International Electronic Journal of Algebra
Volume 15 (2014) 13-25

MINIMAL GROUPS WITH ISOMORPHIC TABLES OF MARKS

Margarita Martínez-López, Alberto G. Raggi-Cárdenas, Luis Valero-Elizondo and Eder Vieyra-Sánchez

Received: 4 October 2012; Revised: 26 October 2013
Communicated by Arturo Magidin

Dedicated to the memory of Professor Efraim P. Armendariz

Abstract. We prove that most groups of order less than 96 cannot have isomorphic tables of marks unless the groups are isomorphic.

Mathematics Subject Classification (2010): 19A22
Keywords: Table of marks, group

1. Introduction

We constructed two nonisomorphic groups of order 96 with isomorphic tables of marks in [3]. Using the Table of Marks Library, we wrote a program in GAP ([1]) to verify that they are the smallest groups with that property (in [6] you may find the code we wrote). We are now trying to prove this explicitly, that is, that if two groups of order less than 96 have isomorphic tables of marks, then they are isomorphic groups. In Section 2 we define tables of marks and list some of their properties. In Section 3 we list the possible number of nonabelian groups for each order less than 96 (we used GAP for this computation). In the next sections, we prove our claim for all but 5 of these orders.

In Section 9, we list the remaining cases, namely; 32, 48, 64, 72 and 80.

2. Tables of marks

Let G be a finite group. Let $\mathcal{C}(\mathrm{G})$ be the family of all conjugacy classes of subgroups of G. We usually assume that the elements of $\mathcal{C}(\mathrm{G})$ are ordered non-decreasingly. The matrix whose H, K-entry is $\#(G/K)^H$ (that is, the number of fixed points of the set G/K under the action of H) is called the **table of marks** of G (where H, K run through all the elements in $\mathcal{C}(\mathrm{G})$).

The **Burnside ring** of G, denoted $B(G)$, is the subring of $\mathbb{Z}^{\mathcal{C}(G)}$ spanned by the columns of the table of marks of G.

This research is partially supported by CIC's Project *Mínimos grupos con tablas de marcas isomorfas*, CONACYT's Project *Funtores de tipo Burnside* and PAPIIT's Project IN104312 *Functores de Biconjuntos y anillos de Burnside*.

Definition 1. Let G and Q be finite groups. Let ψ be a function from $\mathcal{C}(\mathrm{G})$ to $\mathcal{C}(\mathrm{Q})$. Given a subgroup H of G, we denote by $[H]$ its conjugacy class, and by H' any representative of $\psi([H])$. We say that ψ is an *isomorphism between the tables of marks of* G *and* Q if ψ is a bijection and if $\#(Q/K')^{H'} = \#(G/K)^{H}$ for all subgroups H, K of G. We usually refer to H' as the image of H under the isomorphism of table of marks.

An isomorphism between tables of marks preserves the order of the subgroups, the order of their normalizers, the number of elements of a given order, the number of (conjugacy classes of) subgroups of a given order, the number of normal subgroups of a given order, it sends cyclic groups to cyclic groups and elementary abelian groups to elementary abelian groups. It also sends the derived subgroup of G to the derived subgroup of Q, maximal subgroups of G to maximal subgroups of Q, Sylow p-subgroups to Sylow p-subgroups (same p), and the Frattini subgroup of G to the Frattini subgroup of Q.

Assume now that G and Q are finite groups with isomorphic tables of marks. As we mentioned, G and Q must have the same order. It is also easy to check that if G is abelian or simple, then G and Q must be isomorphic groups. If G is a direct product, so is Q, and their corresponding factors have isomorphic tables of marks. If G is a semidirect product $N \rtimes H$ then Q is a semidirect product $N' \rtimes H'$ where H and H' have isomorphic tables of marks (although we cannot say much about N and N', other than they correspond under the isomorphism of tables of marks). Proofs of these claims can be found in [2].

However, an isomorphism of tables of marks may not preserve abelian subgroups, and it may not send the centre of G to the centre of Q. This can be seen in two nonisomorphic groups of order 96 which have isomorphic tables of marks (see [3]).

3. Proving their minimality

Let $A(n)$ denote the number of non-abelian groups of order n up to isomorphism. Using GAP we can list the values of n and $A(n)$ for n from 2 to 95 (we omit the cases with zeroes and ones):

n	8	12	16	18	20	24	27	28	30	32	36	n
$A(n)$	2	3	9	3	3	12	2	2	3	44	10	$A(n)$

n	40	42	44	48	50	52	54	56	60	63	64	66
$A(n)$	11	5	2	47	3	3	12	10	11	2	256	3

n	68	70	72	76	78	80	81	84	88	90	92	n
$A(n)$	3	3	44	2	5	47	10	13	9	8	2	$A(n)$

$A(n) = 0$ for the following 40 values of n: 2, 3, 4, 5, 7, 9, 11, 13, 15, 17, 19, 23, 25, 29, 31, 33, 35, 37, 41, 43, 45, 47, 49, 51, 53, 59, 61, 65, 67, 69, 71, 73, 77, 79, 83, 85, 87, 89, 91, 95.

$A(n) = 1$ for the following 20 values of n: 6, 10, 14, 21, 22, 26, 34, 38, 39, 46, 55, 57, 58, 62, 74, 75, 82, 86, 93, 94.

$A(n) = 2$ for the following 7 values of n: 8, 27, 28, 44, 63, 76, 92

$A(n) = 3$ for the following 9 values of n: 12, 18, 20, 30, 50, 52, 66, 68, 70.

$A(n) = 5$ for $n = 42$ and $n = 78$.

$A(n) = 8$ for $n = 90$.

$A(n) = 9$ for $n = 16$ and $n = 88$.

$A(n) = 10$ for $n = 36$, $n = 56$ and $n = 81$.

$A(n) = 11$ for $n = 40$ and $n = 60$.

$A(n) = 12$ for $n = 24$ and $n = 54$.

$A(n) = 13$ for $n = 84$.

$A(n) = 44$ for $n = 32$ and $n = 72$.

$A(n) = 47$ for $n = 48$ and $n = 80$.

$A(n) = 256$ for $n = 64$.

4. Some cases are easy

Theorem 2. *Let n be a prime number, or the square of a prime number, or a number of the form pq where $p > q$ are primes and q does not divide $p - 1$. Then all groups of order n are abelian.*

This accounts for all the values n such that $A(n) = 0$ except for $n = 45$, which is easy to prove directly.

Theorem 3. *Let n be a number of the form pq where $p > q$ are primes and q divides $p - 1$. Then there is exactly one isomorphism class of non-abelian groups of order n.*

This accounts for all the values n such that $A(n) = 1$ except for $n = 75$, which is easy to prove directly, since the only non-abelian group of order 75 must be the only non-trivial semidirect product $(C_5 \times C_5) \rtimes C_3$.

5. The case $A(n) = 2$

The seven possible values of n are: 8, 27, 28, 44, 63, 76, 92.

Here we must not only count all possible isomorphism classes of non-abelian groups, but we must also prove they have non-isomorphic tables of marks.

The case $n = 8$ is well-known (we can only have the quaternions and the dihedral group). Their tables of marks are not isomorphic, because all the subgroups of the quaternions are normal, which is not true of the dihedral group.

The case $n = 27$ is in the literature (see [5]). One of the groups has a cyclic subgroup of order 9, but the other group has no cyclic subgroups of that order.

The cases when n equals 28, 44, 76 and 92 are all of the form $4p$ with p a prime number larger than 4 and congruent to 3 modulo 4 (namely, 7, 11, 19 and 23). Here we have that the group must be a semidirect product, either $C_p \rtimes C_4$ or $C_p \rtimes (C_2 \times C_2)$. The Sylow 2-subgroups cannot correspond under an isomorphism of tables of marks.

The case $n = 63$ must be a semidirect product, either $C_7 \rtimes C_9$ or $C_7 \rtimes (C_3 \times C_3)$. The Sylow 3-subgroups cannot correspond under an isomorphism of tables of marks.

6. The case $A(n) = 3$

The nine possible values of n are: 12, 18, 20, 30, 50, 52, 66, 68, 70.

A group G of order 12 must have either 1 or 3 Sylow 2-subgroups (isomorphic to either C_4 or $C_2 \times C_2$) and either 1 or 3 Sylow 3-subgroups (isomorphic to C_3). It is easy to show (by hand or using GAP) that if G is not abelian, it can be written as a semidirect product $H \rtimes K$ where the order of H is either 6 (and G is isomorphic to $S_3 \times C_2$), 4 (where G is isomorphic to A_4) or 3 (where G is the only non-trivial semidirect product $C_3 \rtimes C_4$). The group $C_3 \rtimes C_4$ has a Sylow 2-subgroup isomorphic to C_4, and $S_3 \times C_2$ has a normal subgroup of order 6, so no two of these three groups have isomorphic tables of marks.

The cases when $n = 18$ and $n = 50$ are of the form $2p^2$ with $p = 3$ and $p = 5$. The group has to be either the only non-trivial semidirect product $C_{p^2} \rtimes C_2$ or one of the two non-trivial semidirect products $(C_p \times C_p) \rtimes C_2$. The first group has a cyclic Sylow p-subgroup. In the case $(C_p \times C_p) \rtimes C_2$, since $C_p \times C_p$ is a vector space over the field with p elements, its automorphisms of order 2 are easily computed: one of them has an invariant one-dimensional subspace, and the other one does not. One possible semidirect product has C_p as a direct summand, and the other one does not, so their tables of marks cannot be isomorphic.

The cases when n equals 20, 52 and 68 are all of the form $4p$ with p a prime number larger than 4 and congruent to 1 modulo 4 (namely, 5, 13 and 17). Here the possible groups are the only non-trivial semidirect product $C_p \rtimes (C_2 \times C_2)$, and the only two non-trivial semidirect products $C_p \rtimes C_4$. The Sylow 2-subgroup of the first group cannot correspond to C_4 under an isomorphism of tables of marks. In one of the semidirect products $C_p \rtimes C_4$, C_4 acts on C_p as the involution $x \mapsto x^{-1}$, so C_2 centralizes C_p, so the group has a normal subgroup of order 2; but in the other semidirect product, C_4 acts by an automorphism of C_P of order 4, so C_2 cannot centralize C_p, so this groups has no normal subgroup of order 2.

The cases when n equals 30, 66 and 70 are all of the form $2pq$ with p, q primes, $p > q > 2$ and q does not divide $p - 1$. First we observe that all groups of order less than 100 are soluble (except for A_5, which has order 60). By P. Hall's Theorem (see [4], Theorem 5.23), a soluble group G of order $2pq$ has precisely one normal subgroup of order pq, so G is a semidirect product $H \rtimes C_2$ where H is a group of order pq. Since q does not divide $p-1$, H is abelian, so G is $(C_p \times C_q) \rtimes C_2$. The automorphism group of $C_p \times C_q$ has three elements of order 2, so there are at most three non-abelian choices for the group G. Precisely one of these groups has a direct factor isomorphic to C_q, and precisely another of these groups has a direct factor isomorphic to C_p, so neither two of the three groups can have isomorphic tables of marks.

7. The case $A(n) = 5$

There are two possible values for n, namely, 42 and 78. Both are numbers of the form $2pq$ with $p > q$ primes and q divides $p-1$ (actually, $q = 3$ in both cases). Since these groups are soluble, by Hall's Theorem there is precisely one normal subgroup H of order pq. Here we have two possibilities: H could be the only non-abelian group of order pq, or H could be C_{pq} (groups from these two cases cannot have isomorphic tables of marks). If H is cyclic, there are three elements of order two in its automorphism group, so there are three possible non-trivial semidirect products $H \rtimes C_2$: one has C_p as a direct factor (but not C_q), the other has C_q as a direct factor (but not C_p), and the other has no such direct factors, so they cannot have isomorphic tables of marks.

Now assume that H is non-abelian. Note that H has a normal subgroup C_p and p subgroups C_q. An automorphism σ of H of order two must fix C_p setwise, and permute the p different C_q, so it fixes one of the C_q setwise, and acts here and on C_p either as the involution $x \mapsto x^{-1}$ or the identity. Moreover, the fixed points under σ (which form a normal subgroup of H) are trivial unless σ equals the identity map. Since H can be generated by a generator of C_p and a generator of one of the C_q's, there is only one automorphism of H of order 2, so one possible group is $H \times C_2$, and the other is the only non-trivial semidirect product $H \rtimes C_2$, and these two groups cannot have isomorphic tables of marks.

8. Orders 16, 24, 36, 40, 54, 56, 60, 81, 84, 88, 90

In this section we prove that non-isomorphic groups of the given orders cannot have isomorphic tables of marks. In order to do this, for each order we list the isomorphism classes of non-abelian groups (using GAP's StructureDescription) and find invariants which do not correspond but should be preserved by isomorphisms of tables of marks. Note that since normal subgroups and quotients are preserved under isomorphisms of tables of marks, we can eliminate all groups which are **direct products**, since these could only be isomorphic to direct products whose components would also have isomorphic tables of marks. Such groups are therefore excluded.

To differentiate some of these groups, we use the number of elements and subgroups of a given order: we write $EO(n) = m$ to say that there are m elements of order n, $CS(n) = m$ to say that there are m conjugacy classes of subgroups of order n, and $NS(n) = m$ to say that there are m normal subgroups of order n (all of these are invariants preserved under an isomorphism of tables of marks). All these numbers were later verified using GAP.

For each group, we first give the group's Small Group Library number in GAP, then its Structure Description, then one possible presentation, and then invariants which characterize its table of marks.

8.1. Order 16.

```
9 Q16
 [ b^2*a^-2, a^4, a*b^2*a, a*(b*a^-1)^2*b^-1*a^-1*b^-1 ]
EO(2)=1

4 C4 : C4
 [ b^4, a^4, a^-1*b*a*b, a^2*b^-1*a^2*b, (b^-1*a^-2*b^-1)^2 ]
EO(2)=3, EO(4)=12

6 C8 : C2
 [ b^2, a^-1*(a^-1*b)^2*a^-1, a^-2*b*a^2*b, (b*a*b*a^-1)^2 ]
EO(2)=3, EO(4)=4

8 QD16
 [ b^2, a^4, a^-1*b*a^2*b*a^-1, b*a*(b*a^-1)^3 ]
EO(2)=5

3 (C4 x C2) : C2
 [ b^2, a^4, (a*b*a)^2, (b*a^-1)^4, (b*a*b*a^-1)^2 ]
EO(2)=7, NS(2)=3

13 (C4 x C2) : C2
 [ a^2, b^2, c^4, c^-1*a*c*a, c^-1*b*c*b, b*c^2*a*b*a ]
EO(2)=7, NS(2)=1

7 D16
 [ a^2, b^2, (a*b)^8 ]
EO(2)=9
```

8.2. Order 24.

```
1 C3 : C8
 [ d^3, a^-1*d*a*d, a^8 ]
EO(2)=1, EO(4)=2

3 SL(2,3)
 [ a^3, b^4, b^-1*a*b*a*b^-1*a, a^-1*b^-1*(a^-1*b)^2, (a^-1*b^-1)^3 ]
EO(2)=1, EO(4)=6

4 C3 : Q8
 [ d^3, b^2*a^2, b*a*b*a^-1, a^4, a^-1*d*a*d, d^-1*b^-1*d*b,
  (a^-1*d)^2*a^-2 ]
EO(2)=1, EO(4)=14

8 (C6 x C2) : C2
 [ a^2, b^2, d^3, (a*d)^2, d^-1*b*d*b, (b*a)^4 ]
EO(2)=9, EO(3)=2

12 S4
 [ a^2, c^2, b^3, (a*b)^2, (b^-1*c*a)^2, a*c*b*c*b^-1*a*c ]
EO(2)=9, EO(3)=8

6 D24
 [ a^2, d^3, b^4, (a*b)^2, (a*d)^2, d^-1*b^-1*d*b ]
EO(2)=13
```

8.3. Order 36.

```
1 C9 : C4
 [ a^4, a^-1*c*a*c, c^9 ]
EO(2)=1, EO(3)=2

7 (C3 x C3) : C4
 [ c^3, d^3, a^4, a^-1*c*a*c, a^-1*d*a*d, d^-1*c^-1*d*c,
  (a^-1*c)^2*a^-2, c^-1*d*a^-1*d*c^-1*a ]
EO(2)=1, EO(3)=8

3 (C2 x C2) : C9
 [ c^2, (a^-1*c)^2*a^2*c, a^9 ]
EO(2)=3
```

```
9 (C3 x C3) : C4
 [ c^3, a^4, a^-1*c*a^2*c*a^-1, a^-1*c*a*c*a^-1*c^-1*a*c^-1 ]
EO(2)=9

4 D36
 [ a^2, b^2, (a*c^-1)^2, (b*a)^2, c^-1*b*c*b, c^9 ]
EO(2)=19
```

8.4. Order 40.

```
1 C5 : C8
 [ a^-1*d*a*d, d^5, a^8 ]
EO(2)=1, EO(4)=2

3 C5 : C8
 [ a^-1*d^2*a*d, d^-1*a^-1*d*a*d^-1, d^5, a^8, d^-1*a^4*d*a^4 ]
EO(2)=1, EO(4)=10

4 C5 : Q8
 [ b^2*a^2, b*a*b*a^-1, a^4, a^-1*d*a*d, d^-1*b^-1*d*b, d^5,
  (a^-1*d)^2*a^-2 ]
EO(2)=1, EO(4)=22

8 (C10 x C2) : C2
 [ a^2, b^2, (a*d)^2, d^-1*b*d*b, d^5, (b*a)^4 ]
EO(2)=13

6 D40
 [ a^2, b^4, (a*b)^2, (a*d)^2, d^-1*b^-1*d*b, d^5 ]
EO(2)=21
```

8.5. Order 54.

```
6 (C9 : C3) : C2
 [ a^2, b^3, (a*c^-1)^2, b^-1*a*b*a, c^2*b^-1*c*b, c^-1*b^-1*c^4*b ]
EO(2)=9, EO(3)=8

8 ((C3 x C3) : C3) : C2
 [ a^2, b^3, c^3, (a*b)^2, (a*c)^2, (c^-1*b)^3, (c^-1*b^-1)^3 ]
EO(2)=9, EO(3)=26, CS(3)=4

5 ((C3 x C3) : C3) : C2
```

```
 [ a^2, b^3, c^3, (a*c)^2, b^-1*a*b*a, (c^-1*b)^3, (c^-1*b^-1)^3,
  (b*a*c^-1*b^-1*c)^2 ]
EO(2)=9, EO(3)=26, CS(3)=5

1 D54
 [ a^2, (a*b)^2, b^27 ]
EO(2)=27, EO(3)=2

7 (C9 x C3) : C2
 [ a^2, c^3, (a*b^-1)^2, (a*c)^2, c^-1*b^-1*c*b, b^9 ]
EO(2)=27, EO(3)=8

14 (C3 x C3 x C3) : C2
 [ a^2, b^3, c^3, d^3, (a*b)^2, (a*c)^2, (a*d)^2, c^-1*b^-1*c*b,
  d^-1*b^-1*d*b, d^-1*c^-1*d*c ]
EO(2)=27, EO(3)=26
```

8.6. Order 56.

```
1 C7 : C8
 [ a^-1*d*a*d, d^7, a^8 ]
EO(2)=1, EO(4)=2

3 C7 : Q8
 [ b^2*a^2, b*a*b*a^-1, a^4, a^-1*d*a*d, d^-1*b^-1*d*b,
  (a^-1*d)^2*a^-2, d^7 ]
EO(2)=1, EO(4)=30

11 (C2 x C2 x C2) : C7
 [ c^2, a^7, (c*a*c*a^-1)^2, a^-2*(c*a)^2*a*c*a^-1 ]
EO(2)=7

7 (C14 x C2) : C2
 [ a^2, b^2, (a*d)^2, d^-1*b*d*b, d^7, (b*a)^4 ]
EO(2)=17

5 D56
 [ a^2, b^4, (a*b)^2, (a*d)^2, d^-1*b^-1*d*b, d^7 ]
EO(2)=29
```

8.7. Order 60.

```
3 C15 : C4
 [ c^3, a^4, a^-1*c*a*c, a^-1*d*a*d, d^-1*c^-1*d*c, d^5,
  (a^-1*c)^2*a^-2, c^-1*d*a^-1*d*c^-1*a ]
EO(2)=1

7 C15 : C4
 [ c^3, a^4, a^-1*c*a*c, d^-1*c^-1*d*c, a^-1*d*a*d^2, d^5,
  (a^2*d)^2, (a^-1*c)^2*a^-2 ]
EO(2)=5

5 A5
 [ a*b^-4*a*b, a^4*b^-1*a^-1*b^-1, (a^-2*b^2)^2 ]
EO(2)=15

12 D60
 [ a^2, b^2, c^3, (a*c)^2, (a*d)^2, (b*a)^2, c^-1*b*c*b, d^-1*b*d*b,
  d^-1*c^-1*d*c, d^5 ]
EO(2)=31
```

8.8. Order 81.

```
6 C27 : C3
 [ b^3, a^3*b^-1*a^-3*b, b^-1*a^-4*b*a^-5 ]
EO(3)=8, EO(9)=18

10 C3 . ((C3 x C3) : C3) = (C3 x C3) . (C3 x C3)
 [ b^3*a^3, b*a*b*a^-2*b*a, b*a^2*(b*a^-1)^2, a^3*b^-6 ]
EO(3)=8, EO(9)=72, NS(3)=1

4 C9 : C9
 [ b*a^-1*b^2*a, b^-4*a^-1*b*a, a^-2*b^-1*a^2*b^-2, a^9 ]
EO(3)=8, EO(9)=72, NS(3)=4

8 (C9 x C3) : C3
 [ a^3, (a^-1*b^-2)^2*a^-1*b, b^-3*a^-1*b^3*a,
  b*a^-1*b^-1*a*b^-1*a^-1*b*a, b^9 ]
EO(3)=26, CS(3)=3

14 (C9 x C3) : C3
```

```
 [ a^3, b^3, c^-1*a^-1*c*a, c^-1*b^-1*c*b, b^-1*a^-1*c^-3*b*a, c^9 ]
EO(3)=26, CS(3)=5

3 (C9 x C3) : C3
 [ b^3, a^-1*b^-1*a^2*b^-1*a^-1*b^-1, a^-2*(b*a)^2*b,
  a^-1*(b^-1*a)^2*b^-1*a^-1, b^-1*a*b*a^-1*b^-1*a^-1*b*a, a^9 ]
EO(3)=26, CS(3)=7

7 (C3 x C3 x C3) : C3
 [ a^3, b^3, b*a^-1*b^-1*a*b^-1*a^-1*b*a, (b^-1*a^-1*b*a^-1)^3,
  (a*b)^9 ]
EO(3)=44

9 (C9 x C3) : C3
 [ a^3, (b^-1*a)^3, (b*a)^3, b^3*a^-1*b^-3*a, b^9 ]
EO(3)=62
```

8.9. Order 84.

```
5 C21 : C4
 [ c^3, a^4, a^-1*c*a*c, a^-1*d*a*d, d^-1*c^-1*d*c, (a^-1*c)^2*a^-2,
  c^-1*d*a^-1*d*c^-1*a, d^7 ]
EO(2)=1, EO(3)=2

1 (C7 : C4) : C3
 [ b^3, a^4, a^-1*d*a*d, b^-1*a^-1*b*a, d^-1*b^-1*d^2*b,
  d^3*b^-1*d*b ]
EO(2)=1, EO(3)=14

11 (C14 x C2) : C3
 [ b^2, a^3, d^-1*b*d*b, d^-1*a^-1*d*a*d^-1, (b*a)^3, (a^-1*b)^3,
  d^7, d^-1*a^-1*b*a*d*a^-1*b*a ]
EO(2)=3

14 D84
 [ a^2, b^2, c^3, (a*c)^2, (a*d)^2, (b*a)^2, c^-1*b*c*b, d^-1*b*d*b,
  d^-1*c^-1*d*c, d^7 ]
EO(2)=43
```

8.10. Order 88.

```
1 C11 : C8
```

```
 [ a^-1*d*a*d, a^8, d^11 ]
EO(2)=1, EO(4)=2

3 C11 : Q8
 [ b^2*a^2, b*a*b*a^-1, a^4, a^-1*d*a*d, d^-1*b^-1*d*b,
  (a^-1*d)^2*a^-2, d^11 ]
EO(2)=1, EO(4)=46

7 (C22 x C2) : C2
 [ a^2, b^2, (a*d)^2, d^-1*b*d*b, (b*a)^4, d^11 ]
EO(2)=25

5 D88
 [ a^2, b^4, (a*b)^2, (a*d)^2, d^-1*b^-1*d*b, d^11 ]
EO(2)=45
```

8.11. Order 90.

```
3 D90
 [ a^2, (a*b)^2, (a*c)^2, c^-1*b^-1*c*b, c^5, b^9 ]
EO(3)=2

9 (C15 x C3) : C2
 [ a^2, b^3, c^3, (a*b)^2, (a*c)^2, (a*d)^2, c^-1*b^-1*c*b,
  d^-1*b^-1*d*b, d^-1*c^-1*d*c, d^5 ]
EO(3)=8
```

9. Remaining 5 cases

Five orders remain, namely: 32, 48, 64, 72 and 80.

There are 44 isomorphism classes of non-abelian groups of order 32, 47 non-abelian groups of order 48, 256 non-abelian groups of order 64, 44 non-abelian groups of order 72 and 47 non-abelian groups of order 80, for a total of 438 non-abelian groups which we have not tested yet. We shall write a second article where we shall consider these 438 remaining groups.

References

[1] The GAP Group, GAP (Groups, Algorithms and Programming), Version 4.6.2, (http:www.gap-system.org), 2013.

[2] L. M. Huerta-Aparicio, A. Molina-Rueda, A.G. Raggi-Cárdenas and L. Valero-Elizondo, *On some invariants preserved by isomorphisms of tables of marks*, Rev. Colombiana Mat., 43(2) (2009), 165–174.

[3] A. G. Raggi-Cárdenas and L. Valero-Elizondo, *Two non-isomorphic groups of order 96 with isomorphic tables of marks and non-corresponding centres and abelian subgroups*, Comm. Algebra, 37 (2009), 209–212.
[4] J. Rotman, An Introduction to the Theory of Groups, Wm. C. Brown Publishers, 1988.
[5] M. Suzuki, Group Theory II, Volume 248 of Grundlehren der mathematischen Wissenschaften, Springer-Verlag, 1986.
[6] L. Valero-Elizondo, www.fismat.umich.mx/∼valero/gapcode, Gap Software.

M. Martínez, L. Valero and E. Vieyra
Facultad de Ciencias Fsico-Matemticas
Universidad Michoacana de San Nicols de Hidalgo
Edifico Alfa, Ciudad Universitaria
58140, Morelia, Michoacan, Mexico
e-mails: mangelfismat@gmail.com (M. Martínez)
valero@fismat.umich.mx (L. Valero)
edervs@gmail.com (E. Vieyra)

A.G. Raggi
Centro de Ciencias Matemticas, UNAM
Km 8 Antigua Carretera a Patzcuaro Num 8701
Col Ex Hacienda de San Jos de la Huerta
58089, Morelia, Michoacan, Mexico
e-mail: graggi@matmor.unam.mx