)$;

(2) *completely prime* if for all nonzero elements m of M, $(0 : m) = (0 : M)$;

(3) *co-prime* [19] if for all nonzero submodules N of M, $(N : M) = (0 : M)$.

These definitions motivate us to define completely co-prime modules.

Definition 4.1. An R-module M for which $RM \neq \{0\}$ is *completely co-prime* if for all submodules N of M and all elements $m \in M \setminus N$, $(N : m) = (0 : M)$.

Proposition 4.2. *For any R-module M, we have the following implications:*

$$completely\ co\text{-}prime \quad \Rightarrow \quad completely\ prime \quad \Rightarrow \quad prime.$$
$$\Downarrow$$
$$co\text{-}prime$$

Proof. For $\{0\} \neq N \leq M$ and $m \in M \setminus N$, we have $(0 : M) \subseteq (N : M) \subseteq (N : m)$ and $(0 : M) \subseteq (0 : m) \subseteq (N : m)$. If M is completely co-prime, $(0 : M) = (N : m)$ so that we respectively obtain $(N : M) = (0 : M)$ and $(0 : M) = (0 : m)$ for all $0 \neq m \in M$. Thus, M is respectively co-prime and completely prime. To prove completely prime implies prime, let $\{0\} \neq N \leq M$ and $0 \neq m \in N$. Then $(0 : M) \subseteq (0 : N) \subseteq (0 : m)$. If M is completely prime, $(0 : M) = (0 : m)$ so that $(0 : M) = (0 : N)$. This is true for all $\{0\} \neq N \leq M$. Thus, M is prime. $\square$

Proposition 4.3. *The following statements are equivalent for any R-module M with $RM \neq \{0\}$:*

(1) *M is completely co-prime;*

(2) *the set $\{(N : m)\}$ is a singleton for all $N \leq M$ and $m \in M \setminus N$;*

(3) *M is fully completely prime, i.e., every submodule of M is a completely prime submodule;*

(4) *M is completely prime and $(0 : m) = (N : m)$ for all $N \leq M$ and $m \in M \setminus N$;*

(5) *M is co-prime and $(N : M) = (N : m)$ for all $N \leq M$ and $m \in M \setminus N$;*

(6) M is completely prime and for all $a \in R$, $N \leq M$ and $m \in M \setminus N$, $am \in N$
implies $am = 0$;

(7) for all $N \leq M$ and $m \in M \setminus N$, $am \in N$ implies $aM = \{0\}$;

(8) the set $\{Zd_R(M/N) : N \leq M\}$ is a singleton;

(9) $(0 : M) = Zd_R(M/N)$ for all $N \leq M$.

Proof. Elementary. $\square$

From Proposition 4.3(1) and Proposition 4.3(3) we see that the notion of completely co-prime modules coincides with that of fully completely prime modules. From now onwards we use the two interchangeably.

A module is *fully prime* if all its submodules are prime submodules.

Example 4.4. *A fully prime module over a left-duo ring is fully completely prime. For if $am \in P$ for some $a \in R$, $m \in M$ and $P \leq M$, we get $aR \subseteq (P : m)$ since $(P : m)$ is a two sided ideal as R is left-duo.[1] So, $aRm \subseteq P$. By hypothesis, P is a prime submodule of M, hence $m \in P$ or $aM \subseteq P$ which proves that P is a completely prime submodule.*

If R is a commutative ring, then fully prime R-modules are indistinguishable from fully completely prime modules. Fully prime modules over commutative rings were studied in [4].

Example 4.5. *If M is a module such that every factor module of M is torsion-free, then M is completely co-prime and faithful. Observe that a factor module M/N is torsion-free if $(N : m) = \{0\}$ for all $m \in M \setminus N$. Take for instance $M := \mathbb{Z}_4 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$ the group of integers modulo 4 and $R := \mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ the ring of integers modulo 2. M is an R-module with only one nonzero submodule $N := 2\mathbb{Z}_4 = \{\bar{0}, \bar{2}\}$. For any $m \in M \setminus N$ and $a \in R$, $am \in N$ implies $a = 0$, i.e., $(N : m) = \{0\}$ for all $m \in M \setminus N$. Now, for the zero submodule, if $am = 0$ with $a \in R$ and $m \in M \setminus \{0\}$, we still get $a = 0$. So that $(0 : m) = \{0\}$. Hence, M is fully (completely) prime.*

Example 4.6. *Fully completely prime rings were studied by Hirano in [11]. If R is a fully completely prime ring such that R has no one sided left ideals, then the module $_RR$ is a fully completely prime module.*

A module is *fully IFP* if all its submodules are IFP submodules.

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[1]A ring is said to be *left-duo* if every left ideal of that ring is a two sided ideal.

Proposition 4.7. *A cyclic module over a fully completely prime ring is fully completely prime.*

Proof. We use the fact that a fully completely prime ring is fully IFP. Let $M = Rm_0$, $N \leq M$ and $am \in N$ for some $a \in R$ and $m \in M$. Then $arm_0 \in N$ for some $r \in R$ where $m = rm_0$. $ar \in (N : m_0)$. Since R is fully IFP, $(N : m_0)$ is a two sided ideal. Thus, $a \in (N : m_0)$ or $r \in (N : m_0)$ by hypothesis so that $aRm_0 \subseteq N$ or $rm_0 \in N$. From which we obtain $aM \subseteq N$ or $m \in N$. $\qquad\square$

Proposition 4.8. *Let R be a left-duo ring such that for every submodule P of an R-module M, $(P : M)$ is a maximal ideal of R, then M is a fully completely prime module.*

Proof. If $P \leq M$ and $m \in M \setminus P$, then $(P : M) \subseteq (P : m)$. Since R is left-duo, $(P : m)$ is a two sided ideal. $(P : M)$ maximal implies $(P : M) = (P : m)$, i.e., P is a completely prime submodule of M. Since P was arbitrary, M is a fully completely prime module. $\qquad\square$

Proposition 4.9. *Each of the following statements implies that the R-module M is completely co-prime:*

    (1) $(0 : m)$ *is a maximal left ideal of R for all $0 \neq m \in M$,*

    (2) $(N : m)$ *is a minimal left ideal of R for all $N \leq M$ and every $m \in M \setminus N$.*

Proof. We know that $(0 : m) \subseteq (N : m)$ for any $N \leq M$ and $m \in M \setminus N$. If $(0 : m)$ is maximal as a left ideal of R for all $0 \neq m \in M$, then $(0 : m) = (N : m)$ for all $m \in M \setminus N$. On the other hand, if $(N : m)$ is minimal as a left ideal of R for all $N \leq M$ and $m \in M \setminus N$, then $(0 : m) = (N : m)$ for all $m \in M \setminus N$. Thus, both cases imply that M is a completely co-prime module. $\qquad\square$

A ring is said to be a *chain ring* if its ideals are linearly ordered by inclusion. A chain ring is sometimes called a *uniserial ring*.

Theorem 4.10. [11, Theorem 1] *The following statements are equivalent:*

    (1) R *is a fully completely prime ring;*

    (2) R *is a chain ring satisfying $(a) = (a^2)$ for all elements $a \in R$.*

For modules, we get Theorem 4.11 and Corollary 4.12. A module is *fully symmetric* if all its submodules are symmetric submodules.

Theorem 4.11. *If M is a fully completely prime R-module such that $a, b \in R$ and $m \in M$, then*

(1) $abm = bam$,

(2) $am = a^k m$ for all positive integers k,

(3) either $am = abm$ or $bm = abm$.

Proof. Note first that a fully completely prime module is both fully symmetric and fully completely semiprime.

(1) Since $abm \in Rabm$, M fully symmetric implies $bam \in Rabm$ such that $Rbam \subseteq Rabm$. Similarly, $Rabm \subseteq Rbam$. Thus, $Rabm = Rbam$. So, $abm - bam \in R(abm - bam) = Rabm - Rbam = \{0\}$ such that $abm = bam$.

(2) $am \in Ram$. So, $a^k m \in Ram$ and $Ra^k m \subseteq Ram$ for any positive integer k. For the reverse inclusion, we know that $a^k m \in Ra^k m$. M is fully completely semiprime, therefore $am \in Ra^k m$ such that $Ram \subseteq Ra^k m$. Then, $Ram = Ra^k m$ for all positive integers k. It follows that $am - a^k m \in Ram - Ra^k m = \{0\}$. Hence, $am = a^k m$ as required.

(3) From $abm \in Rbm$, we get $Rabm \subseteq Rbm$. Similarly, we obtain $Rbam \subseteq Ram$. Since by 1, $Rabm = Rbam$ we have $Rabm \subseteq Ram$. We now seek to get reverse inclusions. $abm \in Rabm$, if $m \in Rabm$, $Ram \subseteq Rabm$ and $Rbm \subseteq Rabm$ and we are through. Suppose $m \notin Rabm$. Since M is fully completely prime $abm \in Rabm$ implies $aM \subseteq Rabm$ or $bm \in Rabm$ such that $Ram \subseteq Rabm$ or $Rbm \subseteq Rabm$ which are the required inclusions. Hence, either $Ram = Rabm$ or $Rbm = Rabm$ so that either $am = abm$ or $bm = abm$. $\square$

Corollary 4.12. *Suppose an R-module M is torsion-free and fully completely prime, then*

(1) *R is potent and hence it is commutative and fully completely prime,*

(2) *for all $a, b \in R$, $a = ab$ or $b = ab$.*

Proof. Elementary. $\square$

Remark 4.13. *Corollary 4.12 generalizes Example 4.5.*

5. Torsion theories induced

A *torsion theory* in the category R-mod of R-modules is a pair $(\mathcal{T}, \mathcal{F})$ of classes of modules in R-mod such that:

(1) $\operatorname{Hom}(T, F) = \{0\}$ for all $T \in \mathcal{T}$, $F \in \mathcal{F}$;

(2) if $\operatorname{Hom}(C, F) = \{0\}$ for all $F \in \mathcal{F}$, then $C \in \mathcal{T}$;

(3) if $\operatorname{Hom}(T, C) = \{0\}$ for all $T \in \mathcal{T}$, then $C \in \mathcal{F}$.

A functor $\gamma : R\text{-mod} \to R\text{-mod}$ is called a *preradical* if $\gamma(M)$ is a submodule of M and $f(\gamma(M)) \subseteq \gamma(N)$ for each homomorphism $f : M \to N$ in $R\text{-mod}$. A *radical* γ is a preradical for which $\gamma(M/\gamma(M)) = \{0\}$ for all $M \in R\text{-mod}$. A radical γ is *hereditary* if $N \cap \gamma(M) = \gamma(N)$ for all submodules N of M. In general, $\gamma(N) \subseteq N \cap \gamma(M)$. So, to check for hereditariness of γ, it is enough to show that the reverse inclusion, $N \cap \gamma(M) \subseteq \gamma(N)$ holds. Proposition 5.1 provides a criterion for γ to be a radical and for γ to be a hereditary radical.

Lemma 5.1. [14, Proposition 1] *Let $\mathcal{M}$ be any non-empty class of modules closed under isomorphisms, i.e., if $A \in \mathcal{M}$ and $A \cong B$, then $B \in \mathcal{M}$. For any $M \in \mathcal{M}$ define*

$$\gamma(M) = \cap\{K \ : \ K \leq M, M/K \in \mathcal{M}\}.$$

It is assumed that $\gamma(M) = M$ if $M/K \notin \mathcal{M}$ for all $K \leq M$. Then

(1) *$\gamma(M/\gamma(M)) = \{0\}$ for all modules M;*

(2) *if $\mathcal{M}$ is closed under taking non-zero submodules, γ is a radical;*

(3) *if $\mathcal{M}$ is closed under taking essential extensions, then $\gamma(M) \cap N \subseteq \gamma(N)$ for all $N \leq M$, i.e., γ is hereditary.*

It was shown in [6, Examples 3.5 and 3.6] that the completely prime radical β_{co} on the category $R\text{-mod}$ is in general not hereditary. We define a faithful completely prime radical β_{co}^f as

$$\beta_{co}^f(M) := \cap\{N \leq M \ : \ M/N \text{ is a faithful completely prime module}\}$$

and show that on the class of IFP modules, this faithful completely prime radical is hereditary. Later, in Theorem 5.3, we show that on a class of semisimple modules β_{co} is also hereditary. We write $\beta_{co}(M) = M$ (resp. $\beta_{co}^f(M) = M$) if M has no completely prime submodules (resp. if there are no faithful completely prime modules M/N for all submodules N of M).

Theorem 5.2. *The following statements hold for a class of IFP modules:*

(1) *faithful completely prime modules are closed under taking essential extension,*

(2) *the faithful completely prime radical β_{co}^f is hereditary, i.e.,*

$$\beta_{co}^f(N) = N \cap \beta_{co}^f(M)$$

for any submodule N of M;

(3) *$\tau_{\beta_{co}^f} = \{\mathcal{T}_{\beta_{co}^f}, \mathcal{F}_{\beta_{co}^f}\}$ where*

$$\mathcal{T}_{\beta_{co}^f} = \{M \ : \ M \text{ is an IFP module and } \beta_{co}^f(M) = M\}$$

and

$$\mathcal{F}_{\beta_{co}^f} = \{M \ : \ M \text{ is an IFP module and } \beta_{co}^f(M) = \{0\}\}$$

is a torsion theory;

(4) *the faithful completely prime radical is idempotent, i.e.,* $\left(\beta_{co}^f\right)^2 = \beta_{co}^f$.

Proof. (1) Suppose N is an essential submodule of an R-module M such that N is a faithful completely prime module. We show that M is also faithful and completely prime. Let $a \in R$ and $m \in M$ such that $am = 0$. If $m = 0$, M is completely prime. Suppose $m \neq 0$. Since N is an essential submodule of M, there exists $r \in R$ such that $0 \neq rm \in N$. $am = 0$ implies $arm = 0$ since by hypothesis we have a class of IFP modules. N completely prime together with the fact that $0 \neq rm \in N$ lead to $a \in (0 : rm) = (0 : N)$. In general, $(0 : M) \subseteq (0 : N)$. N faithful implies $(0 : M) = (0 : N) = \{0\}$ so that $a \in (0 : M) = \{0\}$. Then $a = 0$ such that $aM = \{0\}$ and M is faithful.

(2) Since faithful completely prime modules are closed under taking essential extension, $\beta_{co}^f(N) = N \cap \beta_{co}^f(M)$ by Lemma 5.1(3) since in general $\beta_{co}^f(N) \subseteq N \cap \beta_{co}^f(M)$.

(3) Follows from [17, Proposition 3.1] and paragraph between Propositions 2.2 and 2.3 of [17].

(4) Follows from [17, Proposition 2.3]. $\hspace{2cm}\square$

Theorem 5.3. *For a class of semisimple R-modules, the following statements hold:*

(1) *the completely prime radical is hereditary, i.e.,* $\beta_{co}(N) = N \cap \beta_{co}(M)$ *for any submodule N of M;*

(2) $\tau_{\beta_{co}} = \{\mathcal{T}_{\beta_{co}}, \mathcal{F}_{\beta_{co}}\}$ *where*

$$\mathcal{T}_{\beta_{co}} = \{M \ : \ M \text{ is semisimple and } \beta_{co}(M) = M\}$$

and

$$\mathcal{F}_{\beta_{co}} = \{M \ : \ M \text{ is semisimple and } \beta_{co}(M) = \{0\}\}$$

is a torsion theory;

(3) *the completely prime radical is idempotent, i.e.,* $\beta_{co}^2 = \beta_{co}$.

Proof. (1) If M is a semisimple R-module, then every submodule N of M is a direct summand. From [6, Corollary 3.8], $\beta_{co}(N) = N \cap \beta_{co}(M)$ for every direct summand N of M.

(2) Follows from [17, Proposition 3.1] and paragraph between Propositions 2.2 and 2.3 of [17].

(3) Follows from [17, Proposition 2.3]. □

Corollary 5.4. *If R is a semisimple Artinian ring, then each of the statements in Theorem 5.3 holds.*

Proof. A module over a semisimple Artinian ring is semisimple. The rest follows from Theorem 5.3. □

Remark 5.5. *Theorem 5.3 and Corollary 5.4 still hold when "completely prime radical" is replaced with any one of the following radicals: prime radical, s-prime radical, l-prime radical, weakly prime radical and classical completely prime radical. The module radicals: s-prime radical (also called Köthe upper nil radical), l-prime radical (also called Levitzki radical), weakly prime radical (also called classical prime radical) and classical completely prime radical were respectively defined and studied in [3], [8], [9] and [10].*

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David Ssevviiri
Department of Mathematics
Makerere University
P.O Box 7062
Kampala, Uganda
e-mails: ssevviiri@cns.mak.ac.ug,
        ssevviirid@yahoo.com