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ON IDENTITIES IN HOM-MALCEV ALGEBRAS

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Abstract. In an anticommutative multiplicative Hom-algebra, an identity, equivalent to the Hom-Malcev identity, is found.

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1. Introduction and statement of results

Hom-Lie algebras were introduced in [3] as a tool in understanding the structure and constructions of q-deformations of the Witt and the Virasoro algebras within the general framework of quasi-Lie algebras and quasi-Hom-Lie algebras introduced in [6,7,13]. Hom-associative algebras, as an analogue and generalization of associative algebras for Hom-Lie algebras, have been introduced in [9]. Since then, the theory of Hom-type algebras began an intensive development (see, e.g., [2,4,8,9,10,14,16,17,18,19]). Hom-type algebras are defined by twisting the defining identities of some well-known algebras by a linear self-map, and when this twisting map is the identity map, one recovers the original type of considered algebras.

In this setting, a Hom-type generalization of Malcev algebras (called Hom-Malcev algebras) is defined in [19]. Recall that a *Malcev algebra* is a nonassociative algebra $(A, \cdot)$, where the binary operation "$\cdot$" is anticommutative, such that the identity

$$J(x, y, x \cdot z) = J(x, y, z) \cdot x \tag{1}$$

holds for all $x, y, z \in A$ (here $J(x, y, z)$ denotes the Jacobian, i.e. $J(x, y, z) = xy \cdot z + yz \cdot x + zx \cdot y$; here and in the sequel, juxtaposition is used in order to reduce the number of braces i.e., e.g., $xy \cdot z$ means $(x \cdot y) \cdot z$). The identity (1) is known as the *Malcev identity.* Malcev algebras were introduced by A. I. Mal'tsev [11] (calling them Moufang-Lie algebras) as tangent algebras to local smooth Moufang loops, generalizing in this way a result in Lie theory stating that a Lie algebra is a tangent algebra to a local Lie group (in fact, Lie algebras are special case of Malcev algebras). Another approach to Malcev algebras is the one from alternative algebras: every alternative algebra is Malcev-admissible [11]. So one could say that the algebraic theory of Malcev algebras started from Malcev-admissibility of algebras. The foundations of the algebraic theory of Malcev algebras go back to E. Kleinfeld [5], A. A. Sagle [12] and, as mentioned in [12], to A. A. Albert and L. J. Paige. Some twisting of the Malcev identity (1) along any algebra self-map α of A gives rise to the notion of a Hom-Malcev algebra $(A, \cdot, \alpha)$ ([19]; see definitions in section 2). Properties and constructions of Hom-Malcev algebras, as well as the relationships between these Hom-algebras and Hom-alternative or Hom-Jordan algebras are investigated in [19]. In particular, it is shown that a Malcev algebra can be twisted into a Hom-Malcev algebra and that Hom-alternative algebras are Hom-Malcev admissible.

In [19], as for Malcev algebras (see [12,15]), equivalent defining identities of a Hom-Malcev algebra are given. In this note, we mention another identity in a Hom-Malcev algebra that is equivalent to the ones found in [19]. Specifically, we shall prove the following

Theorem. *Let $(A, \cdot, \alpha)$ be a Hom-Malcev algebra. Then the identity*

$$J_\alpha(w \cdot x, \alpha(y), \alpha(z)) = J_\alpha(w, y, z) \cdot \alpha^2(x) + \alpha^2(w) \cdot J_\alpha(x, y, z) - 2J_\alpha(y \cdot z, \alpha(w), \alpha(x)) \tag{2}$$

holds for all w, x, y, z in A, where $J_\alpha(x, y, z) = xy \cdot \alpha(z) + yz \cdot \alpha(x) + zx \cdot \alpha(y)$. Moreover, in any anticommutative multiplicative Hom-algebra $(A, \cdot, \alpha)$, the identity (2) is equivalent to the Hom-Malcev identity

$$J_\alpha(\alpha(x), \alpha(y), x \cdot z) = J_\alpha(x, y, z) \cdot \alpha^2(x) \tag{3}$$

for all x, y, z in A.

Observe that when $\alpha = Id$ (the identity map) in (3), then (3) is (1) i.e. the Hom-Malcev algebra $(A, \cdot, \alpha)$ reduces to the Malcev algebra $(A, \cdot)$ (see [19]).

In Section 2 some instrumental lemmas are proved. Some results in these lemmas are a kind of the Hom-version of similar results by E. Kleinfeld [5] in case of Malcev algebras. The Section 3 is devoted mainly to the proof of the theorem.

Throughout this note we work over a ground field $\mathbb{K}$ of characteristic 0.

2. Definitions and preliminary results

In this section we recall useful notions on Hom-algebras ([9,16,17,19]), as well as the one of a Hom-Malcev algebra [19]. In [5], using an analogue of the Bruck-Kleinfeld function, an identity (see identity (6) in [5]) characterizing Malcev algebras is found. This identity is used in [12] to derive further identities for Malcev algebras (see [12], Proposition 2.23). The main result of this section (Lemma 2.7) proves that the Hom-version of the identity (6) of [5] holds in any Hom-Malcev algebra.

Definition 2.1. A *multiplicative Hom-algebra* is a triple (A, μ, α), in which A is a $\mathbb{K}$-module, $\mu : A \times A \to A$ is a bilinear map (the binary operation), and $\alpha : A \to A$ is a linear map (the twisting map) such that α is an endomorphism of (A, μ). The Hom-algebra (A, μ, α) is said to be *anticommutative* if the operation μ is skew-symmetric, i.e. $\mu(x, y) = -\mu(y, x)$, for all $x, y \in A$.

In the rest of this paper, we will use the abbreviation $x \cdot y = \mu(x, y)$ in a Hom-algebra (A, μ, α).

Remark. *The multiplicativity is not necessary in the definition of a Hom-algebra (see, e.g.,* [8,9]*). The multiplicativity is included here for convenience. In what follows, we assume that all Hom-algebras are multiplicative.*

Definition 2.2. Let $(A, \cdot, \alpha)$ be an anticommutative Hom-algebra.

(i) The *Hom-Jacobian* ([9]) of $(A, \cdot, \alpha)$ is the trilinear map $J_\alpha(x, y, z)$ on A defined by $J_\alpha(x, y, z) = xy \cdot \alpha(z) + yz \cdot \alpha(x) + zx \cdot \alpha(y)$.

(ii) $(A, \cdot, \alpha)$ is called a *Hom-Lie algebra* ([3]) if the *Hom-Jacobi identity* $J_\alpha(x, y, z) = 0$ holds in $(A, \cdot, \alpha)$.

Definition 2.3. ([19]) A *Hom-Malcev algebra* is an anticommutative algebra $(A, \cdot, \alpha)$ such that the *Hom-Malcev identity* (see (3))

$$J_\alpha(\alpha(x), \alpha(y), x \cdot z) = J_\alpha(x, y, z) \cdot \alpha^2(x)$$

holds in $(A, \cdot, \alpha)$.

Remark. *When $\alpha = Id$, then the Hom-Jacobi identity reduces to the usual Jacobi identity $J(x, y, z) := xy \cdot z + yz \cdot x + zx \cdot y = 0$, i.e. the Hom-Lie algebra $(A, \cdot, \alpha)$ reduces to the Lie algebra $(A, \cdot)$. Likewise, when $\alpha = Id$, the Hom-Malcev identity reduces to the Malcev identity (1), i.e. the Hom-Malcev algebra $(A, \cdot, \alpha)$ reduces to the Malcev algebra $(A, \cdot)$.*

The following simple lemma holds in any anticommutative Hom-algebra.

Lemma 2.4. *In any anticommutative Hom-algebra $(A, \cdot, \alpha)$ the following hold:*

(i) *$J_\alpha(x, y, z)$ is skew-symmetric in its three variables.*

(ii) *$\alpha^2(w) \cdot J_\alpha(x, y, z) - \alpha^2(x) \cdot J_\alpha(y, z, w) + \alpha^2(y) \cdot J_\alpha(z, w, x) - \alpha^2(z) \cdot J_\alpha(w, x, y) = J_\alpha(w \cdot x, \alpha(y), \alpha(z)) + J_\alpha(y \cdot z, \alpha(w), \alpha(x)) + J_\alpha(w \cdot y, \alpha(z), \alpha(x)) + J_\alpha(z \cdot x, \alpha(w), \alpha(y)) - J_\alpha(z \cdot w, \alpha(x), \alpha(y)) - J_\alpha(x \cdot y, \alpha(z), \alpha(w))$, for all w, x, y, z in A.*

Proof. The skew-symmetry of $J_\alpha(x, y, z)$ in x, y, z follows from the skew-symmetry of the operation "$\cdot$".

Expanding the expression in the left-hand side of (ii) and then rearranging terms, we get (by the skew-symmetry of "$\cdot$")

$$\begin{aligned}
&\alpha^2(w) \cdot J_\alpha(x, y, z) - \alpha^2(x) \cdot J_\alpha(y, z, w) + \alpha^2(y) \cdot J_\alpha(z, w, x) \\
&-\alpha^2(z) \cdot J_\alpha(w, x, y) \\
&= -\alpha^2(z) \cdot (wx \cdot \alpha(y)) + \alpha^2(y) \cdot (wx \cdot \alpha(z)) \\
&-\alpha^2(x) \cdot (yz \cdot \alpha(w)) + \alpha^2(w) \cdot (yz \cdot \alpha(x)) \\
&-\alpha^2(x) \cdot (wy \cdot \alpha(z)) - \alpha^2(z) \cdot (yw \cdot \alpha(x)) \\
&+\alpha^2(w) \cdot (zx \cdot \alpha(y)) + \alpha^2(y) \cdot (xz \cdot \alpha(w)) \\
&-\alpha^2(x) \cdot (zw \cdot \alpha(y)) + \alpha^2(y) \cdot (zw \cdot \alpha(x)) \\
&+\alpha^2(w) \cdot (xy \cdot \alpha(z)) - \alpha^2(z) \cdot (xy \cdot \alpha(w)).
\end{aligned}$$

Next, adding and subtracting $\alpha(yz) \cdot \alpha(wx)$ (resp. $\alpha(wx) \cdot \alpha(yz)$, $\alpha(zx) \cdot \alpha(wy)$, $\alpha(wy) \cdot \alpha(zx)$, $\alpha(xy) \cdot \alpha(zw)$ and $\alpha(zw) \cdot \alpha(xy)$) in the first (resp. second, third, fourth, fifth, and sixth) line of the right-hand side expression in the last equality above, we come to the equality (ii) of the lemma. □

In a Hom-Malcev algebra $(A, \cdot, \alpha)$ we define the multilinear map G by

$$G(w, x, y, z) = J_\alpha(w \cdot x, \alpha(y), \alpha(z)) - \alpha^2(x) \cdot J_\alpha(w, y, z) - J_\alpha(x, y, z) \cdot \alpha^2(w) \quad (4)$$

for all w, x, y, z in A.

Remark.

(i) *If $\alpha = Id$ in (4), then $G(w, x, y, z)$ reduces to the function $f(w, x, y, z)$ defined in* [5] *which in turn is a variation of the Bruck-Kleinfeld function defined in* [1].

(ii) *If in (4) replace $J_\alpha(t, u, v)$ with the Hom-associator* [9] *$as(t, u, v)$, then one recovers the Hom-Bruck-Kleinfeld function defined in* [19].

Lemma 2.5. *In a Hom-Malcev algebra $(A, \cdot, \alpha)$ the function $G(w, x, y, z)$ defined by (4) is skew-symmetric in its four variables.*

Proof. From the skew-symmetry of "$\cdot$" and $J_\alpha(t, u, v)$ (see Lemma 2.4(i)) it clearly follows that

$G(x,w,y,z) = -G(w,x,y,z)$ and $G(w,x,z,y) = -G(w,x,y,z)$.
Next, using the skew-symmetry of $J_\alpha(t,u,v)$,

$$\begin{aligned} G(y,x,y,z) &= J_\alpha(y\cdot x,\alpha(y),\alpha(z)) - J_\alpha(x,y,z)\cdot\alpha^2(y) \\ &= J_\alpha(\alpha(y),\alpha(z),y\cdot x) - J_\alpha(y,z,x)\cdot\alpha^2(y) \\ &= J_\alpha(y,z,x)\cdot\alpha^2(y) - J_\alpha(y,z,x)\cdot\alpha^2(y) \ \text{(by (3))} \\ &= 0. \end{aligned}$$

Likewise, one checks that $G(w,y,y,z) = 0$. This suffices to prove the skew-symmetry of $G(w,x,y,z)$ in its variables. □

As we shall see below, the following lemma is a consequence of the definition of $G(w,x,y,z)$ and the skew-symmetry of $J_\alpha(t,u,v)$ and $G(w,x,y,z)$.

Lemma 2.6. *Let $(A,\cdot,\alpha)$ be a Hom-Malcev algebra. Then*

$$J_\alpha(w\cdot x,\alpha(y),\alpha(z)) + J_\alpha(x\cdot y,\alpha(z),\alpha(w)) + J_\alpha(y\cdot z,\alpha(w),\alpha(x)) + J_\alpha(z\cdot w,\alpha(x),\alpha(y)) = 0; \tag{5}$$

$$\begin{aligned} &2G(w,y,y,z) - \alpha^2(w)\cdot J_\alpha(x,y,z) + \alpha^2(x)\cdot J_\alpha(w,y,z) - \alpha^2(y)\cdot J_\alpha(z,w,x) \\ &\quad +\alpha^2(z)\cdot J_\alpha(w,x,y) = J_\alpha(w\cdot x,\alpha(y),\alpha(z)) + J_\alpha(y\cdot z,\alpha(w),\alpha(x)), \end{aligned} \tag{6}$$

for all w,x,y,z in A.

Proof. From the definition of $G(w,x,y,z)$ (see (4)) we have

$$\begin{aligned} &J_\alpha(w\cdot x,\alpha(y),\alpha(z)) = G(w,x,y,z) + \alpha^2(x)\cdot J_\alpha(w,y,z) + J_\alpha(x,y,z)\cdot\alpha^2(w), \\ &J_\alpha(x\cdot y,\alpha(z),\alpha(w)) = G(x,y,z,w) + \alpha^2(y)\cdot J_\alpha(x,z,w) + J_\alpha(y,z,w)\cdot\alpha^2(x), \\ &J_\alpha(y\cdot z,\alpha(w),\alpha(x)) = G(y,z,w,x) + \alpha^2(z)\cdot J_\alpha(y,w,x) + J_\alpha(z,w,x)\cdot\alpha^2(y), \\ &J_\alpha(z\cdot w,\alpha(x),\alpha(y)) = G(z,w,x,y) + \alpha^2(w)\cdot J_\alpha(z,x,y) + J_\alpha(w,x,y)\cdot\alpha^2(z). \end{aligned}$$

Therefore, by the skew-symmetry of "$\cdot$", $J_\alpha(x,y,z)$ and $G(w,x,y,z)$, we get

$$\begin{aligned} &J_\alpha(w\cdot x,\alpha(y),\alpha(z)) + J_\alpha(x\cdot y,\alpha(z),\alpha(w)) + J_\alpha(y\cdot z,\alpha(w),\alpha(x)) + J_\alpha(z\cdot w,\alpha(x),\alpha(y)) \\ &= G(w,x,y,z) + G(x,y,z,w) + G(y,z,w,x) + G(z,w,x,y) \\ &= G(w,x,y,z) - G(w,x,y,z) + G(y,z,w,x) - G(y,z,w,x) \\ &= 0, \end{aligned}$$

which proves (5).

Next, again from the expression of $G(w,x,y,z)$,

$$\begin{aligned} &J_\alpha(w\cdot x,\alpha(y),\alpha(z)) + J_\alpha(y\cdot z,\alpha(w),\alpha(x)) \\ &= [G(w,x,y,z) + \alpha^2(x)\cdot J_\alpha(w,y,z) + J_\alpha(x,y,z)\cdot\alpha^2(w)] \\ &+[G(y,z,w,x) + \alpha^2(z)\cdot J_\alpha(y,w,x) + J_\alpha(z,w,x)\cdot\alpha^2(y)] \\ &= 2G(w,x,y,z) - \alpha^2(w)\cdot J_\alpha(x,y,z) + \alpha^2(x)\cdot J_\alpha(y,z,w) - \alpha^2(y)\cdot J_\alpha(z,w,x) \\ &+\alpha^2(z)\cdot J_\alpha(w,x,y), \end{aligned}$$

so that we get (6). □

From Lemma 2.5 and Lemma 2.6, we get the following expression of $G(w,x,y,z)$.

Lemma 2.7. *Let* $(A,\cdot,\alpha)$ *be a Hom-Malcev algebra. Then*

$$G(w,x,y,z) = 2[J_\alpha(w\cdot x,\alpha(y),\alpha(z)) + J_\alpha(y\cdot z,\alpha(w),\alpha(x))] \tag{7}$$

for all w,x,y,z *in* A.

Proof. Set $g(w,x,y,z) = J_\alpha(w\cdot x,\alpha(y),\alpha(z)) + J_\alpha(x\cdot y,\alpha(z),\alpha(w)) + J_\alpha(y\cdot z,\alpha(w),\alpha(x)) + J_\alpha(z\cdot w,\alpha(x),\alpha(y))$. Then (5) says that $g(w,x,y,z)=0$ for all w,x,y,z in A. Now, by adding $g(w,x,y,z)-g(x,w,y,z)$ to the right-hand side of Lemma 2.4(ii), we get

$$\begin{aligned}
\alpha^2(w)\cdot J_\alpha(x,y,z) \;&-\; \alpha^2(x)\cdot J_\alpha(y,z,w)\\
&+\; \alpha^2(y)\cdot J_\alpha(z,w,x) - \alpha^2(z)\cdot J_\alpha(w,x,y)\\
&=\; J_\alpha(w\cdot x,\alpha(y),\alpha(z)) + J_\alpha(y\cdot z,\alpha(w),\alpha(x))\\
&+\; J_\alpha(w\cdot y,\alpha(z),\alpha(x)) + J_\alpha(z\cdot x,\alpha(w),\alpha(y))\\
&-\; J_\alpha(z\cdot w,\alpha(x),\alpha(y)) - J_\alpha(x\cdot y,\alpha(z),\alpha(w))\\
&+\; J_\alpha(w\cdot x,\alpha(y),\alpha(z)) + J_\alpha(x\cdot y,\alpha(z),\alpha(w))\\
&+\; J_\alpha(y\cdot z,\alpha(w),\alpha(x)) + J_\alpha(z\cdot w,\alpha(x),\alpha(y))\\
&-\; J_\alpha(x\cdot w,\alpha(y),\alpha(z)) - J_\alpha(w\cdot y,\alpha(z),\alpha(x))\\
&-\; J_\alpha(y\cdot z,\alpha(x),\alpha(w)) - J_\alpha(z\cdot x,\alpha(w),\alpha(y))\\
&=\; 3J_\alpha(w\cdot x,\alpha(y),\alpha(z)) + 3J_\alpha(y\cdot z,\alpha(w),\alpha(x))
\end{aligned}$$

i.e.

$$\begin{aligned}
&\alpha^2(w)\cdot J_\alpha(x,y,z) - \alpha^2(x)\cdot J_\alpha(y,z,w) + \alpha^2(y)\cdot J_\alpha(z,w,x) - \alpha^2(z)\cdot J_\alpha(w,x,y)\\
&\qquad = 3[J_\alpha(w\cdot x,\alpha(y),\alpha(z)) + J_\alpha(y\cdot z,\alpha(w),\alpha(x))].
\end{aligned} \tag{8}$$

Next, adding (6) and (8) together, we get

$$\begin{aligned}
2G(w,x,y,z) \;&-\; \alpha^2(w)\cdot J_\alpha(x,y,z) + \alpha^2(x)\cdot J_\alpha(y,z,w)\\
&-\; \alpha^2(y)\cdot J_\alpha(z,w,x) + \alpha^2(z)\cdot J_\alpha(w,x,y)\\
&+\; \alpha^2(w)\cdot J_\alpha(x,y,z) - \alpha^2(x)\cdot J_\alpha(y,z,w)\\
&+\; \alpha^2(y)\cdot J_\alpha(z,w,x) - \alpha^2(z)\cdot J_\alpha(w,x,y)\\
&=\; J_\alpha(w\cdot x,\alpha(y),\alpha(z)) + J_\alpha(y\cdot z,\alpha(w),\alpha(x))\\
&+\; 3[J_\alpha(w\cdot x,\alpha(y),\alpha(z)) + J_\alpha(y\cdot z,\alpha(w),\alpha(x))]
\end{aligned}$$

i.e.

$2G(w,x,y,z) = 4[J_\alpha(w\cdot x,\alpha(y),\alpha(z)) + J_\alpha(y\cdot z,\alpha(w),\alpha(x))]$ and (7) follows. □

3. Proof

Relaying on the lemmas of Section 2, we are now in position to prove the theorem.

Proof of the Theorem. First we establish the identity (2) in a Hom-Malcev algebra. We may write (4) in an equivalent form:

$$J_\alpha(w \cdot x, \alpha(y), \alpha(z)) = \alpha^2(x) \cdot J_\alpha(w, y, z) + J_\alpha(x, y, z) \cdot \alpha^2(w) + G(w, x, y, z). \quad (9)$$

Now in (9), replace $G(w, x, y, z)$ with its expression from (7) to get

$$-J_\alpha(w \cdot x, \alpha(y), \alpha(z)) = \alpha^2(x) \cdot J_\alpha(w, y, z) + J_\alpha(x, y, z) \cdot \alpha^2(w) + 2J_\alpha(y \cdot z, \alpha(w), \alpha(x)),$$

which leads to (2).

Now, we proceed to prove the equivalence of (2) with (3) in an anticommutative Hom-algebra $(A, \cdot, \alpha)$.

First assume (3) in $(A, \cdot, \alpha)$. Then, as we have seen just above, Lemmas 2.4, 2.5, 2.6, and 2.7 imply that (2) holds in any Hom-Malcev algebra.

Conversely, assume (2) in $(A, \cdot, \alpha)$. Then, setting $w = y$ in (2), we get, by the skew-symmetry of $J_\alpha(x, y, z)$,

$$J_\alpha(y \cdot x, \alpha(y), \alpha(z)) = \alpha^2(y) \cdot J_\alpha(y, z, x) - 2J_\alpha(\alpha(y), \alpha(x), y \cdot z). \quad (10)$$

Now, the permutation of z with x in (10) gives

$$J_\alpha(y \cdot z, \alpha(y), \alpha(x)) = \alpha^2(y) \cdot J_\alpha(y, x, z) - 2J_\alpha(\alpha(y), \alpha(z), y \cdot x),$$

i.e.

$$2J_\alpha(y \cdot z, \alpha(y), \alpha(x)) = -2\alpha^2(y) \cdot J_\alpha(y, z, x) - 4J_\alpha(\alpha(y), \alpha(z), y \cdot x),$$

or

$$4J_\alpha(\alpha(y), \alpha(z), y \cdot x) = -2\alpha^2(y) \cdot J_\alpha(y, z, x) - 2J_\alpha(y \cdot z, \alpha(y), \alpha(x)). \quad (11)$$

Next, the subtraction of (11) from (10) gives (keeping in mind the skew-symmetry of $J_\alpha(x, y, z)$)

$$-3J_\alpha(\alpha(y), \alpha(z), y \cdot x) = 3\alpha^2(y) \cdot J_\alpha(y, z, x)$$

i.e.

$$J_\alpha(\alpha(y), \alpha(z), y \cdot x) = J_\alpha(y, z, x) \cdot \alpha^2(y)$$

so that we get (3). □

Remark. *If set $\alpha = Id$, then the identity (2) (resp. (3)) reduces to the identity (2.26) (resp. (2.4)) of* [12]. *The equivalence of (2.4) and (2.26) of* [12] *could be infered from the works* [12] *and* [15].

Example 3.1. *There is a 4-dimensional Hom-Malcev algebra* $(A, \cdot, \alpha)$ *with basis* $\{e_1, e_2, e_3, e_4\}$ *and multiplication table given by*

$$\begin{aligned} e_1 \cdot e_2 &= -\alpha(e_2) &&= -e_2 \cdot e_1, \\ e_1 \cdot e_3 &= -e_3 &&= -e_3 \cdot e_1, \\ e_1 \cdot e_4 &= e_4 &&= -e_4 \cdot e_1, \\ e_2 \cdot e_3 &= 2e_4 &&= -e_3 \cdot e_2, \end{aligned}$$

and all other products are 0, *with* α *being defined by*

$$\begin{aligned} \alpha(e_1) &= e_1 + e_3 + e_4, \\ \alpha(e_2) &= 2e_2 + e_3 + 2e_4, \\ \alpha(e_3) &= e_3, \\ \alpha(e_4) &= 2e_4 \end{aligned}$$

(see [19], *Example 2.14; here, for simplicity, we specify this example from* [19] *by taking suitable values for the coefficients in the expression of* α*). So* $(A, \cdot, \alpha)$ *verifies (3). Moreover, in general,* $(A, \cdot, \alpha)$ *is neither a Malcev algebra nor a Hom-Lie algebra and so* $(A, \cdot, \alpha)$ *is a nontrivial Hom-Malcev algebra.*

We proceed to show how the main identities (2) and (7) do work in this example.

First we observe that, by the skew-symmetry of $G(w, x, y, z)$ *(see Lemma 2.5),* $G(e_1, e_2, e_3, e_4)$ *is the only one essential nontrivial expression of the map* G *with respect to* $(A, \cdot, \alpha)$*.*

Now we have

$$\begin{aligned} \alpha^2(e_1) &= e_1 + 2e_3 + 3e_4, \\ \alpha^2(e_2) &= 4e_2 + 3e_3 + 8e_4, \\ J_\alpha(e_1, e_3, e_4) &= 0, \\ J_\alpha(e_2, e_3, e_4) &= 0, \\ J_\alpha(e_1 \cdot e_2, \alpha(e_3), \alpha(e_4)) &= 0, \\ J_\alpha(e_3 \cdot e_4, \alpha(e_1), \alpha(e_2)) &= 0. \end{aligned}$$

Therefore, the definition of G *(see (4)) implies that* $G(e_1, e_2, e_3, e_4) = 0$*. From the other side, the identity (7) gives*

$$G(e_1, e_2, e_3, e_4) = 2[J_\alpha(e_1 \cdot e_2, \alpha(e_3), \alpha(e_4)) + J_\alpha(e_3 \cdot e_4, \alpha(e_1), \alpha(e_2))] = 0.$$

Therefore, this example illustrates the concordance of (4) with (7) in the Hom-Malcev algebra $(A, \cdot, \alpha)$ *defined above.*

For the verification of (2), first we consider (4), i.e.

$$G(w, x, y, z) = J_\alpha(w \cdot x, \alpha(y), \alpha(z)) + J_\alpha(w, y, z) \cdot \alpha^2(x) + \alpha^2(w) \cdot J_\alpha(x, y, z).$$

In the present case, this looks as

$$G(e_1, e_2, e_3, e_4) = J_\alpha(e_1 \cdot e_2, \alpha(e_3), \alpha(e_4)) + J_\alpha(e_1, e_3, e_4) \cdot \alpha^2(e_2) + \alpha^2(e_1) \cdot J_\alpha(e_2, e_3, e_4)$$

which identically holds indeed in $(A, \cdot, \alpha)$ *since* $J_\alpha(e_1 \cdot e_2, \alpha(e_3), \alpha(e_4)) = 0$, $J_\alpha(e_1, e_3, e_4) = 0$, $J_\alpha(e_2, e_3, e_4) = 0$, $G(e_1, e_2, e_3, e_4) = 0$ *(see above). Therefore, by (7), we conclude that the identity (2) works for* $(A, \cdot, \alpha)$ *defined above.*

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