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AN ALTERNATIVE CONSTRUCTION TO THE TRANSITIVE CLOSURE OF A DIRECTED GRAPH

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Abstract. One must add arrows to form the transitive closure of a directed graph. In our construction of a transitive directed graph we add vertices instead of arrows and preserve the transitive relationships formed by distinct vertices in the original directed graph. This has applications in algebra.

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1. Introduction

This work is part of a larger investigation of incidence rings, which are rings of functions defined on sets with relations. A good reference for this subject is [7].

We start by fixing notation and defining compression maps in Section 2. If an incidence set is constructed using a finite set of relations, then it is naturally isomorphic to a blocked matrix ring. In this case the relations set may be replaced by its directed graph and a compression map yields an injective ring homomorphism between the blocked matrix rings. This is treated as an application of compression maps in Section 3.

The class of generalized incidence rings over balanced relations was introduced by G. Abrams in [1]. In Section 4 we give the analogous definition for directed graphs and define stable directed graphs, which form a class between balanced and preordered directed graphs.

Section 5 contains our main result, Theorem 5.2, which provides a necessary and sufficient condition for a reflexive directed graph to be the compression of a preordered directed graph. The proof of Theorem 5.2 takes up all of Section 6. A direct application of Theorem 5.2 is given in [5].

This work is dedicated to the author's dearly departed friend, Martin Erickson, who suggested improvements to the writing in earlier drafts.

2. Compression maps

The directed graphs we consider have a finite number of vertices and no repeated arrows. Loops are allowed (a loop is an arrow from a vertex to itself). The vertex set and the arrow set of a directed graph D are denoted by $V(D)$ and $A(D)$, respectively. An arrow from vertex v to vertex w is denoted by vw. The notation D^* is reserved for the subgraph of D with vertex set $V(D^*) = V(D)$ and arrow set $A(D^*) = \{xy \in A(D) : x, y \in V(D) \text{ and } x \neq y\}$.

We say a directed graph D is *reflexive* if $vv \in A(D)$ for all $v \in V(D)$ and *transitive* if $xy, yz \in A(D)$ implies $xz \in A(D)$ for all $x, y, z \in V(D)$. If D is reflexive and transitive then we say D is *preordered*. A *transitive triple in* D is an ordered triple of vertices contained in $\text{Trans}(D) = \{(a, b, c) : a, b, c \in V(D) \text{ and } ab, bc, ac \in A(D)\}$.

Definition 2.1. Suppose D_1 and D_2 are reflexive directed graphs. A *compression map* is a surjective function $\theta : V(D_2) \to V(D_1)$ which satisfies 1, 2, and 3 below.

(1) $\theta(x)\theta(y) \in A(D_1)$ for all $x, y \in D_2$ such that $xy \in A(D_2)$.
(2) For all $(a_1, a_2, a_3) \in \text{Trans}(D_1)$ there exists $(x_1, x_2, x_3) \in \text{Trans}(D_2)$ such that $\theta(x_i) = a_i$ for $i = 1, 2, 3$.
(3) There is a bijection $\theta^* : A(D_2^*) \to A(D_1^*)$ given by $\theta^*(xy) = \theta(x)\theta(y)$ for all $x, y \in V(D_2)$ with $xy \in A(D_2^*)$.

A figure showing a reflexive directed graph D will only display D^* and will not show the loops. Thus we assume the directed graphs in Figure 1 are both reflexive. In Example 2.2 we show directed graph (b) is a compression of directed graph (a).

FIGURE 1. (a) is transitive and (b) is not transitive.

Example 2.2. *Let D_1 and D_2 be the reflexive directed graphs with $V(D_1) = \{x, y, z\}$, $A(D_1^*) = \{xy, yz\}$, $V(D_2) = \{x, y, z, t\}$, and $A(D_2^*) = \{xy, tz\}$ where x, y, z, t are distinct. In Figure 1 we can match up D_1 with (a) and D_2 with (b). A compression map $\theta : V(D_2) \to V(D_1)$ is given by $\theta(x) = x$, $\theta(y) = y$, $\theta(z) = z$, and $\theta(t) = y$. The effect on the directed graphs is to map the two middle vertices of (a) to the middle vertex of (b).*

Lemma 2.3. *Let D_1 and D_2 be reflexive directed graphs and let $\theta : V(D_2) \to V(D_1)$ be a compression. Suppose $xy, yz \in A(D_2)$ and $(\theta(x), \theta(y), \theta(z)) \in \text{Trans}(D_1)$ for some $x, y, z \in V(D_2)$. Then $(x, y, z) \in \text{Trans}(D_2)$.*

Proof. Choose arbitrary $x, y, z \in V(D_2)$ such that $xy, yz \in A(D_2)$. If x, y, z are not distinct then $xz \in A(D_2)$ follows immediately. If x, y, z are distinct then set $\theta(x) = a$, $\theta(y) = b$, $\theta(z) = c$. Assume $(a, b, c) = (\theta(x), \theta(y), \theta(z)) \in \text{Trans}(D_1)$. Then a, b, c are distinct since $ab, bc, ac \in A(D_1^*)$ by part 3 of Definition 2.1. By part 2 of Definition 2.1 there exist $x', y', z' \in V(D_2)$ such that $(x', y', z') \in \text{Trans}(D)$, $\theta(x') = a$, $\theta(y') = b$, and $\theta(z') = c$. Moreover x', y', z' are distinct since a, b, c are distinct. Then $x'y', y'z' \in A(D_2^*)$ so $\theta^*(xy) = ab = \theta^*(x'y')$ and $\theta^*(yz) = bc = \theta^*(y'z')$. Therefore $x' = x$, $y' = y$, and $z' = z$ since θ^* is bijective by part 3 of Definition 2.1. We assumed $(x', y', z') \in \text{Trans}(D)$ and proved $x' = x$ and $z' = z$ so $xz \in A(D_2)$. Therefore $(x, y, z) \in \text{Trans}(D_2)$ as desired. ∎

Lemma 2.3 shows that if a reflexive directed graph has a preordered compression then it must be a preordered directed graph. Example 2.2 shows a preordered directed graph may have a compression which is not preordered. Figure 2 shows directed graphs that are not compressions of preordered directed graphs.

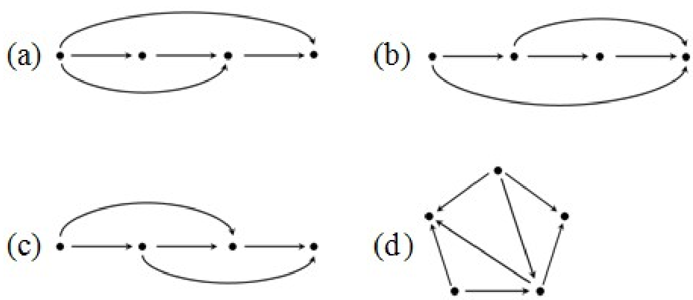

FIGURE 2. Four directed graphs that are not compressions of transitive directed graphs.

3. An application to algebra

In this section we follow conventions set in [6]. Let $M_n(R)$ denote the set of square matrices over R, an arbitrarily chosen ring with unit. If $ab \in A(D)$ then we let E_{ab} denote the standard unit matrix, that is, E_{ab} is the standard $n \times n$-matrix unit whose ab-entry is 1 and all of its other entries are 0. A matrix is *blocked* by D if it is a linear combination of standard matrix units which are indexed by arrows of D. The subset of all blocked matrices in $M_n(R)$ is a free R-bimodule over R and we denote it by $L(D, R)$.

Consider the product of two blocked standard unit matrices B and C. We have $BC = 0$ unless $B = E_{ij}$ and $C = E_{jk}$ for some $i, j, k \in V(D)$ such that

$ij, jk \in A(D)$. Since $E_{ij}E_{jk} = E_{ik}$ we need $ik \in A(D)$. Thus $L(D,R)$ is closed under multiplication if D is transitive. Moreover, $L(D,R)$ is an associative ring with unity if and only if D is preordered. In this case we call $L(D,R)$ a *blocked matrix ring.*

There are many examples of non-isomorphic rings R and S such that their matrix rings $M_n(R)$ and $M_n(S)$ are isomorphic (see book [4] by T. Y. Lam). In particular, a ring R cannot be recovered from its matrix ring $M_n(R)$, if $n > 1$. By contrast, the directed graph can often be recovered from the blocked matrices. If R is a Noetherian semiprime or commutative ring with 1, D_1 and D_2 are preordered directed graphs, and $L(D_1,R)$ and $L(D_2,R)$ are isomorphic as rings then D_1 and D_2 must be isomorphic (see [2, Theorem 2.4]). Theorem 3.1, provides an injective ring homomorphism between blocked matrix rings over the same base ring such that the underlying directed graphs are not isomorphic.

Theorem 3.1. *Suppose R is a ring with unity, C and D are preordered directed graphs, and $\theta : V(D) \to V(C)$ is a compression. There is an injective ring homomorphism $h : L(C,R) \to L(D,R)$ determined by (1) and (2) below.*

(1) *If $a \in V(C)$ then $h(E_{aa}) = \sum_{x \in \theta^{-1}(a)} E_{aa}$.*
(2) *If $\alpha \in A(C^*)$ then $h(E_\alpha) = E_{(\theta^*)^{-1}(\alpha)}$.*

Proof. Blocked standard unit matrices form a basis for $L(C,R)$ so h is automatically an R-linear map. It is easy to see $\ker h = \{0\}$, so h is injective. We only check the multiplication between standard unit matrices is preserved, that is, we prove $h(E_\alpha E_\beta) = h(E_\alpha)h(E_\beta)$ for all $\alpha, \beta \in A(C)$. This is handled by considering cases. We check the case when α and β are both not loops and leave the remaining cases to the reader.

Since α, β are both not loops we may write $\alpha = ab$ and $\beta = cd$ for some $a, b, c, d \in V(C)$ such that $a \neq b$ and $c \neq d$.

$$E_\alpha E_\beta = \begin{cases} E_{ad} & \text{if } b = c \\ 0 & \text{otherwise} \end{cases}$$

We may also write $(\theta^*)^{-1}(\alpha) = wx$ and $(\theta^*)^{-1}(\beta) = yz$ for some $w, x, y, z \in V(D)$ such that $wx, yz \in A(D^*)$. We have $h(E_\alpha)h(E_\beta) = E_{(\theta^*)^{-1}(\alpha)}E_{(\theta^*)^{-1}(\beta)}$, so the following holds.

$$h(E_\alpha)h(E_\beta) = \begin{cases} E_{wz} & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

If $h(E_\alpha)h(E_\beta) \neq 0$ then $x = y$, $\theta(x) = \theta(y)$, $b = c$, $h(E_\alpha E_\beta) = h(E_{ad})$, and $h(E_\alpha)h(E_\beta) = E_{wz}$. This gives $h(E_{ad}) = E_{wz}$ since $\theta(w) = a$ and $\theta(z) = d$; the equality $h(E_\alpha)h(E_\beta) = h(E_\alpha E_\beta)$ follows immediately.

If $h(E_\alpha)h(E_\beta) = 0$ then $x \neq y$. Suppose $x \neq y$ and $b = c$. Then $(a, b, d) \in \mathrm{Trans}(C)$ so there exists $(x_1, x_2, x_3) \in \mathrm{Trans}(D)$ such that $\theta(x_1) = a$, $\theta(x_2) = b$, and $\theta(x_3) = d$. We have $\theta^*(x_1x_2) = \alpha$, $\theta^*(x_2x_3) = \beta$, $\theta^*(wx) = \alpha$, $\theta^*(yz) = \beta$, and θ^* is bijective. This implies $x_1 = w$, $x_2 = x$, $x_2 = y$, and $x_3 = z$, which leads to a contradiction since $x \neq y$. Thus $h(E_\alpha)h(E_\beta) = 0$ implies $b \neq c$ and $h(E_\alpha E_\beta) = 0$. ∎

Example 3.2. *If C and D are the preordered directed graphs with C^* and D^* shown in Figure 3 then C is a compression of D. The compression map $\theta : V(D) \to V(C)$ is given by $\theta(6) = 5$ and $\theta(i) = i$ for all $i \in \{1, 2, 3, 4, 5\}$.*

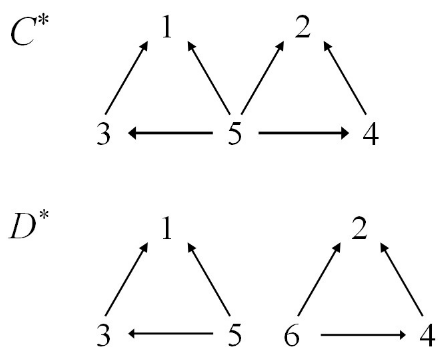

FIGURE 3. Directed graph C is a compression of D.

There is an injective ring homomorphism $h : L(C, R) \to L(D, R)$ described in Theorem 3.1.

$$h\left(\begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 & 0 \\ a_{31} & 0 & a_{33} & 0 & 0 \\ 0 & a_{42} & 0 & a_{44} & 0 \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix}\right) = \begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 & 0 & 0 \\ a_{31} & 0 & a_{33} & 0 & 0 & 0 \\ 0 & a_{42} & 0 & a_{44} & 0 & 0 \\ a_{51} & 0 & a_{53} & 0 & a_{55} & 0 \\ 0 & a_{52} & 0 & a_{54} & 0 & a_{55} \end{bmatrix}$$

We describe the construction of the homomorphism from the compression. The most noticeable change involves vertices 5 and 6. The compression maps both vertices 5 and 6 of D to vertex 5 of C. Thus the image of the standard matrix E_{55} under h is $E_{55} + E_{66}$ since $5, 6 \in \theta^{-1}(5)$. The compression does not relabel vertices 1, 2, *3, or 4. The arrows between any pairs of these four vertices, as well as arrows 51, 53, and 55, are also mapped to themselves. Thus the matrix entries in the positions corresponding to these arrows must be preserved. However, arrows 62, 64, and 66 are mapped to 52, 54, and 55, respectively, so the homomorphism maps the matrix entries in positions 52, 54, and 55 to positions 62, 64, and 66, respectively.*

4. Balanced and stable directed raphs

Definition 4.1. Suppose D is a reflexive directed graph.

(1) D is *balanced* if for all $w, x, y, z \in V(D)$ such that $wx, xy, yz, wz \in A(D)$ there is an arrow from w to y if and only if there is an arrow from x to z.
(2) D is *stable* if D is balanced and $ad \in A(D)$ for all distinct $a, b, c, d \in V(D)$ such that $ab, ac, bc, bd, cd \in A(D)$.

A reflexive directed graph is balanced if and only if does not contain an induced subgraph isomorphic to either (a) or (b) in Figure 2 or either of the directed graphs in Figure 4.

FIGURE 4. Two directed graphs which are not balanced.

A balanced directed graph is not stable if and only if it does not contain an induced subgraph isomorphic to (c) in Figure 2. Directed graph (d) in Figure 2 is stable but not preordered.

Theorem 4.2. *Suppose $\theta : V(D_2) \to V(D_1)$ is a compression map and D_1 and D_2 are reflexive directed graphs.*

(1) *If D_1 is balanced then D_2 is balanced.*
(2) *If D_1 is stable then D_2 is stable.*
(3) *If D_1 is preordered then D_2 is preordered.*

Proof. (1) Suppose D_1 is balanced and $wx, xy, yz, wz \in A(D_2)$ for some $w, x, y, z \in V(D_2)$. Set $a = \theta(w)$, $b = \theta(x)$, $c = \theta(y)$, and $d = \theta(z)$. Then $ab, bc, cd, ad \in A(D_1)$ by property 1 of Definition 2.1.

If $xz \in A(D_2)$ then property 1 of Definition 2.1 gives $bd \in A(D_1)$ so $ac \in A(D_1)$ since D_1 is balanced. Then $(a, b, c) \in \mathrm{Trans}(D_1)$ so $(w, x, y) \in \mathrm{Trans}(D_2)$ by Lemma 2.3. This gives $wy \in A(D_2)$.

For the other direction assume $wy \in A(D_2)$. Then property 1 of Definition 2.1 gives $ac \in A(D_1)$ so $bd \in A(D_1)$ since D_1 is balanced. Then $(b, c, d) \in \text{Trans}(D_1)$ so $(x, y, z) \in \text{Trans}(D_2)$ by Lemma 2.3. This gives $xz \in A(D_2)$.

(2) Suppose D_1 is stable and $wx, wy, xy, xz, yz \in A(D_2)$ for some distinct vertices $w, x, y, z \in V(D_2)$. Set $a = \theta(w)$, $b = \theta(x)$, $c = \theta(y)$, and $d = \theta(z)$. Then $ab, ac, bc, bd, cd \in A(D_1^*)$ by property 3 of Definition 2.1. In particular a, b, c, d are distinct and $ad \in A(D_1^*)$ since D_1 is stable. Then $(a, c, d) \in \text{Trans}(D_1)$ so $(w, y, z) \in \text{Trans}(D_2)$ by Lemma 2.3. This proves $wz \in A(D_2^*)$ as desired. Therefore D_2 is stable if D_1 is stable.

(3) Part 3 follows immediately from Lemma 2.3. ∎

The converse does not hold for every part of Theorem 4.2. Figure 4 shows two directed graphs which are not balanced. However, they are both compressions of preordered directed graphs. The compressions are defined in a similar fashion as Example 2.2 by constructing a directed graph which splits the middle vertex in two.

5. Clasps and soloists

The next definition helps us identify vertices where the transitive relation fails.

Definition 5.1. Suppose D is a reflexive directed graph and $x \in V(D)$.

1. We say x is a *clasp* if there exist $w, y \in V(D) \setminus \{x\}$ such that $wx, xy \in A(D)$ and there is no arrow from w to y.
2. We say x is a *locked clasp* if there exist $u, v, w, y \in V(D) \setminus \{x\}$ such that $(u, x, y), (u, x, v), (w, x, v) \in \text{Trans}(D)$ and there is no arrow from w to y.
3. An *unlocked clasp* is a clasp which is not locked.

Directed graph (d) in Figure 2 contains a locked clasp determined by the vertex in the lower right corner. We are now able to state our main Theorem.

Theorem 5.2. *Let D be a stable directed graph. Then D is the compression of a preordered directed graph if and only if every clasp in D is unlocked.*

Remark 5.3. *Suppose D is a reflexive directed graph. Theorem 5.2 shows D is the compression of a preordered directed graph if D does not contain an induced subgraph isomorphic to one of the directed graphs in Figure 2 or in Figure 4. This is reminiscent of Kuratowki's characterization of planar graphs (see* [3]*). Theorem 5.2 is not a complete classification since both directed graphs in Figure 4 are compressions of preordered directed graphs. But both directed graphs in Figure 4 contain directed cycles so we can give a complete classification if D^* is acyclic.*

Corollary 5.4. *Suppose D is a reflexive directed graph such that D^* is acyclic. Then D is the compression of a preordered directed graph if and only if D does not contain an induced subgraph isomorphic to one of the directed graphs in Figure 2.*

Definition 5.5. Suppose D is a reflexive directed graph. If $r, s \in V(D)$ satisfy $rs, sr \in A(D)$ then r and s are said to be *paired* in $V(D)$. We say an element $s \in V(D)$ is a *soloist* in D if r and s are not paired for all $r \in V(D) \setminus \{s\}$.

Lemma 5.6. *Suppose D is a reflexive directed graph.*

(1) *If $x \in V(D)$ is a clasp then x is a soloist.*

(2) *Suppose D_2 is a stable directed graph and $\theta : V(D_2) \to V(D)$ is a compression map. There is a locked clasp in D_2 if and only if there is a locked clasp in D.*

(3) *If D contains a locked clasp then D is not the compression of a preordered directed graph.*

Proof. (1) Since x is a clasp there exist $w, y \in V(D)$ such that $w, y \in V(D) \setminus \{x\}$, $wx, xy \in A(D)$, and there is no arrow from w to y. Suppose there exists $z \in V(D) \setminus \{x\}$ such that x is paired with z. Applying the balance property to w, x, z, x and to x, z, x, y yields $wz, zy \in A(D)$. This gives $w \neq y$, $y \neq z$, and $w \neq z$ since there is no arrow from w to y. Applying the stable property to w, x, z, y gives $wy \in A(D)$, which is a contradiction.

(2) If $x \in V(D_2)$ is a locked clasp then there exist $u, v, w, y \in V(D_2) \setminus \{x\}$ such that $(u, x, y), (u, x, v), (w, x, v) \in \operatorname{Trans}(D_2)$ and there is no arrow from w to y in D_2. Properties (1) and (3) of Definition 2.1 give $\theta(u), \theta(v), \theta(w), \theta(y) \in V(D) \setminus \{\theta(x)\}$ such that $(\theta(u), \theta(x), \theta(y)), (\theta(u), \theta(x), \theta(v)), (\theta(w), \theta(x), \theta(v)) \in \operatorname{Trans}(D)$, and there is no arrow from $\theta(w)$ to $\theta(y)$ in D. Therefore $\theta(x)$ is a locked clasp in $V(D)$.

If $x_1 \in V(D)$ is a locked clasp then there exist $u_1, v_1, w_1, y_1 \in V(D) \setminus \{x_1\}$ such that $(u_1, x_1, y_1), (u_1, x_1, v_1), (w_1, x_1, v_1) \in \operatorname{Trans}(D)$ and there is no arrow from w_1 to y_1 in D. By part 2 of Definition 2.1 there exist $a, b, c, d, e, f, u_2, x_2, y_2 \in V(D_2)$ such that $(a, b, c), (d, e, f), (u_2, x_2, y_2) \in \operatorname{Trans}(D_2)$, $\theta(a) = u_1$, $\theta(b) = x_1$, $\theta(c) = v_1$, $\theta(d) = w_1$, $\theta(e) = x_1$, $\theta(f) = v_1$, $\theta(u_2) = u_1$, $\theta(x_2) = x_1$, and $\theta(y_2) = y_1$. We have $\theta^*(u_2x_2) = \theta^*(ab)$ and $\theta^*(bc) = \theta^*(ef)$ so $u_2 = a$, $x_2 = b$, $e = b$, and $c = f$ by part 3 of Definition 2.1. Setting $w_2 = a$ and $v_2 = c$ gives $u_2, v_2, w_2, y_2 \in V(D) \setminus \{x_2\}$ such that $(u_2, x_2, y_2), (u_2, x_2, v_2), (w_2, x_2, v_2) \in \operatorname{Trans}(D)$, u_2x_2,$x_2y_2 \in A(D_2)$, and there is no arrow from u_2 to y_2 in D_2. Therefore x_2 is a locked clasp in D_2.

(3) Part 3 follows immediately from part 2. ∎

Lemma 5.7. *Suppose D is a stable directed graph and $s \in V(D)$ is a soloist.*

(1) *Suppose $(a, b, s) \in \text{Trans}(D)$ for some $a, b \in V(D) \setminus \{s\}$ with $a \neq b$.*
   (a) *If $sc \in A(D)$ for some vertex c then $ac \in A(D)$ if and only if $bc \in A(D)$.*
   (b) *If $xa \in A(D)$ for some vertex x then $xs \in A(D)$ if and only if $xb \in A(D)$.*

(2) *Suppose $(a, s, c) \in \text{Trans}(D)$ for some $a, c \in V(D) \setminus \{s\}$.*
   (a) *If $xa \in A(D)$ for some vertex x then $xc \in A(D)$ if and only if $xs \in A(D)$.*
   (b) *If $cd \in A(D)$ for some vertex d then $ad \in A(D)$ if and only if $sd \in A(D)$.*

(3) *Suppose $(s, b, c) \in \text{Trans}(V(D))$ for some $b, c \in V(D) \setminus \{s\}$ with $b \neq c$.*
   (a) *If $cd \in A(D)$ for some vertex d then $bd \in A(D)$ if and only if $sd \in A(D)$.*
   (b) *If $as \in A(D)$ for some vertex a then $ab \in A(D)$ if and only if $ac \in A(D)$.*

Proof. The proofs of parts 1, 2, and 3 are similar. Note that in part 2 we have $a \neq c$ since s is a soloist. We prove part 1. Assume $(a, b, s) \in \text{Trans}(D)$ for some $a, b \in V(D) \setminus \{s\}$ with $a \neq b$.

(a) Suppose $sc \in A(D)$ for some $c \in V(D) \setminus \{s\}$. If $ac \in A(D)$ then applying the balance property to a, b, s, c gives $bc \in A(D)$. On the other hand if $bc \in A(D)$ then $a \neq c$ and $b \neq c$ since s is a soloist. Applying the stable property to a, b, s, c gives $ac \in A(D)$.

(b) Suppose $xa \in A(D)$ for some $x \in V(D) \setminus \{a, b\}$. If $xs \in A(D)$ then applying the balance property to x, a, b, s gives $xb \in A(D)$. On the other hand if $xb \in A(D)$ then $x \neq s$ since s is a soloist. Applying the stable property to x, a, b, s gives $xs \in A(D)$. ∎

The proof of Theorem 5.2 is a constructive algorithm described in Section 6. In each iteration of the algorithm we construct a preordered directed graph with one more vertex and define a compression. The algorithm stops when we arrive at a preordered directed graph and the desired compression is obtained by composition.

We finish this section with an example which covers the steps and constructions given in the proof of Theorem 5.2. The directed graphs in Figure 5 are stable. We may identify (i) as a compression of (ii) by mapping 2 and t_1 to 2. We may also identify (ii) as a compression of (iii) by mapping 4 and t_2 to 4.

Example 5.8. *Let D be reflexive directed graph (i) shown in Figure 5. The clasps are 2 and 4 and we set $x_1 = 2$.*

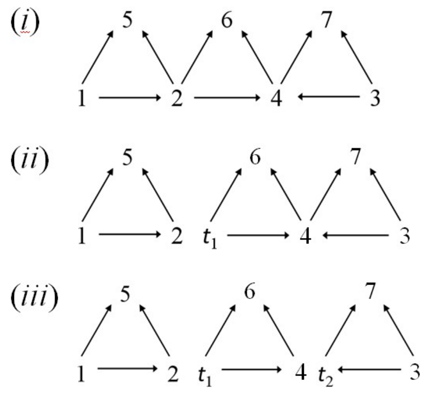

FIGURE 5. A construction using the proof of Theorem 5.2.

- **Step 1:** *$Y_1 = \{4, 6\}$ and A_1 is empty.*
- **Step 2:** *Use construction A since A_1 is empty. Let D_2 be the reflexive directed graph with $V(D_2) = V(D_1) \cup \{t_1\}$ and $A(D_2^*) = \sigma_1 \cup \tau_1$ where $B_1 = \{4, 6\}$, $\sigma_1 = A(D_1^*) \setminus \{24, 26\}$, and $\tau_1 = \{t_1 4, t_1 6\}$. This gives (ii) in Figure 5.*
- **Step 3:** *Define $\theta_1 : V(D_2) \to V(D_1)$ by $\theta_1(t_1) = 2$ and $\theta_1(u) = u$ for all $u \in V(D_1)$.*
- **Step 4:** *We go back to step 1 with $x_2 = 4$ since 4 is the only clasp in D_2.*
- **Step 1:** *We have $Y_2 = \{6, 7\}$ and $A_2 = \{t_1, 3\}$.*
- **Step 2:** *Use construction B with $a_2 = 3$, $b_2 = t_1$, and $y_2 = 6$. Let D_3 be the reflexive directed graph with $V(D_3) = V(D_2) \cup \{t_2\}$ and $A(D_3^*) = \sigma_2 \cup \tau_2$ where $\sigma_2 = A(D_2^*) \setminus \{34, 47\}$, $\tau_2 = \{3t_2, t_2 7\}$. This gives (iii) in Figure 5.*
- **Step 3:** *Define $\theta_2 : V(D_3) \to V(D_2)$ by $\theta_2(t_2) = 4$ and $\theta_2(u) = u$ for all $u \in V(D_2)$.*
- **Step 4:** *(X_3, ρ_3) is preordered so the algorithm stops. The compression map is $\theta_1 \circ \theta_2$.*

6. Proof of Theorem 5.2

If D is the compression of a preordered directed graph then every clasp in D is unlocked by part 3 of Lemma 5.6. We assume D is stable and every clasp in D is unlocked and prove D is the compression of a preorder. We set $D_1 = D$, and describe an algorithm to construct stable directed graphs $D_1, \ldots, D_m$ such that D_i

is a compression of D_{i+1} for each $i < m$. In the last iteration D_m is preordered and the desired compression map is obtained by composition.

Assume D_1 is not preordered and fix a clasp $x_1 \in V(D_1)$. In the first iteration of our algorithm we have $i = 1$.

Step 1. The sets Y_i and A_i are defined below.

- $Y_i = \{y \in V(D_i) \setminus \{x_i\} : wx_i, x_iy \in A(D) \text{ and } wy \notin A(D_i) \text{ for some } w \in V(D_i)\}$
- $A_i = \{a \in V(D_i) \setminus \{x_i\} : (a, x_i, y) \in \text{Trans}(D_i) \text{ for some } y \in Y_i\}$.

Note that Y_i is nonempty since x_i is a clasp.

Step 2. We fix $t_i \notin V(D_i)$ and construct a reflexive directed graph D_{i+1} such that $V(D_{i+1}) = V(D_i) \cup \{t_i\}$, and $A\left(D^*_{i+1}\right) = \sigma_i \cup \tau_i$ where σ_i and τ_i are defined using construction A or construction B. In both constructions $\tau_i = A\left(D^*_{i+1}\right) \setminus \sigma_i$ and $|\tau_i| = |A(D^*_i) \setminus \sigma_i|$ so $|A(D^*_i)| = \left|A\left(D^*_{i+1}\right)\right|$. The arrows in τ_i will all contain t_i. If an arrow does not contain x_i then it will be in σ_i. Moreover σ_i consists of the arrows belonging to both D_{i+1} and D_i. Depending on the construction, an arrow may be contained in σ_i even if it contains x_i.

Use construction B if there exist $a_i, b_i \in A_i$ such that $(b_i, x_i, y_i) \in \text{Trans}(D_i)$ and $a_iy_i \notin A(D_i)$ for some $y_i \in Y_i$. Otherwise use construction A.

Construction A.

- $B_i = \{b \in V(D_i) \setminus \{x_i\} : (x_i, b, y) \in \text{Trans}(D_i) \text{ or } (x_i, y, b) \in \text{Trans}(D_i) \text{ for some } y \in Y_i\}$
- $\sigma_i = A(D^*_i) \setminus (\{ax_i : a \in A_i \setminus \{x_i\}\} \cup \{x_ib : b \in B_i\})$
- $\tau_i = \{at_i : a \in A_i\} \cup \{t_ib : b \in B_i\}$

If $y \in Y_i$ then $y \in B_i$ since $(x_i, y, y) \in \text{Trans}(D_i)$. Therefore $Y_i \subseteq B_i$.

Construction B.

- $T_i = \{cz : c \in V(D_i), z \in V(D_i) \setminus \{x_i\}, (c, x_i, z) \in \text{Trans}(V(D_i)), \text{ and } cy_i \notin A(D_i)\}$
- $\sigma_i = A(D^*_i) \setminus \{cx_i, x_iz : c, z \in V(D_i) \text{ and } cz \in T_i\}$
- $\tau_i = \{ct_i, t_iz : c, z \in V(D_i) \text{ and } cz \in T_i\}$

There exists $z \in Y_i$ such that $(a_i, x_i, z) \in \text{Trans}(D_i)$ since $a_i \in A_i$. Moreover $a_iy_i \notin A(D_i)$, $z \in x_i$, and $x_i \notin Y_i$ so $a_iz \in T_i$. Therefore T_i is nonempty.

Step 3. Define $\theta_i : V(D_{i+1}) \to V(D_i)$ so that $\theta_i(t_i) = x_i$ and $\theta_i(u) = u$ for all $u \in V(D_i)$.

Before moving on we prove θ_i is a compression. Routine calculations show part (1) of Definition 2.1 hold and there is a well-defined map $\theta^*_i : A\left(D^*_{i+1}\right) \to A(D^*_i)$ given by $\theta^*_i(uv) = \theta_i(u)\,\theta_i(v)$ for all $u, v \in V(D_i)$ such that $uv \in A\left(D^*_{i+1}\right)$. It is easy to see $\theta^*_i(\sigma_i) \cup \theta^*_i(\tau_i) = A(D^*_i)$ hence θ^*_i is surjective. We have already shown $\left|A\left(D^*_{i+1}\right)\right| = |A(D^*_i)|$ so θ^*_i is a bijection.

The only condition left is part 2 of Definition 2.1. Suppose $d_1, d_2, d_3 \in V(D_i)$ and $(d_1, d_2, d_3) \in \text{Trans}(D_i)$. We must show (d_1, d_2, d_3) is the image of a transitive triple in D_{i+1}. This is easy if d_1, d_2, d_3 are not distinct since every arrow in D_i is the image of an arrow in D_{i+1}. Assume d_1, d_2, d_3 are distinct.

We check every possible case and make repeated use of the fact $uv \in \sigma_i$ if and only if $uv \in A(D_i^*)$ for all $u, v \in V(D_i) \setminus \{x_i\}$. In cases 2, 3, and 4 we have $x_i = d_r$ for some $r \in \{1, 2, 3\}$. We will show either $(d_1, d_2, d_3) \in \text{Trans}(D_i)$ so that $\theta_i(d_j) = d_j$ for $j = 1, 2, 3$ or the desired transitive triple is obtained by replacing d_r with t_i so that $\theta_i(d_j) = d_j$ for $j \neq r$, and $\theta_i(t_i) = d_r$.

We split cases between construction B and construction A when necessary. Note that x_i is a soloist by part 1 of Lemma 5.6.

Case 1 $d_1, d_2, d_3 \in V(D_i) \setminus \{x_i\}$

We have $d_1d_2, d_2d_3, d_1d_3 \in \sigma_i$ thus $(d_1, d_2, d_3) \in \text{Trans}(D_i)$.

Case 2 If $d_1, d_2 \in V(D_i) \setminus \{x_i\}$ and $d_3 = x_i$ then $d_1d_2 \in \sigma_i$ since $d_1d_2 \in A(D_i^*)$.

Check case 2 for construction A. If $x_iy \in A(D_i)$ then $d_1y \in A(D_i)$ if and only if $d_2y \in A(D_i)$ by part 1(a) of Lemma 5.7. This gives $d_1 \in A_i$ if and only if $d_2 \in A_i$ so $d_1t_i \in \tau_i$ if and only if $d_2t_i \in \tau_i$ and either $(d_1, d_2, t_i) \in \text{Trans}(D_i)$ or $(d_1, d_2, x_i) \in \text{Trans}(D_i)$.

Check case 2 for construction B. We have $d_1z \in A(D_i)$ if and only if $d_2z \in A(D_i)$ for all $z \in V(D_i) \setminus \{x_i\}$ such that $x_iz \in A(D_i)$ by part 1(a) of Lemma 5.7. This gives $d_1z \in T_i$ if and only if $d_2z \in T_i$ for all $z \in V(D_i) \setminus \{x_i\}$. Therefore $d_1t_i \in \tau_i$ if and only if $d_2t_i \in \tau_i$ and either $(d_1, d_2, t_i) \in \text{Trans}(D_i)$ or $(d_1, d_2, x_i) \in \text{Trans}(D_i)$.

Case 3 If $d_1, d_3 \in V(D_i) \setminus \{x_i\}$ and $d_2 = x_i$ then $d_1d_3 \in \sigma_i$ since $d_1d_3 \in A(D_i^*)$.

Check case 3 for construction A. If $d_3 \in B_i$ then there exists $y \in Y_i$ such that $(x_i, d_3, y) \in \text{Trans}(D_i)$ or $(x_i, y, d_3) \in \text{Trans}(D_i)$. This gives $d_1y \in A(D_i)$ by applying either part 2(b) or part 3(b) of Lemma 5.7. Therefore $d_1 \in A_i$.

On the other hand if $d_1 \in A_i$ then $(d_1, x_i, z) \in \text{Trans}(D_i)$ for some $z \in Y_i$. Since $z \in Y_i$ there exists $w \in V(D_i)$ such that $wx_i \in A(D_i)$ and $wz \notin A(D_i)$. If $wd_3 \in A(D_i)$ then $(d_1, x_i, z), (d_1, x_i, d_3), (w, x_i, d_3) \in \text{Trans}(D_i)$ and x_i is a locked clasp, which is a contradiction. We are left with $wd_3 \notin A(D_i)$, $d_3 \in Y_i$, and $d_3 \in B_i$.

We have shown $d_1 \in A_i$ if and only if $d_2 \in B_i$ so $d_1t_i \in \tau_i$ if and only if $t_id_3 \in \tau_i$ and either $(d_1, x_i, d_3) \in \text{Trans}(D_i)$ or $(d_1, t_i, d_3) \in \text{Trans}(D_i)$.

Check case 3 for construction B. If $d_1y_i \notin A(D_i)$ then $d_1d_3 \in T_i$ and $(d_1, t_i, d_3) \in \text{Trans}(D_i)$.

If $d_1y_i \in A(D_i)$ then $d_1x_i \in \sigma_i$ and we must show $x_id_3 \in \sigma_i$. Note that $(d_1,x_i,d_3),(d_1,x_i,y_i),(b_i,x_i,y_i) \in \text{Trans}(D_i)$ so $b_id_3 \in A(D_i)$ since x_i is an unlocked clasp. If $x_id_3 \notin \sigma_i$ then $cd_3 \in T_i$ for some $c \in V(D_i)$ such that $cy_i \notin A(D_i)$. This gives $(b_i,x_i,y_i),(b_i,x_i,d_3),(c,x_i,d_3) \in \text{Trans}(D_i)$ with $cy_i \notin A(D_i)$ and x_i is a locked clasp. This is a contradiction. We are left with $d_1x_i, x_id_3 \in \sigma_i$ and $(d_1,x_i,d_3) \in \text{Trans}(D_i)$.

Case 4 If $d_2,d_3 \in V(D_i)\setminus\{x_i\}$, $d_1 = x_i$ then $d_2d_3 \in \sigma_i$ since $d_2d_3 \in A(D_i^*)$.

Check case 4 for construction A. Suppose $d_j \in B_i$ for $j=2$ or $j=3$ and let $k \in \{2,3\}$ be such that $k \neq j$. If $d_j \in B_i$ then $(x_i,d_j,y) \in \text{Trans}(D_i)$ or $(x_i,y,d_j) \in \text{Trans}(D_i)$ for some $y \in Y_i$. There exists $w \in V(D_i)\setminus\{x_i\}$ such that $wx_i \in A(D_i)$ and $wy \notin A(D_i)$ since $y \in Y_i$. Then $wd_2, wd_3 \notin A(D_i)$ by two applications of part 3(b) of Lemma 5.7. Thus $d_2,d_3 \in Y_i$ and $(t_i,d_2,d_3) \in \text{Trans}(D_i)$.

We have shown $d_2 \in B_i$ or $d_3 \in B_i$ imply $(t_i,d_2,d_3) \in \text{Trans}(D_i)$. On the other hand if $d_2 \notin B_i$ and $d_3 \notin B_i$ then $x_id_2, x_id_3 \in \sigma_i$ and $(x_i,d_2,d_3) \in \text{Trans}(D_i)$.

Check case 4 for construction B. We have $cd_2 \in A(D_i)$ if and only if $cd_3 \in A(D_i)$ for all $c \in V(D_i)$ such that $cx_i \in A(D_i)$ and $cy_i \notin A(D_i)$ by part 3(b) of Lemma 5.7. This gives $cd_2 \in T_i$ if and only if $cd_3 \in T_i$ for all $c \in V(D_i)\setminus\{x_i\}$. Therefore $t_id_2 \in \tau_i$ if and only if $t_id_3 \in \tau_i$ and either $(x_i,d_2,d_3) \in \text{Trans}(D_i)$ or $(t_i,d_2,d_3) \in \text{Trans}(D_i)$.

Step 4. If D_{i+1} is preordered then the algorithm stops and the compression from D_{i+1} to D is determined by composition. Otherwise fix a clasp $x_{i+1} \in V(D_i)$ and go back to step 1 with i replaced by $i+1$.

To study the algorithm we consider a given iteration i. Then $A(D_{i+1})$ is stable by Theorem 4.2 and D_{i+1} contains no unlocked clasps by part 2 of Lemma 5.6. This means we may repeat the algorithm as often as necessary. We must prove the algorithm stops eventually.

In each iteration of the algorithm we are adding a new vertex but not adding any arrows other than loops. The only way this can continue indefinitely is if our algorithm forces vertices to not form arrows with any other elements. We assume every vertex of D_i forms an arrow with some other vertex of D_i and show every vertex of D_{i+1} forms an arrow with some other vertex of D_{i+1}.

Suppose $x,y \in V(D_i)$ satisfy $x \neq y$ and $xy \in A(D_i)$. If $x \neq x_i$ and $y \neq x_i$ then $xy \in \sigma_i$ so $xy \in A(D_{i+1})$. Note that t_i forms an arrow with some other vertex of D_{i+1} by construction. We are left with proving x_i forms an arrow with some other vertex of D_{i+1}.

Assume there is not an arrow formed by x_i and any other vertex of D_{i+1} after using construction A. There exist $b,z \in V(D_i)$ such that $bx_i \in A(D_i)$, $x_iz \in$

$A(D_i)$, and $bz \notin A(D_i)$ since x_i is a clasp. Then $bt_i \in A(D_{i+1})$ and $t_i z \in A(D_{i+1})$ by assumption so $b \in A_i$ and $z \in B_i$. Since $b \in A_i$ there exists $y \in V(D_i)$ such that $(b, x_i, y) \in \mathrm{Trans}(D_i)$. There must also exist $a \in V(D_i)$ such that $ax_i \in A(D_i)$ and $ay \notin A(D_i)$ since $y \in Y_i$. This gives $at_i \in A(D_{i+1})$ by assumption so $a \in A_i$. Thus $a, b \in A_i$ satisfy the conditions in step 2 for construction B. This contradicts our assumption that we used construction A, so there is an arrow formed by x_i and another vertex of D_{i+1}.

In construction B we have $b_i y_i \in A(D_i)$ so $b_i z \notin T_i$ for all $z \in V(D_i) \setminus \{x_i\}$. This gives $b_i x_i \in \sigma_i$ and $b_i x_i \in A(D_{i+1})$ so there is an arrow formed by x_i with another vertex of D_{i+1}.

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