A RELATIONSHIP BETWEEN 2-PRIMAL MODULES AND MODULES THAT SATISFY THE RADICAL FORMULA

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ABSTRACT. The coincidence of the set of all nilpotent elements of a ring with
its prime radical has a module analogue which occurs when the zero submodule
satisfies the radical formula. A ring R is 2-primal if the set of all nilpotent
elements of R coincides with its prime radical. This fact motivates our study
in this paper, namely; to compare 2-primal submodules and submodules that
satisfy the radical formula. A demonstration of the importance of 2-primal
modules in bridging the gap between modules over commutative rings and
modules over noncommutative rings is done and new examples of rings and
modules that satisfy the radical formula are also given.

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1. Introduction

Unless otherwise stated, all rings are unital, associative and not necessarily commutative. The modules are left unital. The set of all positive integers is denoted by $\mathbb{N}$. First, we define key terms and fix notation which we later use in the sequel.

A proper ideal $\mathcal{I}$ of a ring R is *prime* (resp. *completely prime*) if for all ideals $\mathcal{A}, \mathcal{B}$ of R (resp. $a, b \in R$) $\mathcal{A}\mathcal{B} \subseteq \mathcal{I}$ (resp. $ab \in \mathcal{I}$), implies $\mathcal{A} \subseteq \mathcal{I}$ (resp. $a \in \mathcal{I}$) or $\mathcal{B} \subseteq \mathcal{I}$ (resp. $b \in \mathcal{I}$). Any completely prime ideal is prime but not conversely; if R is commutative, there is no distinction between the two notions. We recall a generalization of the above two ring theoretic "primes" to modules.

Definition 1.1. A proper submodule P of an R-module M for which $RM \nsubseteq P$ is

(1) *completely prime* (see [7]) if $am \in P$ implies $m \in P$ or $aM \subseteq P$, for all $a \in R$ and $m \in M$;

(2) *prime* (see [6]) if for all ideals $\mathcal{A}$ of R and submodules N of M, $\mathcal{A}N \subseteq P$ implies $N \subseteq P$ or $\mathcal{A}M \subseteq P$.

Any completely prime submodule is prime but not conversely in general. If R is commutative, the two notions coincide. A simple module is always prime but it need not be completely prime. Let P be a submodule of an R-module M and S a subset of M such that $S \nsubseteq P$. By $(P : S)$ we denote the set $\{r \in R \ : \ rS \subseteq P\}$. If P is a completely prime submodule of an R-module M, then $(P : \{m\})$ is a completely prime ideal of R such that $(P : \{m\}) = (P : M)$ for all $m \in M \setminus P$, see [7, Proposition 2.5]. On the other hand, if P is a prime submodule of an R-module M, then $(P : \{m\})$ need not be a two sided ideal of R but $(P : N)$ coincides with $(P : M)$ for all submodules N of M and it is a two sided prime ideal of R. Evidently, notions of completely prime submodules and prime submodules are distinct. A module is completely prime (resp. prime) if its zero submodule is a completely prime (resp. prime) submodule.

The intersection of all completely prime (resp. prime) submodules of an R-module M containing the submodule N is called the *completely prime* (resp. *prime*) *radical of N* and is denoted by $\beta_{co}(N)$ (resp. $\beta(N)$). If $N = 0$, we call it the *completely prime* (resp. *prime*) *radical of M* and write $\beta_{co}(M)$ (resp. $\beta(M)$) instead of $\beta_{co}(0)$ (resp. $\beta(0)$). If M has no completely prime (resp. prime) submodules containing a submodule N, we write $\beta_{co}(N) = M$ (resp. $\beta(N) = M$).

Definition 1.2. A proper submodule P of an R-module M for which $RM \nsubseteq P$ is *completely semiprime* (resp. *semiprime*) if $a^2 m \in P$ (resp. $aRam \subseteq P$) implies $am \in P$, for all $a \in R$ and $m \in M$.

A module is completely semiprime (resp. semiprime) if its zero submodule is a completely semiprime (resp. semiprime) submodule. Any completely semiprime submodule is semiprime. The converse does not hold, see [19, p. 45].

1.1. Submodules that satisfy the radical formula. For commutative rings, the set of all nilpotent elements of a ring R coincides with the prime radical $\beta(R)$ of R which is the intersection of all prime ideals of R. In general, if $\mathcal{I}$ is an ideal of a ring R and

$$\sqrt{\mathcal{I}} := \{a \in R \ : \ a^n \in \mathcal{I} \text{ for some } n \in \mathbb{N}\},$$

then for any ideal $\mathcal{I}$ of a commutative ring R we have

$$\sqrt{\mathcal{I}} = \beta(\mathcal{I}), \tag{1}$$

where $\beta(\mathcal{I})$ is the intersection of all prime ideals of R containing $\mathcal{I}$. In [16], McCasland and Moore have extended this notion to modules over commutative rings

by defining the radical formula of a submodule. The envelope $E_M(N)$ of a submodule N of an R-module M is the set

$$E_M(N) := \{rm \ : \ r \in R, \ m \in M \text{ and } r^k m \in N \text{ for some } \ k \in \mathbb{N}\}.$$

It is easy to show that if R is a commutative ring and $M = {}_R R$, then $\sqrt{0} = E_M(0)$. Since $E_M(N)$ is in general not a submodule of M, we consider the submodule $\langle E_M(N) \rangle$ of M generated by $E_M(N)$.

We say that a submodule N of an R-module M satisfies the radical formula if

$$\langle E_M(N) \rangle = \beta(N). \tag{2}$$

A module satisfies the radical formula if every submodule of M satisfies the radical formula. If every R-module satisfies the radical formula, then R is also said to satisfy the radical formula. In literature, there has been an intensive study of modules that satisfy the radical formula, see [1,2,9,12,13,17,18] among others. Unlike commutative rings for which $\sqrt{I} = \beta(I)$ for any ideal I, not all modules over commutative rings satisfy the radical formula.

1.2. 2-primal submodules. A not necessarily commutative ring R for which $\sqrt{0} = \beta(R)$ is called a 2-primal ring. This condition forces $\sqrt{0}$ to be an ideal of R. It follows from [5, Proposition 2.1] that a ring R is 2-primal if and only if $\beta_{co}(R) = \beta(R)$, where $\beta_{co}(R)$ denotes the completely prime radical of R. We remind the reader that $\beta_{co}(R)$ is the intersection of all completely prime ideals of R and it is called also the generalized nil radical. Similarly, if $\mathcal{I}$ is any ideal of R, then the symbol $\beta_{co}(\mathcal{I})$ stands for the intersection of all completely prime ideals of R containing $\mathcal{I}$. That intersection is called the completely prime radical of $\mathcal{I}$. The 2-primal rings were studied by many authors (see, for example, [5,10,14,15]). An ideal $\mathcal{I}$ of a ring R is called 2-primal if

$$\beta_{co}(R/\mathcal{I}) = \beta(R/\mathcal{I}). \tag{3}$$

In [8], a generalization of 2-primal rings was done to modules. A submodule N of an R-module M is 2-primal if

$$\beta_{co}(M/N) = \beta(M/N). \tag{4}$$

A module is 2-primal if its zero submodule is 2-primal, i.e., if $\beta_{co}(M) = \beta(M)$. Any module over a commutative ring is 2-primal and a projective module over a 2-primal ring is 2-primal [8, Theorem 2.1]. As 2-primal rings bridge the gap between commutative rings and noncommutative rings, 2-primal modules also bridge the

gap between modules over commutative rings and modules over noncommutative rings.

1.3. Questions to investigate. Since a ring R is 2-primal if and only if $\sqrt{0} = \beta(R) = \beta_{co}(R)$, it is natural to ask whether a module M is also 2-primal if and only if $\langle E_M(0) \rangle = \beta(M) = \beta_{co}(M)$. The answer is no, all submodules of modules defined over commutative rings are 2-primal but they need not satisfy the radical formula, i.e., it is possible that $\langle E_M(0) \rangle \neq \beta(M)$ for a 2-primal module M. Against this background, we pose the following questions which form the basis of our study in this paper:

(1) What is (are) the condition(s) for a module to be 2-primal if and only if $\langle E_M(0) \rangle = \beta(M)$?

(2) When does a 2-primal submodule satisfy the radical formula?

(3) When does a submodule that satisfies the radical formula become 2-primal?

(4) Whenever an ideal I of a ring R is 2-primal, the set $\sqrt{I}$ is an ideal of R; when does the set $E_M(N)$ become a submodule of M for a given submodule N of M?

(5) Can we get modules over noncommutative rings which satisfy the radical formula?

(6) Can we get noncommutative rings which satisfy the radical formula?

Note that, if N is a 2-primal submodule of M, $E_M(N)$ is not necessarily a submodule of M. Take for instance modules over a commutative ring, where each submodule is 2-primal.

In Corollary 2.21, we give a necessary and sufficient condition for a module to be 2-primal if and only if $\langle E_M(0) \rangle = \beta(M)$. In Propositions 2.7 and 2.9 which have Lemmas 2.13 and 2.12 respectively as special cases, we give situations for which 2-primal submodules satisfy the radical formula. Using these lemmas we are able to obtain modules and rings that satisfy the radical formula (see Theorems 2.18 and 2.15, respectively). In Corollaries 2.3 and 2.5 we give conditions on modules M and their submodules N for the equality $E_M(N) = \langle E_M(N) \rangle$.

2. Main results

Lemma 2.1. *If N is a submodule of an R-module M, then*

$$\langle E_M(N) \rangle \subseteq \beta_{co}(N).$$

Proof. Let $m \in E_M(N)$. Then $m = rn$ for some $r \in R$ and $n \in M$. Moreover, there exists $k \in \mathbb{N}$ such that $r^k n \in N$. So, $r^k n \in \beta_{co}(N)$. Since $\beta_{co}(N)$ is a

completely semiprime submodule of M, we have $m = rn \in \beta_{co}(N)$. Thus $E_M(N) \subseteq \beta_{co}(N)$ and finally $\langle E_M(N) \rangle \subseteq \beta_{co}(N)$. $\qquad\square$

Proposition 2.2. *If N is a completely semiprime submodule of an R-module M, then*

$$E_M(N) = N.$$

Proof. Obviously, $N \subseteq E_M(N)$. If $x \in E_M(N)$, then $x = rm$ and $r^k m \in N$ for some $r \in R$, $m \in M$ and $k \in \mathbb{N}$. As N is completely semiprime we get $x = rm \in N$. $\qquad\square$

In [3, Proposition 2.1] Azizi and Nikseresht gave a class of modules M defined over a commutative ring for which $E_M(N)$ is always a submodule of M. This class consists of all modules M such that $\beta(N) = N$ for every submodule N of M. In Corollary 2.4 we give a more general and bigger class of modules M defined over a not necessarily commutative ring for which $E_M(N)$ is a submodule of M for every submodule N of M. The class of modules we provide is that of fully completely semiprime modules. It is easy to check that the class of modules M defined over a commutative ring for which $\beta(N) = N$ for each submodule N of M is a class of fully completely semiprime modules since in such a case semiprime is indistinguishable from completely semiprime. We need Corollary 2.3 first.

Proposition 2.2 implies at once the following:

Corollary 2.3. *For any completely semiprime submodule N of a module M, $E_M(N)$ is a submodule of M.*

Corollary 2.4. *If all submodules of a module M are completely semiprime, then $E_M(N)$ is a submodule of M for any submodule N of M.*

Corollary 2.5. *If M is a 2-primal module, then $E_M(\beta(M)) = \beta(M)$. In particular, $E_M(\beta(M))$ is a submodule of M.*

Proof. As M is 2-primal, we get $\beta(M) = \beta_{co}(M)$. Moreover, $\beta_{co}(M)$ is a completely semiprime submodule of M so the assertion follows from Proposition 2.2. $\qquad\square$

Corollary 2.6. *If N is a 2-primal submodule of M, then*

$$\langle E_M(N) \rangle / N \subseteq \langle E_M(\beta(N)) \rangle / N = \beta(N)/N.$$

In particular,

$$\langle E_M(0) \rangle \subseteq \langle E_M(\beta(M)) \rangle = \beta(M)$$

for any 2-primal module M.

Proof. Suppose N is a 2-primal submodule of M. Since $\beta_{co}(M/N)$ is a completely semiprime submodule of M/N and $\beta(M/N) = \beta_{co}(M/N)$, Proposition 2.2 implies $\langle E_{M/N}(\beta(M/N)) \rangle = \beta(M/N)$. But $\langle E_{M/N}(\beta(M/N)) \rangle = \langle E_M(\beta(N)) \rangle / N$ and $\beta(M/N) = \beta(N)/N$. Hence, $\langle E_M(\beta(N)) \rangle / N = \beta(N)/N$. As $N \subseteq \beta(N)$, we get $\langle E_M(N) \rangle \subseteq \langle E_M(\beta(N)) \rangle$ and consequently $\langle E_M(N) \rangle / N \subseteq \langle E_M(\beta(N)) \rangle / N$. The second statement follows at once from the first one if we put $N = (0)$. $\square$

The next result is a direct consequence of Corollary 2.6.

Proposition 2.7. *Any 2-primal submodule N of an R-module M for which $\beta(N) = N$ satisfies the radical formula.*

Notice that for any 2-primal submodule N of an R-module M, the conditions: $\beta(N) = N$, $\beta_{co}(N) = N$, $\beta_{co}(M/N) = \{\bar{0}\}$ and $\beta(M/N) = \{\bar{0}\}$ are equivalent.

Proposition 2.8. *For any R-module M, the following statements hold:*
  (i) *if R is commutative, then every prime submodule N of M satisfies the radical formula;*
  (ii) *a completely prime submodule of M satisfies the radical formula.*

Proof. If R is commutative, then prime submodules are completely prime. If a submodule N of M is completely prime, then it is 2-primal and prime. Hence $\beta(N) = N$ and the assertion follows directly from Proposition 2.7. $\square$

Proposition 2.9. *If M is a 2-primal R-module such that $\beta(M) = \beta(R)M$ or $\beta_{co}(M) = \beta_{co}(R)M$, then the zero submodule of M satisfies the radical formula.*

Proof. Suppose that $\beta(M) = \beta(R)M$. If $x \in \beta(M)$, then $x = \sum_{i=1}^{n} a_i m_i$ with $a_i \in \beta(R)$ and $m_i \in M$. Since $\beta(R)$ is nil, each a_i is nilpotent and $a_i m_i \in E_M(0)$. Hence, $x \in \langle E_M(0) \rangle$. Since M is 2-primal, Corollary 2.6 implies $\langle E_M(0) \rangle \subseteq \beta(M)$. A similar proof works if we assume that $\beta_{co}(M) = \beta_{co}(R)M$. $\square$

Example 2.10. *Projective modules satisfy the equations: $\beta(M) = \beta(R)M$ and $\beta_{co}(M) = \beta_{co}(R)M$, see [4, Proposition 1.1.3].*

Remark 2.11. *If we consider a module M over a commutative ring, then $\langle E_M(0) \rangle \subseteq \beta(M)$. We see in Corollary 2.6 that, this is still the case when M is 2-primal. Propositions 2.7 and 2.9 and Corollary 2.5 still hold if we replace "N 2-primal" (resp. "M 2-primal") by "R is commutative". This highlights (together with the results obtained in [8]) the importance of 2-primal submodules in bridging the gap between modules over commutative rings and modules over noncommutative rings.*

According to Lee and Zhou in [11], an R-module M is reduced if for all $a \in R$ and every $m \in M$, $am = 0$ implies $Rm \cap aM = 0$. An R-module is reduced in this sense if and only if for all $a \in R$ and every $m \in M$, $a^2m = 0$ implies $aRm = 0$ if and only if for all $a \in R$ and every $m \in M$, $am = 0$ implies $aRm = 0$ and $a^2m = 0$ implies $am = 0$, see [19, p.25–26]. This implies that any reduced module in the sense of Lee and Zhou is completely semiprime. A module M is symmetric if $abm = 0$ implies $bam = 0$ for $a, b \in R$ and $m \in M$. An R-module M is IFP (i.e., it has the insertion-of-factor-property) if whenever $am = 0$ for $a \in R$ and $m \in M$, we have $aRm = 0$. An R-module M is semi-symmetric if for all $a \in R$ and every $m \in M$, $a^2m = 0$ implies $(a)^2m = 0$ where (a) is the ideal of R generated by $a \in R$. A submodule N of an R-module M is Lee-Zhou completely semiprime (resp. symmetric, IFP, semi-symmetric) if in the definition of reduced (resp. symmetric, IFP, semi-symmetric) we have N in the place of "0" and "$\in$" or "$\subseteq$" (whatever is appropriate) in the place of "=". For a detailed account of the origin of symmetric modules, IFP modules and semi-symmetric modules together with their examples, see [8].

The following chart of implications is used in the proof of Lemmas 2.12 and 2.13; it follows from [8, Theorems 2.2 and 2.3]. For any submodule P of an R-module M,

$$
\begin{array}{ccccccc}
& & R \text{ commutative} & & & & 2\text{-primal.} \\
& & \Downarrow & & & & \Uparrow \\
\text{Lee-Zhou} & \Rightarrow & \text{symmetric} & \Rightarrow & \text{IFP} & \Rightarrow & \text{semi-symmetric} \\
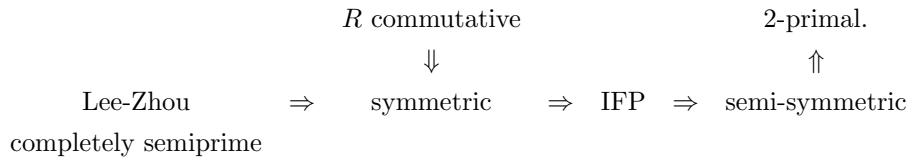\text{completely semiprime} & & & & & &
\end{array}
$$

Chart 1

Lemma 2.12. *For an R-module M, any one of the following statements implies that the zero submodule of M satisfies the radical formula:*

(1) *M is 2-primal and free,*

(2) *M is semi-symmetric and free,*

(3) *M is semi-symmetric and projective,*

(4) *M is IFP and projective,*

(5) *M is IFP and free,*

(6) *M is symmetric and projective,*

(7) *M is symmetric and free,*

(8) *M is reduced and projective,*

(9) *M is reduced and free,*

(10) *R is commutative and M is projective,*

(11) *R is commutative and M is free.*

Proof. From the chart of implications above it follows that any of the following implies that M is 2-primal: R is commutative, M is reduced, M is IFP, M is symmetric and M is semi-symmetric. Secondly, every free module is projective. The rest follows from Proposition 2.9 and Example 2.10. □

Lemma 2.12 recovers [9, Corollary 8] which says that a zero submodule of a projective module over a commutative ring satisfies the radical formula.

Lemma 2.13. *If a submodule N of a module M is completely semiprime (in the sense of Lee-Zhou), IFP, symmetric or semi-symmetric such that $\beta(N) = N$, then N satisfies the radical formula.*

Proof. This follows from Proposition 2.7 and the fact that Lee-Zhou completely semiprime, IFP, symmetric or semi-symmetric submodules are 2-primal. □

The following lemma was proved by McCasland and Moore in [16]. Note that, although they were working with modules over commutative rings, the proof they used still works even when the modules are not defined over a commutative ring.

Lemma 2.14. [16, Theorem 1.5] *Let $\phi\colon M \to M'$ be an R-module epimorphism and let N be a submodule of M such that $N \supseteq Ker\,\phi$.*

(i) *If $\beta(N) = \langle E_M(N) \rangle$, then $\beta(\phi(N)) = \langle E(\phi(N)) \rangle$;*

(ii) *If N' is a submodule of M' and $\beta(N') = \langle E(N') \rangle$, then $\beta(\phi^{-1}(N')) = \langle E(\phi^{-1}(N')) \rangle$.*

Theorem 2.15. *If the R-module M is any one of the modules given in Lemma 2.12 or it is 2-primal and projective, then M satisfies the radical formula.*

Proof. Let N be a submodule of M. For the modules given in Lemma 2.12, apply Lemma 2.14(ii) and Lemma 2.12 by letting $M' = M/N$ and $N' = N$. We know that for a 2-primal and projective module $\beta(M) = \langle E_M(0) \rangle$. When we apply Lemma 2.14(ii) by letting $M' = M/N$ and $N' = N$, we get the desired result. □

An alternative proof can be given for the six (6) R-modules M in Lemma 2.12 which are free. Recall that every R-module M is the image of a free R-module. This together with Lemma 2.14(i) shows that M satisfies the radical formula.

Corollary 2.16. *If R is a semisimple ring such that the R-module M is 2-primal, then M satisfies the radical formula.*

Proof. If R is semisimple, then the R-module M is projective. The rest follows from Theorem 2.15. □

Corollary 2.17. *If R is a semisimple and commutative ring, then the R-module M satisfies the radical formula.*

Proof. If R is semisimple and commutative, then M is 2-primal and projective and it is sufficient to apply Theorem 2.15. □

A ring R is absolutely radical if for all R-modules M, we have $\beta(N) = N$ for each submodule N of M.

Theorem 2.18. *If R is an absolutely radical ring such that each submodule N of the R-module M is one of the following: Lee-Zhou completely semiprime, IFP, symmetric or semi-symmetric, then R satisfies the radical formula.*

Proof. Notice that R is an absolutely radical ring if and only if $\beta(N) = N$ for each submodule N of M. The rest follows from Lemma 2.13. □

Proposition 2.19. *If a submodule N of an R-module M satisfies the radical formula and $\beta_{co}(N) \subseteq \langle E_M(N) \rangle$, then N is 2-primal. On the other hand, if a submodule N of an R-module M is 2-primal and $\beta_{co}(N) \subseteq \langle E_M(N) \rangle$, then the zero submodule of the R-module M/N satisfies the radical formula.*

Proof. By hypothesis, $\beta_{co}(N) \subseteq \langle E_M(N) \rangle = \beta(N)$ and in general, $\beta(N) \subseteq \beta_{co}(N)$. It follows that $\beta_{co}(N) = \beta(N)$ such that $\beta_{co}(M/N) = \beta_{co}(N)/N = \beta(N)/N = \beta(M/N)$. For the second part, suppose $\beta(M/N) = \beta_{co}(M/N)$ and $\beta_{co}(N) \subseteq \langle E_M(N) \rangle$. Then $\beta(N)/N = \beta_{co}(N)/N \subseteq \langle E_M(N) \rangle/N$. From Lemma 2.1, $\langle E_M(N) \rangle/N \subseteq \beta_{co}(N)/N$. This implies $\beta(N)/N = \langle E_M(N) \rangle/N$, i.e., $\beta(M/N) = \langle E_{M/N}(\{\bar{0}\}) \rangle$. □

Remark 2.20. *The conditions: (1) $\beta_{co}(N) = N$ (which for example holds when N is a completely prime submodule) and (2) $\langle E_M(N) \rangle = M$ (which for example holds when M is cyclic and R is nil or M is cyclic and its generator is contained in N) always guarantee existence of the inclusion $\beta_{co}(N) \subseteq \langle E_M(N) \rangle$.*

Corollary 2.21. *The necessary and sufficient condition for the zero submodule of an R-module M to satisfy the radical formula if and only if M is 2-primal is*

$$\beta_{co}(M) \subseteq \langle E_M(0) \rangle.^1 \tag{5}$$

Proof. It follows from Proposition 2.19. □

The following example shows that containment (5) in Corollary 2.21 does not hold in general.

Example 2.22. *Define $R = \mathbb{Z}[x]$, $F = R \oplus R$, $f = (2, x) \in F$ and $\mathcal{P} = 2R + Rx$ (which is a maximal ideal of R). If $N = \mathcal{P}f$ and $M = F/N$, then M is completely semiprime and $\beta(M) = Rf/N \neq 0$, see [9, p. 3600]. This shows that $\langle E_M(0) \rangle = 0$ (see Proposition 2.2) and $\beta_{co}(M) \neq 0$ since for modules over a commutative ring, there is no distinction between completely prime (resp. completely semiprime) submodules and prime (resp. semiprime) submodules.*

All submodules of a module defined over a commutative ring are 2-primal but they need not satisfy the radical formula. We do not know of an example of a submodule which satisfies the radical formula but not 2-primal, although we suspect these examples exist. The motivation of our suspicion is that, for any module M, $\beta(M) \subseteq \beta_{co}(M)$ and $\langle E_M(0) \rangle \subseteq \beta_{co}(M)$ and these inclusions are in general strict. Hence, it is probably possible that $\beta(M) = \langle E_M(0) \rangle \not\subseteq \beta_{co}(M)$, in which case the zero submodule of M satisfies the radical formula but not 2-primal. An affirmative answer to any one of the following questions gives us the desired example(s).

Question 2.1. *Is there a prime module M which is not completely prime and $E_M(0) = 0$?*

Question 2.2. *Can we get a completely semiprime module M which is not completely prime and $\beta(M) = 0$?*

References

[1] A. Azizi, *Radical formula and prime submodules*, J. Algebra, 307(1) (2007), 454–460.

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[1] To have $\beta_{co}(M) \subseteq \langle E_M(0) \rangle$ is equivalent to having $\beta_{co}(M) = \langle E_M(0) \rangle$ since the reverse inclusion always holds.

[2] A. Azizi, *Radical formula and weakly prime submodules*, Glasg. Math. J., 51(2) (2009), 405–412. DOI:10.1017/S0017089509005072.

[3] A. Azizi and A. Nikseresht, *Simplified radical formula in modules*, Houston J. Math., 38(2) (2012), 333-344.

[4] L. Bican, T. Kepka and P. Nemec, Rings, Modules and Preradicals, Lecture Notes in Pure and Applied Mathematics, 75. Marcel Dekker, Inc., New York, 1982.

[5] G. F. Birkenmeier, H. E. Heatherly and E. K. Lee, *Completely prime ideals and associated radicals*, Ring theory (Granville, OH, 1992), 102-129, World Sci. Publ., River Edge, NJ, 1993.

[6] J. Dauns, *Prime modules*, J. Reine Angew. Math., 298 (1978), 156-181.

[7] N. J. Groenewald and D. Ssevviiri, *Completely prime submodules*, Int. Electron. J. Algebra, 13 (2013), 1–14.

[8] N. J. Groenewald and D. Ssevviiri, *2-primal modules*, J. Algebra Appl., 12(5) (2013), 12 pp. DOI: 10.1142/S021949881250226X.

[9] J. Jenkins and P. F. Smith, *On the prime radical of a module over a commutative ring*, Comm. Algebra, 20(12) (1992), 3593–3602.

[10] Y. Lee, C. Huh and H. K. Kim, *Questions on 2-primal rings*, Comm. Algebra, 26(2) (1998), 595–600.

[11] T. K. Lee and Y. Zhou, *Reduced Modules*, Rings, modules, algebras, and abelian groups, 365-377, Lecture Notes in Pure and Appl. Math., 236, Dekker, New York, 2004.

[12] K. H. Leung and S. H. Man, *On commutative Noetherian rings which satisfy the radical formula*, Glasgow Math. J., 39(3) (1997), 285–293.

[13] S. H. Man, *On commutative Noetherian rings which satisfy the generalized radical formula*, Comm. Algebra, 27(8) (1999), 4075–4088.

[14] G. Marks, *On 2-primal ore extensions*, Comm. Algebra, 29(5) (2001), 2113–2123.

[15] G. Marks, *A taxonomy of 2-primal rings*, J. Algebra, 266(2) (2003), 494–520.

[16] R. L. McCasland and M. E. Moore, *On radicals of submodules*, Comm. Algebra, 19(5) (1991), 1327–1341.

[17] A. Nikseresht and A. Azizi, *On radical formula in modules*, Glasg. Math. J., 53(3) (2011), 657–668. DOI:10.1017/S0017089511000243.

[18] H. Sharif, Y. Sharifi and S. Namazi, *Rings satisfying the radical formula*, Acta Math. Hungar., 71 (1996), 103–108.

[19] D. Ssevviiri, A Contribution to the Theory of Prime Modules, PhD Thesis, Nelson Mandela Metropolitan University, 2013.

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