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THE AFFINENESS CRITERION FOR WEAK YETTER-DRINFEL'D MODULES

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Dedicated to the memory of Professor Efraim P. Armendariz

Abstract. In this paper, we introduce the concept of total quantum integrals in the case of weak Hopf algebras and study the affineness criterion for weak Yetter-Drinfel'd modules, which is a generalization of the results studied by Menini and Militaru (J. Algebra, 247 (2002), 467-508).

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1. Introduction

The integrals for a Hopf algebra H and the more general ones were introduced by Doi ([7]), stating that the existence of an integral is the necessary and sufficient condition for the existence of a natural transformation between two functors linking the categories of relative Hopf-modules $\mathcal{M}_A^H$ and right H-comodules $\mathcal{M}^H$. The categorical point of view towards integrals associated to a Doi-Koppinen datum (H, A, C) was introduced by Caenepeel et al. ([4]) to prove separability theorems. In [9], the authors weakened the conditions for a total A-integral in the sense of Caenepeel. The integrals cover the integrals introduced by Doi and the classic integral by Larson and Sweedler [8]. As a major application, the quantum integrals associated to quantum Yetter-Drinfel'd modules ${}^H\mathcal{YD}_A$ were introduced.

Weak bialgebras and weak Hopf algebras given in [2] generalize the ordinary bialgebras and Hopf algebras by weakening the comultiplication of the unit and the multiplication of the counit. Comultiplication is allowed to be non-unital, $\Delta(1_H) = 1_1 \otimes 1_2 \neq 1_H \otimes 1_H$ but the comultiplication is coassociative. In exchange for coassociativity, the multipicativity of the counit is repaced by a weaker condition: $\varepsilon(hg) = \varepsilon(h1_1)\varepsilon(1_2 g)$, implying that the unit representation is not necessarily one-dimensional and irreducible. Weak Hopf algebras provide a good framework for studying symmetries of certain quantum field theories. The examples of weak

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Hopf algebras are groupoid algebras, face algebras and generalized Kac algebras. It has turned out that many classical results of bialgebras and Hopf algebras can be generalized to weak bialgebras and weak Hopf algebras. The main purpose of this paper is to define the more general concept of an integral associated to weak Yetter-Drinfel'd modules, which generalizes the integral introduced by Menini and Militaru in ordinary Hopf algebra setting.

The paper is organized as follows: In Section 3, we introduce weak Yetter-Drinfel'd modules and prove that there exists some equivalence between the category ${}^H\mathcal{YD}_A$ of weak Yetter-Drinfel'd modules and the category $\mathcal{M}_B$ of left B-module, where $B = A^{coH} = \{a \in A | \tilde{\rho}(a) = S^{-1}(1_{(1)})1_{(-1)} \otimes a1_{(0)}\}$ (see Proposition 3.5), and study the affineness criterion for weak Yetter-Drinfel'd modules (see Theorem 3.7).

2. Preliminaries

We always work over a fixed field k and follow Sweedler's book [10] for the terminologies on coalgebras and comodules. For the comultiplication Δ in a coalgebra C, we use the Sweedler-Heyneman's notation, $\Delta(c) = c_1 \otimes c_2$, for all $c \in C$. All algebras, linear spaces etc. will be over k. All maps are k-linear and $\otimes$ means $\otimes_k$ unless otherwise specified, etc.

Definition 2.1. Let H be both an algebra and a coalgebra. Then H is called a *weak bialgebra* in [2] if it satisfies the following conditions:

$$\begin{aligned}
&\triangle(xy) = \triangle(x)\triangle(y),\\
&\triangle^2(1) = (\triangle(1) \otimes 1)(1 \otimes \triangle(1)) = (1 \otimes \triangle(1))(\triangle(1) \otimes 1),\\
&\varepsilon(xyz) = \varepsilon(xy_1)\varepsilon(y_2 z),\ \varepsilon(xyz) = \varepsilon(xy_2)\varepsilon(y_1 z),
\end{aligned}$$

for any $x, y, z \in H$, where $\triangle(1) = 1_1 \otimes 1_2$ and $\triangle^2 = (\triangle \otimes id_H) \circ \triangle$.

Moreover, if there exists a k-linear map $S : H \longrightarrow H$ called antipode, satisfying the following axioms for all $x \in H$,

$$x_1 S(x_2) = \varepsilon(1_1 x)1_2,\ S(x_1)x_2 = 1_1\varepsilon(x1_2),\ S(x_1)x_2 S(x_3) = S(x).$$

Then the weak bialgebra H is called a *weak Hopf algebra.*

For any weak Hopf algebra H, we define two maps $\varepsilon_t, \varepsilon_s : H \to H$ by the formulas

$$\varepsilon_t(x) = \varepsilon(1_1 x)1_2,\ \varepsilon_s(x) = 1_1\varepsilon(x1_2)$$

and denote by H_t the image $\varepsilon_t(H)$, and denote by H_s the image $\varepsilon_s(H)$.

Let H be a weak Hopf algebra. Recall from [2] that the following properties hold, for all $h, g \in H$,

(W1) H_t and H_s are two sub-algebras of H,

(W2) $\triangle(1) = 1_1 \otimes 1_2 \in H_s \otimes H_t,\ \varepsilon_t(h)\varepsilon_s(g) = \varepsilon_s(g)\varepsilon_t(h)$,

(W3) $\triangle(\varepsilon_t(h)) = 1_1\varepsilon_t(h) \otimes 1_2,\ \triangle(\varepsilon_s(g)) = 1_1 \otimes \varepsilon_s(g)1_2$,

(W4) $h_1 \otimes \varepsilon_s(h_2) = h1_1 \otimes S(1_2),\ \varepsilon_t(h_1) \otimes h_2 = S(1_1) \otimes 1_2 h$,

(W5) $h_1 \otimes \varepsilon_t(h_2) = 1_1 h \otimes 1_2,\ \varepsilon_s(h_1) \otimes h_2 = 1_1 \otimes h1_2$,

(W6) $\varepsilon_t \circ \varepsilon_t = \varepsilon_t,\ \varepsilon_s \circ \varepsilon_s = \varepsilon_s$,

(W7) $\varepsilon_t \circ S = \varepsilon_t \circ \varepsilon_s = S \circ \varepsilon_s,\ \varepsilon_s \circ S = \varepsilon_s \circ \varepsilon_t = S \circ \varepsilon_t$,

(W8) $S(hg) = S(g)S(h)$, $S(h_2) \otimes S(h_1) = S(h)_1 \otimes S(h)_2$, and $S(1) = 1,\ \varepsilon \circ S = \varepsilon$,

(W9) $h_1\varepsilon_s(g) \otimes h_2 = h_1 \otimes h_2 S(\varepsilon_s(g)),\ h_1 \otimes \varepsilon_t(g)h_2 = S(\varepsilon_t(g))h_1 \otimes h_2$.

Definition 2.2. Let H be a weak Hopf algebra over the field k. Recall from [3] that a *left H-comodule algebra* is an algebra A together with a multiplicative left H-coaction $\rho^l : A \to H \otimes A$ satisfying the condition

$$\rho^l(1_A) = (\varepsilon_s \otimes id_A) \circ \rho^l(1_A).$$

We use the standard notation $\rho^l(a) = a_{(-1)} \otimes a_{(0)}$, for any $a \in A$. For the coassociativity of the left comodule, we write $((id_H \otimes \rho^l) \circ \rho^l)(a) = a_{(-2)} \otimes a_{(-1)} \otimes a_{(0)} = a_{(-1)1} \otimes a_{(-1)2} \otimes a_{(0)} = ((\Delta_H \otimes id_A) \circ \rho^l)(a)$, for any $a \in A$.

Similarly, a *right H-comodule algebra* is an algebra A together with a multiplicative right H-coaction $\rho^r : A \to A \otimes H$ satisfying the condition

$$\rho^r(1_A) = (id_A \otimes \varepsilon_t) \circ \rho^r(1_A).$$

We use the notation $\rho^r(a) = a_{(0)} \otimes a_{(1)}$. For the coassociativity of the right comodule, we write $((\rho^r \otimes id_H) \circ \rho^r)(a) = a_{(0)} \otimes a_{(1)} \otimes a_{(2)} = a_{(0)} \otimes a_{(1)1} \otimes a_{(1)2} = ((id_A \otimes \Delta) \circ \rho^r)(a)$, for any $a \in A$.

An *H-bicomodule algebra* is an algebra A, that is an H-bimodule, such that A is a left and a right H-comodule algebra.

3. The affineness criterion for weak Yetter-Drinfel'd modules

Recall form [6], a *weak Yetter-Drinfel'd module* is a right A-module and a left H-comodule $(M, \cdot, \rho_M)$ such that

(1) $\rho_M(m) \in H \boxtimes M = \{h1_{(-1)} \otimes m \cdot 1_{(0)} | h \in H, m \in M\}$,

(2)

$$m_{(-1)}a_{(-1)} \otimes m_{(0)} \cdot a_{(0)} = a_{(1)}(m \cdot a_{(0)})_{(-1)} \otimes (m \cdot a_{(0)})_{(0)}. \tag{3. 1}$$

The category of weak Yetter-Drinfel'd modules and k-linear maps that preserve the A-action and H-coaction is denoted ${}^H\mathcal{YD}_A$.

Lemma 3.1. *Let A be an H-bicomodule algebra. For all $a \in A$, we have*

$$\varepsilon_t(a_{(1)}) \otimes a_{(0)} = 1_{(1)} \otimes 1_{(0)}a, \tag{3. 2}$$

$$\varepsilon_s(a_{(1)}) \otimes a_{(0)} = S(1_{(1)}) \otimes a1_{(0)}, \tag{3. 3}$$

$$1_{(-1)} \otimes 1_{(0)(0)} \otimes 1_{(0)(1)} = 1_{(-1)} \otimes 1_{(0)}1_{(0)} \otimes 1_{(1)}. \tag{3. 4}$$

Proof. Similar to [10]. □

Lemma 3.2. *Let M be a right A-module and a left H-comodule. Then the compatibility relation Eq. (3.1) is equivalent to*

$$\rho_M(m \cdot a) = S^{-1}(a_{(1)})m_{(-1)}a_{(-1)} \otimes m_{(0)} \cdot a_{(0)}. \tag{3. 5}$$

Proof. Assume first that Eq.(3.5) holds. Then for $a \in A, m \in M$,

$$\begin{aligned}
& a_{(1)}(m \cdot a_{(0)})_{(-1)} \otimes (m \cdot a_{(0)})_{(0)} \\
\overset{(3.5)}{=} \ & a_{(2)}S^{-1}(a_{(1)})m_{(-1)}a_{(0)(-1)} \otimes m_{(0)} \cdot a_{(0)(0)} \\
= \ & a_{(1)2}S^{-1}(a_{(1)1})m_{(-1)}a_{(0)(-1)} \otimes m_{(0)} \cdot a_{(0)(0)} \\
= \ & S^{-1}(\varepsilon_t(a_{(1)}))m_{(-1)}a_{(0)(-1)} \otimes m_{(0)} \cdot a_{(0)(0)} \\
\overset{(3.2)}{=} \ & S^{-1}(1_{(1)})m_{(-1)}(1_{(0)}a)_{(-1)} \otimes m_{(0)} \cdot (1_{(0)}a)_{(0)} \\
= \ & S^{-1}(1_{(1)})m_{(-1)}1_{(-1)}a_{(-1)} \otimes m_{(0)} \cdot 1_{(0)}a_{(0)} \\
\overset{(3.5)}{=} \ & m_{(-1)}a_{(-1)} \otimes m_{(0)} \cdot a_{(0)}.
\end{aligned}$$

Conversely, if Eq.(3.1) holds, then

$$\begin{aligned}
& S^{-1}(a_{(1)})m_{(-1)}a_{(-1)} \otimes m_{(0)} \cdot a_{(0)} \\
\overset{(3.1)}{=} \ & S^{-1}(a_{(2)})a_{(1)}(m \cdot a_{(0)})_{(-1)} \otimes (m \cdot a_{(0)})_{(0)} \\
= \ & S^{-1}(a_{(1)2})a_{(1)1}(m \cdot a_{(0)})_{(-1)} \otimes (m \cdot a_{(0)})_{(0)} \\
= \ & S^{-1}(\varepsilon_s(a_{(1)}))(m \cdot a_{(0)})_{(-1)} \otimes (m \cdot a_{(0)})_{(0)} \\
\overset{(3.3)}{=} \ & 1_{(1)}(m \cdot a1_{(0)})_{(-1)} \otimes (m \cdot a1_{(0)})_{(0)} \\
\overset{(3.1)}{=} \ & (m \cdot a)_{(-1)}1_{(-1)} \otimes (m \cdot a)_{(0)} \cdot 1_{(0)} \\
= \ & (m \cdot a)_{(-1)} \otimes (m \cdot a)_{(0)} = \rho_M(m \cdot a).
\end{aligned}$$

□

An important object of ${}^H\mathcal{YD}_A$ is the Verma structure $(A, \cdot, \tilde{\rho})$, where $\cdot$ is the multiplication on A and the left H-coaction $\tilde{\rho}$ is given by

$$\tilde{\rho} : A \to H \otimes A, \quad a \mapsto S^{-1}(a_{(1)})a_{(-1)} \otimes a_{(0)},$$

for all $a \in A$.

Recall from [1], see also [5], that ${}^H\mathcal{YD}_A$ can be viewed as a category of weak Doi-Koppinen modules associated to the weak Doi-Koppinen datum $(H \otimes H^{op}, A, H)$, where

(1) A is a weak left $H \otimes H^{op}$-comodule algebra via

$$a \mapsto (a_{(-1)} \otimes S^{-1}(a_{(1)})) \otimes a_{(0)},$$

for all $a \in A$,

(2) H is a weak right $H \otimes H^{op}$-module coalgebra via

$$g \cdot (h \otimes k) = kgh,$$

for all $g, h, k \in H$. Then ${}^H\mathcal{YD}_A = {}^H\mathcal{M}(H \otimes H^{op})_A$. Here we shall check

$$g \cdot \varepsilon_t(h \otimes k) = \varepsilon(g_1 \cdot (h \otimes k))g_2,$$

which is a condition for being a module coalgebra. Indeed,

$$\begin{aligned}
\varepsilon(g_1 \cdot (h \otimes k))g_2 \quad &= \varepsilon(hg_1k))g_2 \\
&= \varepsilon(\varepsilon_s(h)g_1\varepsilon_t(k))g_2 \\
&\overset{(W3)}{=} \varepsilon(\varepsilon_s(h)(g\varepsilon_t(k))_1)(g\varepsilon_t(k))_2 \\
&\overset{(W9)}{=} \varepsilon((g\varepsilon_t(k))_1)S^{-1}(\varepsilon_s(h))(g\varepsilon_t(k))_2 \\
&= S^{-1}(\varepsilon_s(h))(g\varepsilon_t(k)) \\
&= g \cdot (\varepsilon_t(k) \otimes S^{-1}(\varepsilon_s(h))) \\
&= g \cdot \varepsilon_t(h \otimes k).
\end{aligned}$$

Definition 3.3. Let H be a weak Hopf algebra with a bijective antipode S and A an H-bicomodule algebra. A k-linear map $\gamma : H \to \mathrm{Hom}(H, A)$ is called a *quantum integral*, if

$$g_1 \otimes \gamma(g_2)(h) = S^{-1}((\gamma(g)(h_1))_{(1)})h_2(\gamma(g)(h_1))_{(-1)} \otimes (\gamma(g)(h_1))_{(0)}, \qquad (3.\ 6)$$

for all $g, h \in H$. A quantum integral $\gamma : H \to \mathrm{Hom}(H, A)$ is called total, if

$$\gamma(h_1)(h_2) = \varepsilon(S^{-1}(1_{(1)})h1_{(-1)})1_{(0)}, \qquad (3.\ 7)$$

for all $h \in H$.

Proposition 3.4. *Let H be a weak Hopf algebra with a bijective antipode S and A an H-bicomodule algebra. Assume that there exists a total quantum integral $\gamma : H \to Hom(H, A)$. Then $\tilde{\rho} : A \to H \boxtimes A$ splits in ${}^H\mathcal{YD}_A$.*

Proof. We can prove that the map

$$\lambda : H \boxtimes A \to A,$$

$$S^{-1}(1_{(1)})h1_{(-1)} \otimes a1_{(0)} \mapsto a_{(0)}\gamma(1_1^{'}h1_1)(1_2^{'}S^{-1}(a_{(1)})a_{(-1)}1_2),$$

for all $h \in H, a \in A$, is a left H-colinear retraction of $\tilde{\rho}$. In particular,

$$\lambda(S^{-1}(1_{(1)})1_{(-1)} \otimes 1_{(0)}) = 1.$$

Since

$$\begin{aligned}
& \lambda(S^{-1}(1_{(1)})1_{(-1)} \otimes 1_{(0)}) \\
= \; & 1_{(0)}\gamma(1_1^{'}1_1)(1_2^{'}S^{-1}(1_{(1)})1_{(-1)}1_2) \\
= \; & 1_{(0)}\gamma(1_1^{'}1_1)(1_2^{'}1_2S^{-1}(1_{(1)})1_{(-1)}) \\
= \; & 1_{(0)}\gamma(1_1)(1_2S^{-1}(1_{(1)})1_{(-1)}) \\
= \; & 1_{(0)}\gamma((S^{-1}(1_{(1)})1_{(-1)})_1)((S^{-1}(1_{(1)})1_{(-1)})_2) \\
= \; & 1_{(0)}\varepsilon(S^{-1}(1_{(1)}^{'})S^{-1}(1_{(1)})1_{(-1)}1_{(-1)}^{'})1_{(0)}^{'} \\
= \; & 1_{(0)}\varepsilon(S^{-1}(1_{(1)}1_{(1)}^{'})1_{(-1)}1_{(-1)}^{'})1_{(0)}^{'} \\
= \; & 1_{(0)}\varepsilon(S^{-1}(1_{(1)})1_{(-1)}),
\end{aligned}$$

we have

$$1_{(0)}\varepsilon(S^{-1}(1_{(1)})1_{(-1)}) = 1.$$

Since λ is an H-colinear map, we have

$$\begin{aligned}
& g_1 \otimes \lambda(S^{-1}(1_{(1)})g_21_{(-1)} \otimes a1_{(0)}) \\
= \; & S^{-1}(\lambda(S^{-1}(1_{(1)})g1_{(-1)} \otimes a1_{(0)})_{(1)})\lambda(S^{-1}(1_{(1)})g1_{(-1)} \otimes a1_{(0)})_{(-1)} \\
& \qquad \otimes \lambda(S^{-1}(1_{(1)})g1_{(-1)} \otimes a1_{(0)})_{(0)},
\end{aligned}$$

for all $g \in H$ and $a \in A$. We define now

$$\lambda : H \boxtimes A \to A,$$

$$S^{-1}(1_{(1)})h1_{(-1)} \otimes a1_{(0)} \mapsto \lambda(S^{-1}(1_{(1)})S^{-2}(a_{(1)})hS(a_{(-1)})1_{(-1)} \otimes 1_{(0)})a_{(0)},$$

for all $h \in H, a \in A$. Then, for $a \in A$, we have

$$\begin{aligned}
& (\lambda \circ \tilde{\rho})(a) \\
= \; & \lambda(S^{-1}(a_{(1)})a_{(-1)} \otimes a_{(0)}) \\
= \; & \lambda(S^{-1}(1_{(1)})S^{-2}(a_{(0)(1)})S^{-1}(a_{(1)})a_{(-1)}S(a_{(0)(-1)})1_{(-1)} \otimes 1_{(0)})a_{(0)(0)} \\
= \; & \lambda(S^{-1}(1_{(1)})S^{-2}(a_{(0)(1)1})S^{-1}(a_{(0)(1)2})a_{(-1)}S(a_{(0)(-1)})1_{(-1)} \otimes 1_{(0)})a_{(0)(0)} \\
= \; & \lambda(S^{-1}(1_{(1)})S^{-2}(\varepsilon_t(a_{(0)(1)}))a_{(-1)}S(a_{(0)(-1)})1_{(-1)} \otimes 1_{(0)})a_{(0)(0)}
\end{aligned}$$

$$\begin{aligned}
&\overset{(3.2)}{=} && \lambda(S^{-1}(1_{(1)})S^{-2}(1'_{(1)})a_{(-1)}S((1'_{(0)}a_{(0)})_{(-1)})1_{(-1)}\otimes 1_{(0)})(1'_{(0)}a_{(0)})_{(0)}\\
&= && \lambda(S^{-1}(1_{(1)})S^{-2}(1'_{(1)})a_{(-2)}S(1'_{(-1)}a_{(-1)})1_{(-1)}\otimes 1_{(0)})1'_{(0)}a_{(0)}\\
&= && \lambda(S^{-1}(1_{(1)})S^{-2}(1'_{(1)})a_{(-1)1}S(a_{(-1)2})S(1'_{(-1)})1_{(-1)}\otimes 1_{(0)})1'_{(0)}a_{(0)}\\
&= && \lambda(S^{-1}(1_{(1)})S^{-2}(1'_{(1)})\varepsilon_t(a_{(-1)})S(1'_{(-1)})1_{(-1)}\otimes 1_{(0)})1'_{(0)}a_{(0)}\\
&= && \lambda(S^{-1}(1_{(1)})S^{-2}(1'_{(1)})S(1''_{(-1)})S(1'_{(-1)})1_{(-1)}\otimes 1_{(0)})1'_{(0)}1''_{(0)}a\\
&= && \lambda(S^{-1}(1_{(1)})S^{-2}(1'_{(1)})S(1'_{(-1)}1''_{(-1)})1_{(-1)}\otimes 1_{(0)})1'_{(0)}1''_{(0)}a\\
&= && \lambda(S^{-1}(1_{(1)})S^{-2}(1'_{(1)})S(1'_{(-1)})1_{(-1)}\otimes 1_{(0)})1'_{(0)}a\\
&= && 1_{(0)}\gamma(1'_1S^{-2}(1'_{(1)})S(1'_{(-1)})1_1)(1'_2S^{-1}(1_{(1)})1_{(-1)}1_2)1'_{(0)}a\\
&= && 1_{(0)}\gamma(S^{-2}(1'_{(1)})S(1'_{(-1)})1_1)(S^{-1}(1_{(1)})1_{(-1)}1_2)1'_{(0)}a\\
&= && 1_{(0)}\gamma((S^{-2}(1'_{(1)})S(1'_{(-1)})S^{-1}(1_{(1)})1_{(-1)})_1)\\
& && \qquad\qquad ((S^{-2}(1'_{(1)})S(1'_{(-1)})S^{-1}(1_{(1)})1_{(-1)})_2)1'_{(0)}a\\
&= && 1_{(0)}\varepsilon(S^{-1}(1''_{(1)})S^{-2}(1'_{(1)})S(1'_{(-1)})S^{-1}(1_{(1)})1_{(-1)}1''_{(-1)})1''_{(0)}1'_{(0)}a\\
&= && 1_{(0)}\varepsilon(S^{-1}(1''_{(1)})S^{-1}(1_{(1)})S^{-2}(1'_{(1)})S(1'_{(-1)})1_{(-1)}1''_{(-1)})1''_{(0)}1'_{(0)}a\\
&= && 1_{(0)}\varepsilon(S^{-1}(1_{(1)}1''_{(1)})S^{-2}(1'_{(1)})S(1'_{(-1)})1_{(-1)}1''_{(-1)})1''_{(0)}1'_{(0)}a\\
&= && 1_{(0)}\varepsilon(S^{-1}(1_{(1)})S^{-2}(1'_{(1)})S(1'_{(-1)})1_{(-1)})1'_{(0)}a\\
&= && 1_{(0)}\varepsilon(S^{-2}(1'_{(1)})S^{-1}(1_{(1)})1_{(-1)}S(1'_{(-1)}))1'_{(0)}a\\
&= && 1_{(0)}\varepsilon(\varepsilon_s(\varepsilon_t(S^{-2}(1'_{(1)})))S^{-1}(1_{(1)})1_{(-1)}\varepsilon_t(1'_{(-1)}))1'_{(0)}a\\
&= && 1_{(0)}\varepsilon(S^{-1}(1'_{(1)})S^{-1}(1_{(1)})1_{(-1)}1'_{(-1)})1'_{(0)}a\\
&= && 1_{(0)}\varepsilon(S^{-1}(1_{(1)})1_{(-1)})a=a,
\end{aligned}$$

i.e., λ still is a retraction of $\tilde{\rho}$. Now, for $h\in H, a,b\in A$, we have

$$\begin{aligned}
& && \lambda((S^{-1}(1_{(1)})h1_{(-1)}\otimes a1_{(0)})\cdot b)\\
&= && \lambda(S^{-1}(b_{(1)})hb_{(-1)}\otimes ab_{(0)})\\
&= && \lambda(S^{-1}(1_{(1)})S^{-2}((ab_{(0)})_{(1)})S^{-1}(b_{(1)})hb_{(-1)}\\
& && \qquad\qquad S((ab_{(0)})_{(-1)})1_{(-1)}\otimes 1_{(0)})(ab_{(0)})_{(0)}\\
&= && \lambda(S^{-1}(1_{(1)})S^{-2}(a_{(1)})S^{-2}(b_{(0)(1)})S^{-1}(b_{(1)})hb_{(-1)}\\
& && \qquad\qquad S(b_{(0)(-1)})S(a_{(-1)})1_{(-1)}\otimes 1_{(0)})a_{(0)}b_{(0)(0)}\\
&= && \lambda(S^{-1}(1_{(1)})S^{-2}(a_{(1)})S^{-2}(b_{(0)(1)1})S^{-1}(b_{(0)(1)2})hb_{(-1)}\\
& && \qquad\qquad S(b_{(0)(-1)})S(a_{(-1)})1_{(-1)}\otimes 1_{(0)})a_{(0)}b_{(0)(0)}\\
&= && \lambda(S^{-1}(1_{(1)})S^{-2}(a_{(1)})S^{-2}(\varepsilon_t(b_{(0)(1)}))hb_{(-1)}S(b_{(0)(-1)})
\end{aligned}$$

$$
\begin{aligned}
&\qquad S(a_{(-1)})1_{(-1)}\otimes 1_{(0)})a_{(0)}b_{(0)(0)}\\
=\;& \lambda(S^{-1}(1_{(1)})S^{-2}(a_{(1)})S^{-2}(1'_{(1)})hb_{(-1)}S((1'_{(0)}b_{(0)})_{(-1)})\\
&\qquad S(a_{(-1)})1_{(-1)}\otimes 1_{(0)})a_{(0)}(1'_{(0)}b_{(0)})_{(0)}\\
=\;& \lambda(S^{-1}(1_{(1)})S^{-2}(a_{(1)})S^{-2}(1'_{(1)})hb_{(-1)1}S(b_{(-1)2})S(1'_{(-1)})\\
&\qquad S(a_{(-1)})1_{(-1)}\otimes 1_{(0)})a_{(0)}1'_{(0)}b_{(0)}\\
=\;& \lambda(S^{-1}(1_{(1)})S^{-2}(a_{(1)})S^{-2}(1'_{(1)})h\varepsilon_t(b_{(-1)})S(1'_{(-1)})\\
&\qquad S(a_{(-1)})1_{(-1)}\otimes 1_{(0)})a_{(0)}1'_{(0)}b_{(0)}\\
=\;& \lambda(S^{-1}(1_{(1)})S^{-2}(a_{(1)})S^{-2}(1'_{(1)})hS(1''_{(-1)})S(1'_{(-1)})S(a_{(-1)})\\
&\qquad 1_{(-1)}\otimes 1_{(0)})a_{(0)}1'_{(0)}1''_{(0)}b\\
=\;& \lambda(S^{-1}(1_{(1)})S^{-2}(a_{(1)})hS(1''_{(-1)})S(a_{(-1)})1_{(-1)}\otimes 1_{(0)})a_{(0)}1''_{(0)}b\\
=\;& \lambda(S^{-1}(1_{(1)})S^{-2}(a_{(1)})hS(a_{(-1)})1_{(-1)}\otimes 1_{(0)})a_{(0)}b\\
=\;& \lambda(S^{-1}(1_{(1)})h1_{(-1)}\otimes a1_{(0)})b,
\end{aligned}
$$

hence λ is right A-linear. It remains to prove that λ is also left H-colinear.

$$
\begin{aligned}
&\tilde\rho\circ\lambda(S^{-1}(1_{(1)})h1_{(-1)}\otimes a1_{(0)})\\
=\;& \tilde\rho(\lambda(S^{-1}(1_{(1)})S^{-2}(a_{(1)})hS(a_{(-1)})1_{(-1)}\otimes 1_{(0)})a_{(0)})\\
=\;& S^{-1}(\lambda(S^{-1}(1_{(1)})S^{-2}(a_{(1)})hS(a_{(-1)})1_{(-1)}\otimes 1_{(0)})a_{(0)})_{(1)}\\
& \lambda(S^{-1}(1_{(1)})S^{-2}(a_{(1)})hS(a_{(-1)})1_{(-1)}\otimes 1_{(0)})a_{(0)})_{(-1)}\\
& \otimes\lambda(S^{-1}(1_{(1)})S^{-2}(a_{(1)})hS(a_{(-1)})1_{(-1)}\otimes 1_{(0)})a_{(0)})_{(0)}\\
=\;& S^{-1}(a_{(1)})S^{-1}(\lambda(S^{-1}(1_{(1)})S^{-2}(a_{(2)})hS(a_{(-2)})1_{(-1)}\otimes 1_{(0)})_{(1)})\\
& \lambda(S^{-1}(1_{(1)})S^{-2}(a_{(2)})hS(a_{(-2)})1_{(-1)}\otimes 1_{(0)})_{(-1)}a_{(-1)}\\
& \otimes\lambda(S^{-1}(1_{(1)})S^{-2}(a_{(2)})hS(a_{(-2)})1_{(-1)}\otimes 1_{(0)})_{(0)}a_{(0)}\\
=\;& S^{-1}(a_{(1)})S^{-2}(a_{(2)1})h_1S(a_{(-2)2})a_{(-1)}\\
& \otimes\lambda(S^{-1}(1_{(1)})S^{-2}(a_{(2)2})h_2S(a_{(-2)1})1_{(-1)}\otimes 1_{(0)})a_{(0)}\\
=\;& S^{-1}(a_{(1)1})S^{-2}(a_{(1)2})h_1S(a_{(-2)2})a_{(-1)}\\
& \otimes\lambda(S^{-1}(1_{(1)})S^{-2}(a_{(1)3})h_2S(a_{(-2)1})1_{(-1)}\otimes 1_{(0)})a_{(0)}\\
=\;& S^{-2}(\varepsilon_s(a_{(1)1})h_1S(a_{(-2)2})a_{(-1)}\\
& \otimes\lambda(S^{-1}(1_{(1)})S^{-2}(a_{(1)2})h_2S(a_{(-2)1})1_{(-1)}\otimes 1_{(0)})a_{(0)}\\
=\;& S^{-2}(1_1)h_1S(a_{(-2)2})a_{(-1)}\\
& \otimes\lambda(S^{-1}(1_{(1)})S^{-2}(a_{(1)}1_2)h_2S(a_{(-2)1})1_{(-1)}\otimes 1_{(0)})a_{(0)}
\end{aligned}
$$

$$\begin{aligned}
&= h_1 S(a_{(-2)2})a_{(-1)} \\
&\quad \otimes\lambda(S^{-1}(1_{(1)})S^{-2}(a_{(1)})h_2 S(a_{(-2)1})1_{(-1)}\otimes 1_{(0)})a_{(0)} \\
&= h_1 S(a_{(-1)2})a_{(-1)3}\otimes\lambda(S^{-1}(1_{(1)})S^{-2}(a_{(1)})h_2 S(a_{(-1)1})1_{(-1)}\otimes 1_{(0)})a_{(0)} \\
&= h_1\varepsilon_s(a_{(-1)2})\otimes\lambda(S^{-1}(1_{(1)})S^{-2}(a_{(1)})h_2 S(a_{(-1)1})1_{(-1)}\otimes 1_{(0)})a_{(0)} \\
&= h_1 S(1_2)\otimes\lambda(S^{-1}(1_{(1)})S^{-2}(a_{(1)})h_2 S(a_{(-1)}1_1)1_{(-1)}\otimes 1_{(0)})a_{(0)} \\
&= h_1\otimes\lambda(S^{-1}(1_{(1)})S^{-2}(a_{(1)})h_2 S(a_{(-1)})1_{(-1)}\otimes 1_{(0)})a_{(0)} \\
&= (id\otimes\lambda)\rho_{H\boxtimes A}(S^{-1}(1_{(1)})h1_{(-1)}\otimes a1_{(0)}),
\end{aligned}$$

i.e., it is proved that λ is a retraction of $\tilde{\rho}$ in ${}^H\mathcal{YD}_A$. □

We can define the coinvariants of A as

$$\begin{aligned}
B = A^{coH} &= \{a\in A|\tilde{\rho}(a) = S^{-1}(1_{(1)})1_{(-1)}\otimes a1_{(0)}\} \\
&= \{a\in A|S^{-1}(a_{(1)})a_{(-1)}\otimes a_{(0)} = S^{-1}(1_{(1)})1_{(-1)}\otimes a1_{(0)}\},
\end{aligned}$$

then B is a subalgebra of A and will be called the *subalgebra of quantum coinvariants.*

Now, we will construct functors connecting ${}^H\mathcal{YD}_A$ and $\mathcal{M}_B$. First, if $M\in {}^H\mathcal{YD}_A$, then the coinvariants of M

$$M^{coH} = \{m\in M|m_{(-1)}\otimes m_{(0)} = S^{-1}(1_{(1)})1_{(-1)}\otimes m\cdot 1_{(0)}\}$$

is a right B-module. Furthermore, we have a covariant functor

$$(-)^{coH} : {}^H\mathcal{YD}_A\to\mathcal{M}_B.$$

Now, for $N\in\mathcal{M}_B,\ N\otimes_B A\in {}^H\mathcal{YD}_A$ via the structures

$$(n\otimes_B a)\cdot a' = n\otimes_B aa',$$

$$\rho_{N\otimes_B A}(n\otimes_B a) = S^{-1}(a_{(1)})a_{(-1)}\otimes n\otimes_B a_{(0)},$$

for all $n\in N, a, a'\in A$. In this way, we have constructed a covariant functor called the *induction functor*

$$-\otimes_B A : \mathcal{M}_B\to {}^H\mathcal{YD}_A.$$

We now prove that the above functors are an adjoint pair.

Proposition 3.5. *Let H be a weak Hopf algebra with a bijective antipode S and A an H-bicomodule algebra. Then the induction functor $-\otimes_B A : \mathcal{M}_B\to {}^H\mathcal{YD}_A$ is a left adjoint of the coinvariant functor $(-)^{coH} : {}^H\mathcal{YD}_A\to\mathcal{M}_B$.*

Proof. The unit and the counit of the adjointness are given by

$$\eta_N : N \to (N \otimes_B A)^{coH}, \quad n \mapsto n \otimes_B 1_A,$$

for all $N \in \mathcal{M}_B, n \in N$, and

$$\beta_M : M^{coH} \otimes_B A \to M, \quad m \otimes_B a \mapsto ma,$$

for all $M \in {}^H\mathcal{YD}_A,\ m \in M^{coH}$ and $a \in A$. □

Let A be an H-bicomodule algebra. Notice that

$$H \boxtimes A = \{S^{-1}(1_{(1)})h1_{(-1)} \otimes b1_{(0)} | h \in H,\ a \in A\} \in {}^H\mathcal{YD}_A$$

via the following structures

$$(S^{-1}(1_{(1)})h1_{(-1)} \otimes b1_{(0)}) \cdot a = S^{-1}(a_{(1)})ha_{(-1)} \otimes ba_{(0)}, \tag{3. 8}$$

$$\rho_{H\boxtimes A}(S^{-1}(1_{(1)})h1_{(-1)} \otimes b1_{(0)}) = h_1 \otimes S^{-1}(1_{(1)})h_2 1_{(-1)} \otimes b1_{(0)}. \tag{3. 9}$$

Lemma 3.6. *Let A be an H-bicomodule algebra. Then*

$$(H \boxtimes A)^{coH} \cong A.$$

Proof. We can construct the desired map as follows

$$\theta : A \to (H \boxtimes A)^{coH}, \quad a \mapsto S^{-1}(1_{(1)})1_{(-1)} \otimes a1_{(0)}.$$

Notice that $\theta(a) \in (H \boxtimes A)^{coH}$, we check it as follows

$$\begin{aligned}
\rho_{H\boxtimes A}(\theta(a)) &= \rho_{H\boxtimes A}(S^{-1}(1_{(1)})1_{(-1)} \otimes a1_{(0)}) \\
&= 1_1 \otimes S^{-1}(1_{(1)})1_2 1_{(-1)} \otimes a1_{(0)},
\end{aligned}$$

and

$$\begin{aligned}
& S^{-1}(1'_{(1)})1'_{(-1)} \otimes (S^{-1}(1_{(1)})1_{(-1)} \otimes a1_{(0)}) \cdot 1'_{(0)} \\
=\ & S^{-1}(1'_{(1)})1'_{(-1)} \otimes S^{-1}(1'_{(0)(1)})1'_{(0)(-1)} \otimes a1'_{(0)(0)} \\
=\ & S^{-1}(1'_2)1_1 \otimes S^{-1}(1'_1 1_{(1)})1_2 1_{(-1)} \otimes a1_{(0)} \\
=\ & 1_1 \otimes S^{-1}(1_{(1)})1_2 1_{(-1)} \otimes a1_{(0)} \\
=\ & \rho_{H\boxtimes A}(\theta(a)).
\end{aligned}$$

□

From Lemma 3.6 it follows that the adjunction map $\beta_{H\boxtimes A}$ can be viewed as a map in ${}^H\mathcal{YD}_A$ via

$$\beta_{H\boxtimes A} : A \otimes_B A \to H \boxtimes A, \quad a \otimes_B b \mapsto S^{-1}(b_{(1)})b_{(-1)} \otimes ab_{(0)},$$

for all $a, b \in A$. Here $A \otimes_B A \in^H \mathcal{YD}_A$ via the structures

$$(a \otimes_B b) \cdot a' = a \otimes_B ba',$$
$$\rho_{A \otimes_B A}(a \otimes_B b) = S^{-1}(b_{(1)})b_{(-1)} \otimes a \otimes_B b_{(0)},$$

for all $a, b, a' \in A$.

We will now prove the main result of this section, that is, the affineness criterion for weak Yetter-Drinfel'd modules.

Theorem 3.7. *Let H be a weak Hopf algebra with a bijective antipode S, A an H-bicomodule algebra, and $B = A^{coH}$. Assume that*
(1) there exists a total quantum integral $\gamma : H \to Hom(H, A)$;
(2) the canonical map

$$\beta : A \otimes_B A \to H \boxtimes A, \quad a \otimes_B b \mapsto S^{-1}(b_{(1)})b_{(-1)} \otimes ab_{(0)}$$

is surjective. Then the induction functor $- \otimes_B A : \mathcal{M}_B \to {}^H\mathcal{YD}_A$ is an equivalence of categories.

Proof. In Proposition 3.5 we have shown that the adjunction map $\eta_N : N \to (N \otimes_B A)^{coH}$ is an isomorphism for all $N \in \mathcal{M}_B$ under the assumption (1). It remains to prove that the other adjunction map, namely $\beta_M : M^{coH} \otimes_B A \to M$ is also an isomorphism for all $M \in {}^H\mathcal{YD}_A$.

Let V be a k-module. Then $A \otimes V \in {}^H\mathcal{YD}_A$ via the structures induced by A, i.e.,

$$(a \otimes v) \cdot b = ab \otimes v,$$
$$\rho_{A \otimes V}(a \otimes v) = S^{-1}(a_{(1)})a_{(-1)} \otimes a_{(0)} \otimes v,$$

for all $a, b \in A$ and $v \in V$. In particular, for $V = A$, $A \otimes A \in {}^H\mathcal{YD}_A$ via

$$(a \otimes a') \cdot b = ab \otimes v, \tag{3. 10}$$
$$\rho_{A \otimes A}(a \otimes a') = S^{-1}(a_{(1)})a_{(-1)} \otimes a_{(0)} \otimes a', \tag{3. 11}$$

for all $a, b, a' \in A$. We will first prove that the adjunction map $\beta_{A \otimes V} : (A \otimes V)^{coH} \otimes_B A \to A \otimes V$ is an isomorphism for any k-module V.

First, $V \otimes B$ and $B \otimes V \in \mathcal{M}_B$ via the usual B-actions $(v \otimes a) \cdot b = v \otimes ab$, and $a' \cdot (b' \otimes v') = a'b' \otimes v'$ for all $a, b, a', b' \in B$ and $v, v' \in V$. The flip map $\tau : V \otimes B \to B \otimes V$, $\tau(v \otimes b) = b \otimes v$, for all $b \in B$ and $v \in V$, is an isomorphism in $\mathcal{M}_B$. On the other hand $V \otimes A \in {}^H\mathcal{YD}_A$ via the structures induced by A, i.e.

$$(v \otimes a) \cdot b = v \otimes ab, \tag{3. 12}$$
$$\rho_{V \otimes A}(v \otimes a) = S^{-1}(a_{(1)})a_{(-1)} \otimes v \otimes a_{(0)}, \tag{3. 13}$$

It is easy to see that the flip map $\tau : A\otimes V \to V\otimes A,\ \tau(a\otimes v) = v\otimes a$ is an isomorphism in ${}^H\mathcal{YD}_A$.

Applying Proposition 3.5 for $N = V\otimes B \cong B\otimes V$, we obtain the following isomorphisms in $\mathcal{M}_B$:

$$B\otimes V \cong V\otimes B \cong (V\otimes B\otimes_B A)^{coH} \cong (V\otimes A)^{coH} \cong (A\otimes V)^{coH}.$$

Hence, $(A\otimes V)^{coH}\otimes_B A \cong A\otimes V$.

Let us define

$$\widetilde{\beta} : A\otimes_B A \to H\boxtimes A,\quad a\otimes_B b \mapsto S^{-1}(b_{(1)})b_{(-1)}\otimes ab_{(0)},$$

for all $a, b \in A$. As β is surjective, $\widetilde{\beta}$ is surjective, because $\widetilde{\beta} = \beta\circ can$, where $can : A\otimes A \to A\otimes_B A$ is the canonical surjection.

Let us define now

$$\xi : A\otimes A \to H\boxtimes A,\ \xi(a\otimes b) = (\widetilde{\beta}\circ\tau)(a\otimes b) = S^{-1}(a_{(1)})a_{(-1)}\otimes ba_{(0)}$$

for all $a, b\in A$. The map ξ is surjective, as $\widetilde{\beta}$ and τ are. We will prove that ξ is a morphism in ${}^H\mathcal{YD}_A$. where $A\otimes A$ and $H\boxtimes A$ are weak Yetter-Drinfel'd modules via Eq.(3.8), Eq.(3.9), and Eq.(3.10), Eq.(3.11), respectively. Indeed,

$$\begin{aligned}
\xi((a\otimes b)c) = \xi(ac\otimes b) &= S^{-1}(c_{(1)})S^{-1}(a_{(1)})a_{(-1)}c_{(-1)}\otimes ba_{(0)}c_{(0)}\\
&= (S^{-1}(a_{(1)})a_{(-1)}\otimes ba_{(0)})c\\
&= \xi(a\otimes b)c,
\end{aligned}$$

and

$$\begin{aligned}
\rho_{H\boxtimes A}\xi(a\otimes b) &= \rho_{H\boxtimes A}(S^{-1}(a_{(1)})a_{(-1)}\otimes ba_{(0)})\\
&= S^{-1}(a_{(1)2})a_{(-1)1}\otimes S^{-1}(a_{(1)1})a_{(-1)2}\otimes ba_{(0)}\\
&= (id\otimes\xi)(S^{-1}(a_{(1)})a_{(-1)}\otimes a_{(0)}\otimes b)\\
&= (id\otimes\xi)\rho_{A\otimes A}(a\otimes b).
\end{aligned}$$

Hence, ξ is a surjective morphism in ${}^H\mathcal{YD}_A$. $H\boxtimes A$ is projective as a right A-module, where $H\boxtimes A$ is a right A-module in the usual way, i.e. $(h\boxtimes a)b = (S^{-1}(1_{(1)})h1_{(-1)}\otimes a1_{(0)})b = S^{-1}(1_{(1)})h1_{(-1)}\otimes ab1_{(0)} = h\boxtimes ab$, for all $h\in H$ and $a, b\in A$. On the other hand, the map

$$u : H\boxtimes A \to A\boxtimes H,\quad h\otimes a \mapsto S^{-1}(a_{(1)})ha_{(-1)}\otimes a_{(0)}$$

is a splitting surjection of right A-module, where the first $H\boxtimes A$ has the usual right A-module structure and the second $H\boxtimes A$ has the right A-module given in

Eq.(3.10). The right inverse of u is given by

$$v : H \boxtimes A \to A \boxtimes H, \quad h \otimes a \mapsto S^{-2}(a_{(1)})hS(a_{(-1)}) \otimes a_{(0)}.$$

Hence, we can view the second $H \boxtimes A$ as a right A-module direct summand of the first $H \boxtimes A$. So we obtain that $H \boxtimes A$, with the right A-module structure given in Eq.(3.10), is still projective as a right A-module. It follows that there exits $\zeta : H \boxtimes A \to A \otimes A$ such that $\xi \circ \zeta = id_{H\boxtimes A}$ since $A \otimes A \to H \boxtimes A$ is surjective. Hence, ξ splits in the category of right A-modules. In particular ξ is a k-split epimorphism in ${}^H\mathcal{YD}_A$.

Let now $M \in {}^H\mathcal{YD}_A$. Then $A \otimes A \otimes M \in {}^H\mathcal{YD}_A$ via the structures arising from $A \otimes A$, that is,

$$(a \otimes b \otimes m) \cdot c = ac \otimes b \otimes m;$$
$$\rho_{A\otimes A\otimes M}(a \otimes b \otimes m) = S^{-1}(a_{(1)})a_{(-1)} \otimes a_{(0)} \otimes b \otimes m,$$

for all $a, b, c \in A$ and $m \in M$. On the other hand, $H \boxtimes A \otimes M \in {}^H\mathcal{YD}_A$ via the structures arising from $H \boxtimes A$, that is,

$$(h \boxtimes a \otimes m) \cdot b = S^{-1}(b_{(1)})hb_{(-1)} \otimes ab_{(0)} \otimes m;$$
$$\rho_{H\boxtimes A\otimes M}(h \boxtimes a \otimes m) = h_1 \otimes h_2 \otimes a \otimes m,$$

for all $a, b \in A, h \in H$ and $m \in M$. We obtain that

$$\xi \otimes id_M : A \otimes A \otimes M \to H \boxtimes A \otimes M$$

is a k-split epimorphism in ${}^H\mathcal{YD}_A$.

Applying ${}^H\mathcal{YD}_A = {}^H\mathcal{M}(H \otimes H^{op})_A$, we obtain that the map

$$f : H \boxtimes A \otimes M \to M, \quad h \boxtimes a \otimes m \mapsto m_{(0)}\gamma(S^{-2}(a_{(1)})hS(a_{(-1)}))(m_{(-1)})a_{(0)}$$

is a k-split epimorphism in ${}^H\mathcal{YD}_A$. Hence, the composition

$$g = f \circ (\xi \otimes id_M) : A \otimes A \otimes M \to M,$$

$$a \otimes b \otimes m \mapsto m_{(0)}\gamma(S^{-2}(b_{(1)})S(b_{(-1)}))(m_{(-1)})b_{(0)}a$$

is a k-split epimorphism in ${}^H\mathcal{YD}_A$. We note that the structure of $A \otimes A \otimes M$ as an object in ${}^H\mathcal{YD}_A$ is of the form $A \otimes V$, for the k-module $V = A \otimes M$.

To conclude, we have constructed a k-split epimorphism in ${}^H\mathcal{YD}_A$

$$A \otimes A \otimes M = M_1 \xrightarrow{g} M \longrightarrow 0$$

such that the adjunction map β_{M_1} for M_1 is bijective. As g is k-split and there exists a total quantum integral $\gamma : H \to Hom(H, A)$, we obtain that g also splits in ${}^H\mathcal{M}$. In particular, the sequence

$$M_1^{coH} \xrightarrow{g^{coH}} M^{coH} \longrightarrow 0$$

is exact. Continuing the resolution with $Ker(g)$ instead of M, we obtain an exact sequence in ${}^H\mathcal{YD}_A$

$$M_2 \longrightarrow M_1 \longrightarrow M \longrightarrow 0$$

which splits in ${}^H\mathcal{M}$ and the adjunction maps for M_1 and M_2 are bijective. Using the Five lemma we obtain that the adjunction map for M is bijective. $\square$

Finally we consider a special case. In this case if setting $A = H$, then A is an H-bicomodule algebra in a natural way. And we can define the coinvariants of H as

$$\begin{aligned} B = H^{coH} &= \{h \in H | \tilde{\rho}(h) = S^{-1}(1_3)1_1 \otimes a1_2\} \\ &= \{h \in H | S^{-1}(h_3)h_1 \otimes h_{(2)} = S^{-1}(1_3)1_1 \otimes h1_2\}, \end{aligned}$$

then B is a subalgebra of H. Hence we can obtain the following result.

Corollary 3.8. *Let H be a weak Hopf algebra with a bijective antipode S, $B = H^{coH}$. Assume that*
(1) there exists a total quantum integral $\gamma : H \to Hom(H, H)$;
(2) the canonical map

$$\beta : H \otimes_B H \to H \boxtimes H, \quad h \otimes_B g \mapsto S^{-1}(g_3)g_1 \otimes hg_2$$

is surjective. Then the induction functor $-\otimes_B H : \mathcal{M}_B \to {}^H\mathcal{YD}_H$ is an equivalence of categories.

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