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PIECEWISE SEMIPRIME RINGS AND SOME APPLICATIONS

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Abstract. We define piecewise semiprime (PWSP) rings R in terms of a set of triangulating idempotents in R. The class of PWSP rings properly contains both the class of semiprime rings and the class of piecewise prime rings. The PWSP property is Morita invariant and it is shared by some important ring extensions. A ring is PWSP if and only if it has a generalized upper triangular matrix representation with semiprime rings on the main diagonal. Another characterization of PWSP rings involves a generalization of the concept of m-systems and is similar to the description of a semiprime ring in terms of the prime radical. Finally we use the PWSP property to determine (right) weak quasi-Baer rings. These are rings in which the right annihilator of every nilpotent ideal is generated as a right ideal by an idempotent.

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1. Introduction

All rings are associative and R denotes a ring with unity 1. The word *ideal* without the adjective right or left means two sided ideal. A ring R is *quasi-Baer* (*Baer*) if the right annihilator of every right ideal (nonempty subset) of R is generated as a right ideal by an idempotent. We now recall a few definitions and results from [1] which motivated our study and serve as the background material for the present work. An idempotent $e \in R$ is a *left semicentral* idempotent if $exe = xe$, for all $x \in R$. Similarly right semicentral idempotent can be defined. The set of all left (right) semicentral idempotents of R is denoted by $\mathrm{S}_l(R)$ ($\mathrm{S}_r(R)$). An idempotent $e \in R$ is *semicentral reduced* if $\mathrm{S}_l(eRe) = \{0, e\}$. If 1 is semicentral reduced, then R is called semicentral reduced. An ordered set $\{e_1, ..., e_n\}$ of nonzero distinct idempotents of R is called a set of *left triangulating idempotents* of R if all the following hold.

(1) $e_1 + \cdots + e_n = 1$,
(2) $e_1 \in \mathrm{S}_l(R)$,
(3) $e_{k+1} \in \mathrm{S}_l(c_k R c_k)$, where $c_k = 1 - (e_1 + \cdots + e_k)$ for $1 \leq k \leq n$.

From part (3) of the above definition, it can be seen that a set of left triangulating idempotents is a set of pairwise orthogonal idempotents. A set $E = \{e_1, ..., e_n\}$ of left triangulating idempotents of R is *complete*, if each e_i is semicentral reduced. A (complete) set of right triangulating idempotents is defined similarly. The cardinalities of complete sets of left triangulating idempotents of R are the same and is denoted by $\tau\dim(R)$ [1, Theorem 2.10]. According to [1, Proposition 1.3] R has a (complete) set of left triangulating idempotents if and only if there exists an isomorphism of rings between R and an n by n upper triangular matrix ring with the (i, j)-entry R_{ij} where each R_{ii} is a (semicentral reduced) ring with identity, and each R_{ij} is a left R_{ii} right R_{jj} bimodule for $i < j$. In this case, the ring R is said to have a (*complete*) *generalized triangular matrix representation* [3]. Following [1, p. 591], a ring R is called *piecewise prime* (abbreviated PWP) if there exists a complete set of left triangulating idempotents $E = \{e_1, ..., e_n\}$ of R such that $xRy = 0$ implies $x = 0$ or $y = 0$ where $x \in e_iRe_j$, and $y \in e_jRe_k$ for $1 \le i, j, k \le n$ (in this case, we say that R is PWP with respect to E). In [1, Theorem 4.11], it is shown that if R is a PWP ring then R is PWP with respect to any complete set of left triangulating idempotents of R; furthermore if $\tau\dim(R) = n$, then R is a PWP ring if and only if it is a quasi-Baer ring. The factor ring of a quasi-Baer ring by its prime radical is considered in [3]. For a comprehensive study of these concepts and results the reader is referred to [1-5]. It should be mentioned that in [7], modules whose endomorphism rings are of finite triangulating dimension are thoroughly investigated and the triangulating dimension is generalized from rings to modules.

In this paper, we define (in Section 2 to follow) piecewise semiprime (abbreviated PWSP) rings with respect to sets of left triangulating idempotents. We show that if R is PWSP with respect to a set of left triangulating idempotents of R, then it is PWSP with respect to any complete set of left triangulating idempotents of R. We prove that R is PWSP if and only if R has a generalized triangular matrix representation with semiprime rings on the main diagonal. The PWSP property is Morita invariant and it is shared between a ring and some of the more important extensions of that ring. Another characterization of PWSP rings is given using the concept of generalized m-systems with respect to sets of triangulating idempotents. Accordingly, a ring R is PWSP with respect to a set of triangulating idempotents E if and only if for every $e \in E$, $eNe = 0$ where N is the prime radical of R. In the final section of this paper the concept of *weak quasi Baer* ring is defined and investigated in terms of PWSP property. Throughout the paper, we shall only deal with rings of finite triangulating dimension in all results.

2. Piecewise Semiprime Rings

In this section, we define a piecewise semiprime (ideal) ring and show that this property is shared between a ring and some of the more important extensions of that ring. Also, piecewise semiprime endomorphism rings are investigated. We begin with the following definitions.

Definition 2.1. Let I be a proper ideal of a ring R.

(1) The ideal I is called *piecewise prime* (*PWP ideal*) if there is a complete set of left triangulating idempotents $E = \{e_1, \cdots, e_n\}$ such that $xRy \subseteq I$ implies $x \in I$ or $y \in I$, where $x \in e_iRe_j$, and $y \in e_jRe_k$ for $1 \leq i, j, k \leq n$.

(2) If $E = \{e_1, \cdots, e_n\}$ is a set of left triangulating idempotents then the ideal I is called a *piecewise semiprime* (*PWSP*) ideal with respect to E if for any $x \in R$ and $i = 1, \cdots, n$,

$$e_ixe_iRe_ixe_i \subseteq I \Rightarrow e_ixe_i \in I$$

The ring R is *piecewise semiprime* with respect to E if 0 is a PWSP ideal with respect to E. Also an ideal J of a ring R is said to be PWSP if there exists a set of left triangulating idempotents E of R such that J is PWSP with respect to E. The ring R is then called *PWSP* if 0 is a PWSP ideal of R.

Remarks 2.2. *If $E = \{e_1, ..., e_n\}$ is a set of left triangulating idempotents of a ring R, then I is a* PWSP *ideal of R with respect to E if and only if R/I is a* PWSP *ring with respect to $\{e_1 + I, ..., e_n + I\}$. Any semiprime ring is* PWSP *with respect to any set of left triangulating idempotents. If R is* PWSP *with respect to the set of left triangulating idempotent $\{1_R\}$, then it is semiprime. In general if R is a* PWSP *ring with respect to a set of left triangulating idempotents then R is not necessarily semiprime. The following non reversible implications hold for rings; see Examples and Remarks 2.12.*

$$\begin{array}{ccc} \text{prime} & \Rightarrow & \text{semiprime} \\ \Downarrow & & \Downarrow \\ \text{piecewise prime} & \Rightarrow & \text{piecewise semiprime} \end{array}$$

Theorem 2.3. *Let $E = \{e_1, ..., e_n\}$ be a set of left triangulating idempotents of a ring R and I an ideal of R. If I is a* PWSP *ideal with respect to E, then it is* PWSP *with respect to any complete set of left triangulating idempotents.*

Proof. Let $F = \{f_1, ..., f_m\}$ be a complete set of left triangulating idempotents of R and $f_ixf_iRf_ixf_i \subseteq I$ for some $f_i \in F$ and $x \in R$. Let d be the smallest in $\{1, ..., n\}$ such that $f_ie_d \neq 0$. Since $f_ixf_iRf_ixf_i \subseteq I$, $e_d(f_ixf_i)e_dRe_d(f_ixf_i)e_d \subseteq I$ which implies that $e_d f_i x f_i e_d \in I$ and then $f_i e_d f_i x f_i e_d f_i \in I$. If $c_d = e_1 + ... + e_d$, then $f_i c_d = f_i e_d \neq 0$. Since $c_d \in \mathrm{S}_l(R)$ by [1, Proposition 1.6], we have $f_i c_d = f_i c_d f_i c_d$. Thus $f_i c_d f_i = f_i e_d f_i \neq 0$. By [1, Lemma 1.4(ii)], $f_i c_d f_i \in \mathrm{S}_l(f_i R f_i)$. Since f_i is a semicentral reduced idempotent, $f_i e_d f_i = f_i c_d f_i = f_i$. Hence $f_i x f_i \in I$ as desired. □

Corollary 2.4. *Let $E = \{e_1, ..., e_n\}$ be a set of left triangulating idempotents of R. If R is* PWSP *with respect to E, then it is* PWSP *with respect to any complete set of left (right) triangulating idempotents of R.*

Proof. This is evident by Theorem 2.3. □

Proposition 2.5. *Let R be a* PWSP *ring with respect to a set of left triangulating idempotents $E = \{e_1, ..., e_n\}$. Then R is semiprime if and only if $E \subseteq \mathrm{S}_l(R)$.*

Proof. If R is semiprime then each semicentral left idempotent is central. Since for each $1 \leq i \leq n$, $c_i = e_1 + ... + e_i \in \mathrm{S}_l(R)$ by [1, Proposition 1.6], we deduce that for each $1 \leq i \leq n$, c_i is central. This implies that for each $1 \leq i \leq n$, e_i is central. In particular $E \subseteq \mathrm{S}_l(R)$. Conversely if $E \subseteq \mathrm{S}_l(R)$, then $R = \oplus_{i=1}^{n} Re_i = \oplus_{i=1}^{n} e_i Re_i$. Since each $e_i Re_i$ is semiprime by definition, we deduce that R is a semiprime ring. □

Proposition 2.6. *Let $E = \{e_1, ..., e_n\}$ be a set of left triangulating idempotents and I be an ideal of R. Then the following are equivalent.*

(1) *I is a* PWSP *ideal with respect to E.*
(2) *If J is any ideal of R such that $Je_iJ \subseteq I$ for some $e_i \in E$, then $e_iJe_i \subseteq I$.*
(3) *If J is any right ideal of R such that $Je_iJ \subseteq I$ for some $e_i \in E$, then $e_iJe_i \subseteq I$.*
(4) *If J is any left ideal of R such that $Je_iJ \subseteq I$ for some $e_i \in E$, then $e_iJe_i \subseteq I$.*
(5) *If (a) is any principal right (left) ideal of R such that $(a)e_i(a) \subseteq I$ for some $e_i \in E$, then $e_iae_i \subseteq I$.*

Proof. (1) $\Rightarrow$ (2) Let $x \in J$. Since $e_ixe_iRe_ixe_i \subseteq Je_iJ \subseteq I$, $e_ixe_i \in I$. Hence $e_iJe_i \subseteq I$.
(2) $\Rightarrow$ (1) Assume $e_ixe_iRe_ixe_i \subseteq I$ for some $x \in R$ and $e_i \in E$. Thus $(Re_ixe_iR)e_i(Re_ixe_iR) \subseteq I$ which implies that $e_ixe_i \in I$.
(2) $\Rightarrow$ (3) If $Je_iJ \subseteq I$ for some $e_i \in E$ and a right ideal J, then (Re_iJ) is an ideal and $(Re_iJ)e_i(Re_iJ) \subseteq I$ which implies $e_iJe_i \subseteq I$.
(3)$\Rightarrow$(1) Assume $e_ixe_iRe_ixe_i \subseteq I$ for some $x \in R$ and $e_i \in E$. Then $(e_ixe_iR)e_i(e_ixe_iR) \subseteq I$ which implies that $e_ixe_i \in I$.
(1) $\Leftrightarrow$ (4) The proof is similar to the proof of (1) $\Leftrightarrow$ (3).
(2) $\Leftrightarrow$ (5) It is routine. □

Proposition 2.7. *Let $R \subseteq S$ be rings, ${}_RS$ be free with a multiplicatively closed basis set X such that $rx = xr$ for all $x \in X$ and $r \in R$. If $E = \{e_1, ..., e_n\}$ is a set of left triangulating idempotents of R, then*

(1) *E is a set of left triangulating idempotents of S.*
(2) *R is* PWSP *with respect to E if and only if S is* PWSP *with respect to E.*

Proof. By hypothesis $aRb = 0$ if and only if $aSb = 0$ where $a, b \in R$. This proves (1). Also, if $(e_ise_i)S(e_ise_i) = 0$ for some $e_i \in E$, $s \in S$ and $s = \sum_{j=1}^{m} r_jx_j$ with $x_1, ..., x_m \in X$, then since X is a multiplicatively closed set, we can deduce that $(e_ir_je_i)R(e_ir_je_i) = 0$ for each $j = 1, ..., m$, the proof is complete. □

Suppose that R is a ring. The n by n matrix ring on R will be denoted by $\mathrm{M}_n(R)$. Let X be a non empty set of not necessarily commuting indeterminates on R and $x, y \in X$. By $R[[x]]$ and $R < X >$ we mean the ring of formal power series in x and the free R-ring generated by X, respectively. Also if $xy = yx$ and I = the ideal generated by $xy - 1$ in $R[x, y]$, then the ring $R[x, y]/I$, as usual, we be denoted by $R[x, x^{-1}]$; see [8, 1.2, p. 6] for an excellent reference for these rings.

Proposition 2.8. *Let $E = \{e_1, ..., e_n\}$ be a set of left triangulating idempotents of R, $x \in X$ be a non empty set of not necessarily commuting indeterminates on R, and I_m be an identity matrix of size m. Then the following statements are equivalent.*
(i) R is PWSP *with respect to E.*
(ii) $\mathrm{M}_m(R)$ is PWSP *with respect to $\overline{E} = \{e_iI_m\}_{i=1}^{n}$.*
(iii) $R < X >$ is PWSP *with respect to E.*
(iv) $R[[x]]$ is PWSP *with respect to E.*
(v) $R[x, x^{-1}]$ is PWSP *with respect to E.*

Proof. (i) ⇔ (ii). Let $\bar{e_i} = e_iI_m$ for each i. If J is an ideal of R, then it is seen that $\mathrm{M}_m(J)\bar{e_i}\mathrm{M}_m(J) = \mathrm{M}_m(Je_iJ)$ and $\bar{e_i}\mathrm{M}_m(J)\bar{e_i} = \mathrm{M}_m(e_iJe_i)$. Thus the equivalence (i) ⇔ (ii) is true by Proposition 2.6 and the fact that any ideal of $\mathrm{M}_m(R)$ has the form $\mathrm{M}_m(J)$ for some ideal J of R. The proof is now completed by Proposition 2.7. □

Lemma 2.9. *Let $T = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$, where R and S are two rings and ${}_RM_S$ is a nonzero bimodule. If $\{\begin{pmatrix} e_i & m_i \\ 0 & f_i \end{pmatrix}\}_{i=1}^{n}$ is a set of left triangulating idempotents of T, then $E_0 = \{e_1, ..., e_n\} \setminus \{0\}$, and $F_0 = \{f_1, ..., f_n\} \setminus \{0\}$ are sets of left triangulating idempotents of R and S respectively.*

Proof. It is easy to see that for each $1 \leq i \leq n$, e_i and f_i are idempotents with $\sum e_i = 1_R$ and $\sum f_i = 1_S$. Thus [1, Lemma 1.2], implies that E_0 and F_0 are sets of left triangulating idempotents of R and S respectively. □

Proposition 2.10. *Let T be a formal triangular matrix ring as in 2.9. Then T is* PWSP *if and only if R and S are* PWSP*.*

Proof. Let R and S be PWSP with respect to the sets of left triangulating idempotents $E = \{e_1, ..., e_n\}$ and $F = \{f_1, ..., f_m\}$ respectively. Then T is PWSP with respect to $\{\begin{pmatrix} e_1 & 0 \\ 0 & 0 \end{pmatrix}, ..., \begin{pmatrix} e_n & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & f_1 \end{pmatrix}, ..., \begin{pmatrix} 0 & 0 \\ 0 & f_m \end{pmatrix}\}$.

Conversely if T is PWSP with respect to $\{\begin{pmatrix} e_i & m_i \\ 0 & f_i \end{pmatrix}\}_{i=1}^n$, then R, and S are PWSP with respect to E_0 and F_0 as in 2.9. To see this let $e_ixe_iRe_ixe_i = 0$ where $e_i \in E_0$, and $x \in R$. Since $e_ixe_iRe_ixe_i = 0$, If

$$\begin{pmatrix} e_i & m_i \\ 0 & f_i \end{pmatrix}\begin{pmatrix} xe_i & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} e_i & m_i \\ 0 & f_i \end{pmatrix} = \begin{pmatrix} e_ixe_i & e_ixe_im_i \\ 0 & 0 \end{pmatrix} = t$$

then $tTt = 0$. This implies $t = 0$. Hence $e_ixe_i = 0$. A similar argument implies that S is PWSP with respect to F_0. □

Corollary 2.11. *Let $R = \oplus_{i=1}^n R_i$ be a ring decomposition. Then R is* PWSP *(*PWP*) if and only if each R_i is* PWSP *(*PWP*).*

Proof. For the PWSP case use induction and Proposition 2.10. The PWP case has a routine argument. □

Examples and Remarks 2.12. *(1) If R_1 and R_2 are* PWP *rings, then by Corollary 2.11, $R_1 \oplus R_2$ is a* PWP *ring but not a prime ring.*

(2) If $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ with a nonzero bimodule ${}_AM_B$, then R is never a semiprime ring but it is a PWSP *ring provided that A, B are so (Proposition 2.10). Also if A is a prime ring, X is a non-trivial ideal of A and $R = \begin{pmatrix} A & A/X \\ 0 & A \end{pmatrix}$, then R is* PWSP *with respect to the complete set of left triangulating idempotents $\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}$. But the right annihilator of $\begin{pmatrix} 0 & A/X \\ 0 & 0 \end{pmatrix}$ is $\begin{pmatrix} A & A/X \\ 0 & X \end{pmatrix}$ which is not a direct summand of R. This shows that R is not quasi-Baer. Hence by* [1, Theorem 4.11]*, R is not* PWP*.*

(3) A subring of a PWSP *ring need not be* PWSP*. To see this let R be a* PWSP *ring with respect to a set of left triangulating idempotents $E = \{e_1, ..., e_n\}$. Then*

$S = \{\begin{pmatrix} r & a \\ 0 & r \end{pmatrix} \mid r, a \in R\}$ *is a subring of* $T = \begin{pmatrix} R & R \\ 0 & R \end{pmatrix}$. *Now* T *is* PWSP *but* S *is not. For if* $E^* = \{\begin{pmatrix} e_i & r_i \\ 0 & e_i \end{pmatrix} | e_i \in E, r_i \in R, i = 1, ..., n\}$ *is a set of left triangulating idempotents of* S*, then* $\begin{pmatrix} e_i & r_i \\ 0 & e_i \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} e_i & r_i \\ 0 & e_i \end{pmatrix} = \begin{pmatrix} 0 & e_i \\ 0 & 0 \end{pmatrix}$, *and so it can be seen that* $\begin{pmatrix} 0 & e_i \\ 0 & 0 \end{pmatrix} S \begin{pmatrix} 0 & e_i \\ 0 & 0 \end{pmatrix} = 0$.

(4) Part (2) above shows that if R *is* PWSP *with respect to a set of left triangulating idempotents* $E = \{e_1, ..., e_n\}$*, then* R *need not be* PWSP *with respect to another set of left triangulating idempotents* E' *of* R *say* $E' = \{1_R\}$.

(5) If R *is a* PWSP *ring and* α *is a ring endomorphism of* R*, then the skew polynomial ring* $S = R[x, \alpha]$ *of* R *need not be* PWSP. *Note that by definition,* S *is a ring extension of* R *such that* ${}_RS$ *is free with a basis of the form* $1, x, x^2....$ *and* $xr = \alpha(r)x$ *for all* $r \in R$. *Now consider* $S = R[x, \alpha]$*, where* $R = k[y]$ *for some field* k *and* $\alpha : R \to R$ *is the evaluation map at zero. If* S *were* PWSP *for a set of left triangulating idempotents then by Corollary 2.4, it would be* PWSP *with respect to any complete set of left triangulating idempotents. However the singleton* $\{1_R\}$*, by* [2, Theorem 4.4]*, is such a set, hence* S *would be semiprime. But* S *is not semiprime because* $yxR[x, \alpha]yx = 0$. *It implies that* S *is not* PWSP.

Following [9], an R-module M is called *duo* if every submodule of M is fully invariant in M.

Proposition 2.13. *Let* M *be an* R*-module,* $S = \mathrm{End}_R(M)$ *and* $F = \{b_1, ..., b_n\}$ *be a set of left triangulating idempotents of* S.

(1) S *is* PWSP *with respect to* F *if and only if* $b_igb_iN = 0$ *implies* $b_igb_i = 0$ *where* $b_i \in F$*,* N *any fully invariant submodule of* M*, and* $g \in \mathrm{Hom}_R(M, N)$.
(2) *If* M *is a duo module then* S *is* PWSP *with respect to* F *if and only if for each* $g \in S$*,* $(b_ig)^2 = 0$ *implies* $b_igb_i = 0$.

Proof. Let S be PWSP with respect to F. If $b_igb_iN = 0$, then we have $b_igb_iSb_igb_iM \subseteq b_igb_iN = 0$, which implies $b_igb_i = 0$. Conversely let $b_igb_iSb_igb_i = 0$. Then $N = Sb_igb_iM$ is a fully invariant submodule of M where $b_igb_iN = 0$. Thus $b_igb_i = 0$. For second part let $N = g(M)$ for $g \in S$. □

Let M be a nonzero R-module. A decomposition $M = \oplus_{i=1}^n M_i$ is a *left (right) orthogonal decomposition* if $\mathrm{Hom}_R(M_j, M_i) = 0$ for each $i < j$ $(j < i)(i, j = 1, ..., n)$ [7].

Proposition 2.14. *Let M be an R-module and $S = \mathrm{End}_R(M)$. Then the following statements are equivalent.*

(1) *S is* PWSP.
(2) *There exists a left orthogonal decomposition $M = \oplus_{i=1}^n M_i$ such that for each i, $\mathrm{End}_R(M_i)$ is semiprime.*

Proof. (1) $\Rightarrow$ (2) Let S be PWSP with respect to a set of left triangulating idempotents $F = \{f_1, ..., f_n\} \subseteq S$. Then $M = \oplus_{i=1}^n f_i M$ is an orthogonal decomposition for M and for each i, $\mathrm{End}_R(f_i M) \simeq f_i S f_i$ which is semiprime by Definition 2.1.
(2) $\Rightarrow$ (1) Let $X = \oplus_{i=2}^n M_i$. Since M has a left orthogonal decomposition $M = M_1 \oplus X$, the ring S is isomorphic to $\begin{pmatrix} A & N \\ 0 & C \end{pmatrix}$ where $A = \mathrm{End}_R(M_1)$, $N=$ $\mathrm{Hom}_R(X, M_1)$ and $C = \mathrm{End}_R(X)$. Thus the result follows by Proposition 2.10, using an induction argument on n. $\square$

Corollary 2.15. *The following statements are equivalent.*

(1) *R is a* PWSP *ring.*
(2) *There exists a set of left triangulating idempotents $E = \{e_1, ..., e_n\}$ of R such that for each i, $e_i R e_i$ is semiprime.*
(3) *R has a triangular matrix ring representation with semiprime rings on the main diagonal.*

Proposition 2.16. *If R is a* PWSP *ring and e an idempotent in R, then eRe is a* PWSP *ring.*

Proof. Let R be a ring and e an idempotent in R. Suppose that R is a PWSP ring. By Corollary 2.15, there exists a ring isomorphism ϕ from R to a generalized triangular matrix ring T with semiprime rings on its main diagonal. Since $\phi(e)$ is an idempotent in T then its main diagonal elements are idempotent and the restriction of ϕ is a ring isomorphism from eRe onto $\phi(e)T\phi(e)$, where the rings on the main diagonal of $\phi(e)T\phi(e)$ are also semiprime. Thus again by Corollary 2.15, eRe is PWSP. $\square$

Theorem 2.17. *The property* PWSP *is Morita invariant.*

Proof. If R is Morita equivalent to a ring S, then it is known that S is isomorphic to the ring $e\mathrm{M}_n(R)e$ for some natural number n and suitable idempotent e in $\mathrm{M}_n(R)$. The result is now obtained by Propositions 2.8(2) and 2.16. $\square$

Proposition 2.18. *Let R be a* PWSP *ring with respect to a set of left triangulating idempotents $E = \{e_1, ..., e_n\}$. Then $R = A \oplus B$ such that $A = \oplus_{i=1}^k A_i$ is a direct* *sum of semiprime rings and there is a ring isomorphism*

$$B \simeq \begin{pmatrix} B_1 & B_{12} & ... & B_{1l} \\ 0 & B_2 & ... & B_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & B_l \end{pmatrix} \quad (*)$$

such that each B_i is a semiprime ring, B_{ij} is a left B_i- right B_j bimodule, and $k+l=n$.

Proof. Let $\{a_1, ..., a_k\}$ be the set of central idempotents of E. Since $e = a_1+...+a_k$ is a central idempotent, $R \simeq eR \oplus (1-e)R = A \oplus B$ is a ring decomposition in which $A = \oplus_{i=1}^{k} A_i = \oplus_{i=1}^{k} a_i R a_i$. By definition, for each i, A_i is semiprime. It can be seen that $\{b_1, ..., b_l\} = E \backslash \{a_1, ..., a_k\}$ is a set of left triangulating idempotents for B. Thus there is an isomorphism as in $(*)$ such that for each $1 \leq j \leq l$, B_j is semiprime. Also $k+l=n$. □

Let Q be a ring. A *right order* in Q is any subring R such that

(1) Every regular element b of R (i.e., $\mathrm{r.ann}_R(b) = \mathrm{l.ann}_R(b) = 0$) is invertible in Q.
(2) Every element of Q has the form ab^{-1} for some $a \in R$ and some regular $b \in R$.

Proposition 2.19. *Let R be a right order in a semiprimary ring Q. If R is* PWSP, *then so is Q.*

Proof. Let $E = \{e_1, ..., e_n\}$ be a set of left triangulating idempotents of R and $xQx = 0$ where $x \in e_i Q e_i$. By [6, Theorem A], $e_i R e_i$ is a right order in $e_i Q e_i$. Thus $x = ab^{-1}$ for some $a, b \in e_i R e_i$. Hence $ab^{-1}Qab^{-1} = 0$ and $ab^{-1}Qa = 0$. Since $aRa \subseteq ab^{-1}Qa = 0$, $a = 0$ which implies that $x = 0$. □

3. Piecewise m-Systems

It is common knowledge that m-systems play a useful role in the study of prime and semiprime rings. A nonempty subset S of R is called *m-system* if for every x and y in S there exists $r \in R$ such that $xry \in S$. It is well known that for every ideal of R, the set $\{r \in R \mid \text{ every m-system containing } r \text{ meets } I\}$ is equal to $\sqrt{I} := \cap \mathcal{P}$ where $\mathcal{P}$ is the set of all prime ideals of R containing I. The purpose of this section is to develop an analogous concept for the study of PWSP rings.

Definition 3.1. Let R be a ring and $E = \{e_1, ..., e_n\}$ be a set of left triangulating idempotents of R. A nonempty subset S of R is a *piecewise m-system with respect to E*, if for each $e_i a e_j$ and $e_j b e_k \in S$, where $e_i, e_j, e_k \in E$ and $a, b \in R$, there exists $r \in R$ such that $e_i a e_j r e_j b e_k \in S$. The intersection of all PWP ideals of R containing I will be denoted by $\sqrt[\mathrm{p}]{I}$.

Lemma 3.2. *Let R be a ring and E be a complete set of left triangulating idempotents of R.*

(1) *If P is an ideal of R then $R \setminus P$ is a piecewise m-system with respect to E, if and only if P is a* PWP *ideal.*
(2) *Let P be an ideal of R such that $e_i P e_i \cap S = \varnothing$ for some $e_i \in E$ and some piecewise m-system S. If P is maximal with respect to this property then P is prime.*

Proof. (1) It is easy to check.
(2) Let I_1 and I_2 be ideals such that $I_1 I_2 \subseteq P$ and $a_j \in I_j \setminus P$ $(j = 1, 2)$. Since P is maximal with respect to the mentioned property, there exist $s_j \in (e_i(P + (a_j))e_i) \cap S$ $(j = 1, 2)$. Thus $e_i s_1 e_i r e_i s_2 e_i \in (e_i P e_i) \cap S$ which is a contradiction. Thus P is prime. □

The following example shows that m-systems and piecewise m-systems are different in general.

Example 3.3. *Let $R = \begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$ where k is a field. Then R is a* PWP *ring. Thus by Lemma 3.2, $R \setminus \{0\}$ is a piecewise m-system with respect to the complete set of left triangulating idempotents $\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}$. However that set is not an m-system.*

Proposition 3.4. *Let R be a ring, $E = \{e_1, ..., e_n\}$ a set of left triangulating idempotents, and I an ideal in R. If*

$$I_{\mathrm{p}} = \{e_i r e_i \mid r \in R, \textit{every piecewise } m\textit{−system containing } e_i r e_i \textit{ meets } e_i I e_i\}$$

then $I_{\mathrm{p}} = \sqrt[\mathrm{p}]{I} = \sqrt{I}$.

Proof. Suppose that $e_i r e_i \in I_{\mathrm{p}}$. We shall show that $e_i r e_i \in \sqrt[\mathrm{p}]{I}$. If there exists a PWP ideal P of R containing I such that $e_i r e_i \notin P$, then $e_i r e_i \in R \setminus P := S$ and $S \cap e_i I e_i = \emptyset$. On the other hand, S is a piecewise m-system by Lemma 3.2 and by definition of I_{p}, S meets $e_i I e_i$, a contradiction. Thus $e_i r e_i \in P$. This shows that $I_{\mathrm{p}} \subseteq \sqrt[\mathrm{p}]{I}$. Since each prime ideal of R is PWP, we have $\sqrt[\mathrm{p}]{I} \subseteq \sqrt{I}$. It is enough to show that $\sqrt{I} \subseteq I_{\mathrm{p}}$. Let $e_i r e_i \notin I_{\mathrm{p}}$. Then there exists a piecewise m-system T such that $e_i r e_i \in T$ and $T \cap e_i I e_i = \emptyset$. Let $\mathcal{A} = \{J \trianglelefteq R \mid I \subseteq J \textit{ and } e_i J e_i \cap T = \emptyset\}$. By Zorn's Lemma, $\mathcal{A}$ has a maximal member Q. By Lemma 3.2, Q is a prime ideal of R. Thus $e_i r e_i \notin \sqrt{I}$. □

Theorem 3.5. *Let $E = \{e_1, ..., e_n\}$ be a set of left triangulating idempotents of a ring R. Then for any ideal I in R the following are equivalent.*

(1) *I is a* PWSP *ideal with respect to E.*
(2) *For each $e_i \in E$, $e_i I e_i = e_i \sqrt{I} e_i$ where $\sqrt{I}$ is the prime radical of I.*
(3) *For each $e_i \in E$, $e_i I e_i = e_i(\overset{\mathrm{p}}{\sqrt{I}})e_i$.*

Proof. (1) $\Rightarrow$ (2) Let $x_0 = e_i x e_i \notin e_i I e_i$. Then $e_i x_0 e_i R e_i x_0 e_i \not\subseteq I$. Now let $x_1 \in e_i x_0 e_i R e_i x_0 e_i \setminus I$. Continuing this method will yield a set $S = \{x_0, x_1, ...\}$. By Zorn's Lemma there exists an ideal P containing I and disjoint from S. It can be seen that P is prime. Hence $x_0 \notin e_i \sqrt{I} e_i$.
(2) $\Rightarrow$ (1) Let $e_i x e_i R e_i x e_i \subseteq I$ for some $e_i \in E$ and $x \in R$. Then by (2), $e_i x e_i R e_i x e_i \subseteq e_i I e_i = e_i \sqrt{I} e_i \subseteq \sqrt{I}$. Thus $e_i x e_i \in \sqrt{I}$ which implies $e_i x e_i \in e_i \sqrt{I} e_i = e_i I e_i \subseteq I$.
(2) $\Leftrightarrow$ (3) This follows from Proposition 3.4. □

Proposition 3.6. *Let P be a* PWP *(resp.* PWSP*) ideal of R with respect to a complete set of left triangulating idempotents $E = \{e_1, ..., e_n\}$. Then P contains a minimal* PWP *(resp.* PWSP*) ideal with respect to E.*

Proof. Let $\mathcal{A}$ be the set of PWP (resp. PWSP) ideals with respect to E which are contained in P. By Zorn's Lemma it can be seen that $\mathcal{A}$ has a minimal member which is a PWP (resp. PWSP) ideal. Thus P contains a minimal PWP (resp. PWSP) ideal. □

4. Weak Quasi-Baer Rings

As mentioned in Section 1, for a ring R of finite triangulating dimension, it is shown in [1, Theorem 4.11] that R is PWP if and only if R is quasi-Baer. This important result naturally motivates the following question. What modification of the concept of a Baer ring is equivalent to the PWSP property? Although we do not know the precise answer to the above question, here we introduce the concept of *right weak quasi Baer* rings and then find a description of these rings in terms of the PWSP property. If M_R is a module and X is a nonempty subset of M, then the right annihilator of X in R is denoted by $\mathrm{r}_R(X)$.

Definition 4.1. A ring R is called *right weak quasi-Baer*, if the right annihilator of every nilpotent ideal of R is generated as a right ideal by an idempotent.

Similarly, the left weak quasi-Baer property for a ring R is defined (i.e. the left annihilator of every nilpotent ideal of R is generated as a left ideal by an idempotent). Examples 4.8(3) shows that the weak quasi-Baer property is not left-right symmetric.

We recall from the introduction that $\tau\dim(R)$ *is assumed to be finite in the following results.*

Proposition 4.2. *If R is a right weak quasi-Baer ring, then R is* PWSP.

Proof. Let $E = \{e_1, ..., e_n\}$ be a complete set of left triangulating idempotents of R and let $e_ixe_iRe_ixe_i = 0$ for some i and element $x \in R$. Then $I = Re_ixe_iR$ is an ideal of R such that $I^2 = 0$. Since R is right weak quasi-Baer, $\mathrm{r}_R(I) = fR$ for some nonzero idempotent $f \in R$. Since fR is an ideal in R, $zf = fzf$ for all $z \in R$. It follows that e_ife_i is an idempotent element in e_iRe_i, and since $e_iy = ye_i = y$ for all $y \in e_iRe_i$, we can deduce that $e_ife_i \in \mathrm{S}_l(e_iRe_i)$. Therefore $e_ife_i = e_i$ or 0. If $e_ife_i = e_i$ then $e_ixe_i = e_ix(e_ife_i) = 0$ because $If = 0$. Suppose that $e_ife_i = 0$. Since $e_ixe_i \in fR$, we have $e_ixe_i \in (e_if)R = (fe_if)R$. Hence $e_ixe_i = e_i(e_ixe_i) \in (e_ife_i)R = 0$. The proof is complete. $\square$

We are now going to find conditions under which a PWSP ring is right weak quasi-Baer. Since a PWSP ring is isomorphic to a formal triangular matrix ring $T = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ for suitable bimodule ${}_AM_B$, we first state when T is a right weak quasi-Baer ring. In [4], the characterization of T as a quasi-Baer ring is provided. A characterization of formal triangular matrix rings that are weak quasi-Baer is then obtained by methods as in [4, Theorem 3.2] using the fact that nilpotent ideals of T have the form $\begin{pmatrix} I & N \\ 0 & J \end{pmatrix}$ where I and J are nilpotent ideals in A and B respectively, and N is a sub-bimodule of ${}_AM_B$. We record this below.

Proposition 4.3. *Let M be an R-S bimodule and $T = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$. Then the following statements are equivalent.*

(1) *T is right weak quasi-Baer.*

(2) (a) *R is right weak quasi-Baer.*

(b) *For each nilpotent ideal I of R, $\mathrm{r}_M(I) = \mathrm{r}_R(I)M$.*

(c) *If ${}_RN_S \subseteq {}_RM_S$ and J is any nilpotent ideal of S, then $\mathrm{r}_S(N) \cap \mathrm{r}_S(J)$ is a direct summand of S.*

Corollary 4.4. *Let M be an R-S bimodule, R be a semiprime ring and $T = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$. Then T is right weak quasi-Baer if and only if for any ${}_RN_S \subseteq {}_RM_S$ and nilpotent ideal J of S, $\mathrm{r}_S(N) \cap \mathrm{r}_S(J)$ is a direct summand of S.*

Proof. This follows from Proposition 4.3. $\square$

Motivated by Proposition 4.3, we give the following definition.

Definition 4.5. Let e be a semicentral idempotent in a ring R, $T = eRe$, and $S = (1-e)R(1-e)$. We say that R is *right e-weak Baer* if for every left-T, right-S sub-bimodule N of $eR(1-e)$ and nilpotent ideal J of S, the ideal $\mathrm{r}_S(J) \cap \mathrm{r}_S(N)$ is generated as a right ideal by an idempotent. Clearly, right 0-weak quasi-Baer means right weak quasi-Baer.

Lemma 4.6. *Let e be a semicentral reduced idempotent in a ring R. Then there exists a complete set of left triangulating idempotents $E = \{e_1, ..., e_n\}$ such that $e = e_1$.*

Proof. We have $R = eR \oplus (1-e)R$ such that $\mathrm{Hom}_R(eR, (1-e)R) = 0$. Since $\tau\mathrm{dim}(R)$ is assumed to be finite, the R-module $(1-e)R$ has finite triangulating dimension [7, Theorem 2.4]. Thus $(1-e)R$ has a left orthogonal decomposition $(1-e)R = \oplus_{i=1}^{k} M_i$ such that for each $i = 1, ..., k$, $\tau\mathrm{dim}(\mathrm{End}_R(M_i)) = 1$ [7, Proposition 2.7]. It follows that there exists a complete set of left triangulating idempotents $E = \{e_1, ..., e_n\}$ such that $e = e_1$. □

Theorem 4.7. *The following statements are equivalent for a ring R.*

(1) *R is right weak quasi-Baer.*
(2) *R is* PWSP *and right e-weak quasi-Baer for each $e \in \mathrm{S}_l(R)$.*
(3) *R is* PWSP *and right e-weak quasi-Baer for some reduced idempotent e in $\mathrm{S}_l(R)$.*

Proof. (1) $\Rightarrow$ (2) This follows from Propositions 4.2 and 4.3 using the fact that if $e \in \mathrm{S}_l(R)$, then $R \simeq \begin{pmatrix} eRe & eR(1-e) \\ 0 & (1-e)R(1-e) \end{pmatrix}$.

(2) $\Rightarrow$ (3) This is clear.

(3) $\Rightarrow$ (1) Suppose R is PWSP and e is a reduced idempotent in $\mathrm{S}_l(R)$. By Lemma 4.6, there exists a complete set of left triangulating idempotents $E = \{e_1, ..., e_n\}$ with $e = e_1$. By Corollary 2.4, R is PWSP with respect to E. Thus $R \simeq \begin{pmatrix} eRe & eR(1-e) \\ 0 & (1-e)R(1-e) \end{pmatrix}$ where eRe is a semiprime ring. The result is now obtained by e-weak quasi-Baer condition on R and Corollary 4.4. □

Examples 4.8. (1) *Any semiprime ring is clearly right weak quasi-Baer. Further examples can be readily constructed by taking S to be a semisimple ring in Corollary 4.4.*

(2) *Let F be a field, $F_n = F$ for $n = 1, 2, ...$ and let*

$$R = \{(a_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} F_n | \ (a_n)_{n=1}^{\infty} \text{ is eventually constant}\}$$

Then R is a subring of $\prod_{i=1}^{n} F_n$ and $R[[x]]$ is a semiprime ring which is not quasi-Baer; see [2, Example 1.6]. *Let now I be a maximal ideal of $R[[x]]$*

and $S = R[[x]]/I$. *Then by* [4, Theorem 3.2], $T = \begin{pmatrix} R[[x]] & S \\ 0 & S \end{pmatrix}$ *is not quasi-Baer and clearly it is not semiprime but by Corollary 4.4,* T *is a right weak quasi-Baer ring.*

(3) *The ring* $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix}$ *is right weak quasi-Baer, but it is not left weak quasi-Baer because the left annihilator of* $\begin{pmatrix} 0 & \mathbb{Z}_2 \\ 0 & 0 \end{pmatrix}$ *is not a direct summand of* ${}_RR$.

We end by recording a noteworthy fact:

Remark 4.9. *Being right weak quasi-Baer is a Morita invariant for rings. To see this, in* [5, Lemma 2]*, let* I *be a nilpotent ideal.*

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