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EXTENSIONS OF GENERALIZED n-LIKE RINGS

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Abstract. Let n be a fixed positive integer. A ring R is a J-n-like ring provided that $(a-a^n)(b-b^n) \in J(R)$ for all a, b in R. If $(a-a^n)(b-b^n) \in P(R)$ for all $a, b \in R$, then R is called a *P-n-like ring*. If $R = \mathbb{Z}_{(p)} \cap \mathbb{Z}_{(q)} = \{\frac{m}{n} \in \mathbb{Q} \mid m, n \in \mathbb{Z}, p \nmid n, q \nmid n\}$, where p, q are distinct prime integers, then it is shown that R is a J-$\big((p-1)(q-1)+1\big)$-like ring. R is a P-n-like ring if and only if R is a J-n-like ring and $J(R) = P(R)$. Also, if $n = p_1^{r_1} p_2^{r_2} \cdots p_s^{r_s}$, where $p_1, p_2, \cdots, p_s$ are distinct primes, and $r_1, r_2, \cdots, r_s \in \mathbb{N}$, then we prove that $\mathbb{Z}_n$ is a P-$\big(\prod_{i=1}^{s}(p_i-1)+1\big)$-like ring.

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1. Introduction

In [1], Foster introduced the concept of Boolean-like rings. Later, Yaqub concerned with a new class of rings which are called p-like rings ([2]). In fact, Boolean-like rings are easily seen to reduced to 2-like rings. Then, Yaqub extended p-like rings to n-like rings ([3]), where n is assumed to be any integer ($n > 1$), but not necessarily prime. In 1980, Moore continued to introduce generalized n-like rings, as an extension of n-like rings ([4]). In 1981, Tominaga and Yaqub characterized generalized n-like rings ([5]). After that, Yasuyuki Hirano and Takashi Suenaga also investigated generalized n-like rings and their properties ([6]).

In this paper, we introduce J-n-like rings and P-n-like rings, so as to extensions of these preceding rings. Let n be a fixed integer, a ring R is a *J-n-like ring* provided that $(a-a^n)(b-b^n) \in J(R)$ for all $a, b \in J(R)$, and if $(a-a^n)(b-b^n) \in P(R)$, then R is called a *P-n-like ring*. We show if $R = \mathbb{Z}_{(p)} \cap \mathbb{Z}_{(q)} = \{\frac{m}{n} \in \mathbb{Q} \mid m, n \in \mathbb{Z}, p \nmid n, q \nmid n\}$ where p, q are distinct primes, then R is a J-$\big((p-1)(q-1)+1\big)$-like ring. Furthermore, R is a P-n-like ring if and only if R is a J-n-like ring and $P(R) = J(R)$. In particular, let $n = p_1^{r_1} p_2^{r_2} \cdots p_s^{r_s}$ where $p_1, p_2, \cdots, p_s$

are distinct primes, and $r_1, r_2, \cdots, r_s \in \mathbb{N}$. Then $\mathbb{Z}_n$ is a P-$\big(\prod_{i=1}^{s}(p_i - 1) + 1\big)$-like ring. Some related results are also obtained.

In what follows $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_n$, $R[[x]]$, $P(R)$, $Nil(R)$, $U(R)$ and $J(R)$ denote the natural numbers, integers, rational numbers, the ring of integers modulo n, the power series ring over a ring R, the prime radical, the set of nilpotent elements, the set of all invertible elements and the Jacobson radical of R, respectively.

2. J-n-like rings

We begin this section with the following definition.

Definition 2.1. Let n be a fixed integer. A ring R is called a *J-n-like ring* provided that $(a - a^n)(b - b^n) \in J(R)$ for all a, b in R.

Recall that a ring R is called a *generalized n-like ring* if R satisfies the polynomial identity $(xy)^n - xy^n - x^n y + xy = 0$ for an integer $n > 1$. It is well known that R is a generalized n-like ring if and only if R satisfies the polynomial identities $(xy)^n = x^n y^n$ and $(x - x^n)(y - y^n) = 0$ (see [8, Lemma 3]). Clearly, every generalized n-like ring is a J-n-like ring. On the other hand if R is J-n-like, then R need not be generalized n-like ring (see Example 2.17).

Proposition 2.2. *If R is a J-n-like ring, then $R/J(R)$ is commutative.*

Proof. Let $x, y \in R$. Since R is a J-n-like ring, we have $(x - x^n)(y - y^n) \in J(R)$; that is, in $R/J(R)$, we get $\overline{(x - x^n)(y - y^n)} = \overline{0}$. Since $R/J(R)$ is semi-primitive, similar to the proof of [8, Lemma 1(3)], we see that $R/J(R)$ is commutative. □

Corollary 2.3. *Let R be a ring. Then R is a J-n-like ring if and only if $R/J(R)$ is a generalized n-like ring.*

Proof. Assume that R is a J-n-like ring. Then $R/J(R)$ is commutative by Proposition 2.2. So R satisfies the polynomial identities $(\overline{a}\overline{b})^n = \overline{a}^n \overline{b}^n$ and $(\overline{a} - \overline{a}^n)(\overline{b} - \overline{b}^n) = \overline{0}$. Conversely, suppose that $R/J(R)$ is a generalized n-like ring. Then for every element a, b in R, we get $(\overline{a} - \overline{a}^n)(\overline{b} - \overline{b}^n) = \overline{0}$. Therefore $(a - a^n)(b - b^n) \in J(R)$. We readily obtain R is a J-n-like ring. The proof is completed. □

Proposition 2.4. *A ring R is J-n-like if and only if $R[[x]]$ is J-n-like.*

Proof. It is clear by Corollary 2.3 because $R/J(R) \cong R[[x]]/J\big(R[[x]]\big)$. □

Corollary 2.5. *If R is a J-n-like ring, then $Nil(R) \subseteq J(R)$.*

Proof. Let $a \in Nil(R)$ with $a^m = 0$ for some integer m. Then we obtain $\overline{a^m} = \overline{0}$ in $R/J(R)$. It follows, by Proposition 2.2, $\overline{a} \in Nil(R/J(R)) \subseteq J(R/J(R)) = 0$, so $a \in J(R)$. Hence $Nil(R) \subseteq J(R)$. □

Let $J^{\#}(R)$ denote the subset $\{x \in R \mid \exists\ n \in \mathbb{N}$ such that $x^n \in J(R)\}$ of R. The *commutant* of $a \in R$ is defined by $comm(a) = \{x \in R \mid xa = ax\}$. If $R^{qnil} = \{a \in R \mid 1 + ax \in U(R)$ for every $x \in comm(a)\}$ and $a \in R^{qnil}$, then a is said to be *quasinilpotent* [3]. It is obvious that $J(R) \subseteq J^{\#}(R) \subseteq R^{qnil}$ and $Nil(R) \subseteq J^{\#}(R) \subseteq R^{qnil}$.

Lemma 2.6. *If R is a J-n-like ring, then $J^{\#}(R) = R^{qnil} = \{a \in R \mid a^2 \in J(R)\}$.*

Proof. Let $a \in R^{qnil}$. By assumption, we have $(a - a^n)^2 = a^2(1 - a^{n-1})^2 \in J(R)$. Since a is quasinilpotent, we get $1 - a^{n-1} \in U(R)$ and so $a^2 \in J(R)$. This implies that $J^{\#}(R) = R^{qnil} = \{a \in R \mid a^2 \in J(R)\}$. □

Remark 2.7. *By Corollary 2.5 and Lemma 2.6, if R is a J-n-like ring, then we have $Nil(R) \subseteq J(R) \subseteq J^{\#}(R) = R^{qnil}$.*

Proposition 2.8. *A ring R is J-n-like if and only if eRe is J-n-like for all idempotent $e \in R$.*

Proof. Let $eae, ebe \in eRe$. Then $(eae-(eae)^n)(ebe-(ebe)^n) = e[(eae-(eae)^n)(ebe-(ebe)^n)]e$. Since R is a J-n-like ring and $J(R)$ is an ideal, we have $(eae-(eae)^n)(ebe-(ebe)^n) \in J(R)$, therefore $e[(eae-(eae)^n)(ebe-(ebe)^n)]e \in eJ(R)e = J(eRe)$; that is, $(eae-(eae)^n)(ebe-(ebe)^n) \in J(eRe)$. Thus the ring eRe is also a J-n-like ring. The converse is trivial. □

Lemma 2.9. *If R_1, R_2 are two J-n-like rings, then $R = R_1 \oplus R_2$ is a J-n-like ring.*

Proof. Let $(a, b), (c, d) \in R_1 \oplus R_2$. Then

$$((a, b) - (a, b)^n)((c, d) - (c, d)^n) = ((a - a^n)(c - c^n), (b - b^n)(d - d^n)).$$

Since R_1, R_2 are J-n-like rings, we get $(a - a^n)(c - c^n) \in J(R_1), (b - b^n)(d - d^n) \in J(R_2)$, therefore

$$((a - a^n)(c - c^n), (b - b^n)(d - d^n)) \in J(R_1) \oplus J(R_2) = J(R_1 \oplus R_2),$$

so $((a, b) - (a, b)^n)((c, d) - (c, d)^n) \in J(R_1 \oplus R_2)$, thus $R = R_1 \oplus R_2$ is a J-n-like ring. We complete the proof. □

Proposition 2.10. *Every finite direct product of J-n-like rings is also a J-n-like ring.*

Proof. As $J(R_1) \oplus J(R_2) \oplus \cdots \oplus J(R_n) = J(R_1 \oplus R_2 \oplus \cdots \oplus R_n)$, by a similar discussion of Lemma 2.9 and inductive method, we can easily get the conclusion. $\square$

Lemma 2.11. *Suppose R_1, R_2 are two J-n-like rings, then the subdirect product R of R_1, R_2 is also a J-n-like ring.*

Proof. Let $\varphi : R \to R_1$, $\psi : R \to R_2$ be epimorphisms given by hypothesis. Then $R/ker(\varphi) \cong R_1$, $R/ker(\psi) \cong R_2$. Let $I = ker(\varphi)$, $K = ker(\psi)$. For every element a, b in R, we have $\overline{(a - a^n)(b - b^n)} \in J(R/I)$. Hence for any $r \in R$, $\overline{1 - (a - a^n)(b - b^n)r} \in U(R/I)$ then there exists s in R such that

$$1 - (1 - (a - a^n)(b - b^n)r)s \in I. \tag{1}$$

Similarly, there is $\overline{(a - a^n)(b - b^n)} \in J(R/K)$ in R/K, then $\overline{1 - (a - a^n)(b - b^n)r} \in U(R/K)$, and so we have an element t in R such that

$$1 - (1 - (a - a^n)(b - b^n)r)t \in K. \tag{2}$$

Multiply (1) by (2), we get $(1-(1-(a-a^n)(b-b^n)r)s)(1-(1-(a-a^n)(b-b^n)r)t) \in IK \subseteq I \cap K$. Since R is the subdirect product of R_1, R_2, we have $I \cap K = 0$. Clearly, $1-(1-(a-a^n)(b-b^n)r)d = 0$ for a d in R. We infer that $(a-a^n)(b-b^n) \in J(R)$, so R is also a J-n-like ring. $\square$

Proposition 2.12. *Every finite subdirect product of J-n-like rings is also a J-n-like ring.*

Proof. It is obvious by Lemma 2.11. $\square$

Recall that an ideal of a ring is said to be a *nil*, if each of its elements is nilpotent.

Theorem 2.13. *Let I be a nil ideal of R. Then R is a J-n-like ring if and only if R/I is a J-n-like ring.*

Proof. $\Rightarrow$: Let $\varphi : R \to R/I$ denote the natural epimorphism, $\varphi(J(R)) \subseteq J(R/I)$. Let $\overline{x}, \overline{y} \in R/I$, since R is a J-n-like ring, then $(x - x^n)(y - y^n) \in J(R)$, therefore $\overline{(x - x^n)(y - y^n)} \in \varphi(J(R)) \subseteq J(R/I)$. Accordingly R/I is a J-n-like ring.

$\Leftarrow$: For any $a, b \in R$, since R/I is a J-n-like ring, then $\overline{(a - a^n)(a - b^n)} \in J(R/I)$. Thus, for any element $x \in R$, $\overline{1 - (a - a^n)(b - b^n)x} \in U(R/I)$, so there is a $y \in R$ such that $\overline{(1 - (a - a^n)(b - b^n)x)y} = \overline{1}$, hence we have $1-(1-(a-a^n)(b-b^n)x)y \in I$. Let $c = (a - a^n)(b - b^n)$, as I is nil, $(1 - (1 - cx)y)^m = 0$ for some $m \in \mathbb{N}$, it follows that $1 - (1 - cx)d = 0$ for some $d \in R$; that is $(1 - cx)d = 1$, so we get $1 - cx$ is invertible and $c \in J(R)$. As a result $(a - a^n)(b - b^n) \in J(R)$, hence R is a J-n-like ring. $\square$

Proposition 2.14. *Let R be a ring. Then the following are equivalent.*

(1) *R is a J-n-like ring.*

(2) $S = \left\{ \begin{bmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{bmatrix} \mid a, a_{ij} \in R (i < j) \right\}$ *is a J-n-like ring.*

Proof. (1)$\Rightarrow$ (2) It is obvious from Proposition 2.8.

(2)$\Rightarrow$ (1) Choose

$$I = \left\{ \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & 0 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \mid a_{ij} \in R(i < j) \right\}.$$

Then $I^n = 0$ and $S/I \cong R$. Hence, we complete the proof by Theorem 2.13. $\square$

Let S and T be any rings, M an S-T-bimodule and R the formal triangular matrix ring $\begin{bmatrix} S & M \\ 0 & T \end{bmatrix}$. It is well-known that $J(R) = \begin{bmatrix} J(S) & M \\ 0 & J(T) \end{bmatrix}$.

Proposition 2.15. *Let $R = \begin{bmatrix} S & M \\ 0 & T \end{bmatrix}$. Then R is J-n-like if and only if S and T are J-n-like.*

Proof. One direction is obvious from Proposition 2.8. Assume that S and T are J-n-like and let $A = \begin{bmatrix} a & x \\ 0 & b \end{bmatrix}$, $B = \begin{bmatrix} c & y \\ 0 & d \end{bmatrix} \in R$. By assumption, we see that $(a - a^n)(c - c^n) \in J(S)$ and $(b - b^n)(d - d^n) \in J(T)$. Then, by direct calculation one sees that $(A - A^n)(B - B^n) = \begin{bmatrix} (a - a^n)(c - c^n) & * \\ 0 & (b - b^n)(d - d^n) \end{bmatrix} \in J(R)$. Hence R is a J-n-like ring, as desired. $\square$

Theorem 2.16. *Let R be a ring and $T_m(R)$ be $m \times m$ upper triangular matrices over R. Then the following are equivalent.*

(1) *R is J-n-like.*

(2) *$T_m(R)$ is J-n-like for all $m \in \mathbb{N}$.*

Proof. (1)$\Rightarrow$ (2) Let $\alpha = [a_{ij}], \beta = [b_{ij}] \in T_m(R)$ for some $m \geq 2$. It is easy to check that $(\alpha - \alpha^n)(\beta - \beta^n) = [c_{ij}]$ where $c_{ii} = (a_{ii} - a_{ii}^n)(b_{ii} - b_{ii}^n)$ for $i = 1, 2, \ldots, m$. Since R is J-n-like, we get $(a_{ii} - a_{ii}^n)(b_{ii} - b_{ii}^n) \in J(R)$, and so $[c_{ij}] \in J\big(T_m(R)\big)$. This gives $T_m(R)$ is J-n-like for all $m \in \mathbb{N}$.

(2)⇒(1) It is clear. □

Recall that a ring R is called *abelian* if every idempotent is central. There exists a J-n-like ring which is not generalized n-like as the following example shows.

Example 2.17. *It is easy to see that $R = \mathbb{Z}_3$ is a J-3-like ring. By Theorem 2.16, $T_2(R)$ is J-3-like. If $T_2(R)$ is a generalized 3-like ring, then $T_2(R)$ is abelian, a contradiction. Hence $T_2(R)$ is not generalized 3-like.*

We say that B is a *subring* of a ring A if $\emptyset \neq B \subseteq A$ and for any $x, y \in B$, $x - y, xy \in B$ and $1_A \in B$. Let A be a ring and B a subring of A and $R[A, B]$ denote the set $\{(a_1, a_2, \cdots, a_m, b, b, \cdots) \mid a_i \in A, b \in B,\ 1 \leq i \leq m\}$. Then $R[A, B]$ is a ring under the componentwise addition and multiplication. Also $J(R[A, B]) = R[J(A), J(A) \cap J(B)]$.

Proposition 2.18. *Let A be a ring and B a subring of A. The following are equivalent.*

(1) *A and B are J-n-like.*

(2) *$R[A, B]$ is J-n-like.*

Proof. (1) ⇒ (2) Let $\alpha = (a_1, \cdots, a_m, b, b, \cdots), \beta = (c_1, \cdots, c_m, d, d, \cdots) \in R[A, B]$. Then $(\alpha - \alpha^n)(\beta - \beta^n) = (a_1 - a_1^n, \cdots, a_m - a_m^n, b - b^n, b - b^n, \cdots) = (c_1 - c_1^n, \cdots, c_m - c_m^n, d - d^n, d - d^n, \cdots) = \big((a_1 - a_1^n)(c_1 - c_1^n), \cdots, (a_m - a_m^n)(c_m - c_m^n), (b-b^n)(d-d^n), (b-b^n)(d-d^n), \cdots\big)$. By (1), we have $(a_i - a_i^n)(c_i - c_i^n) \in J(A)$ and $(b - b^n)(d - d^n) \in J(A) \cap J(B)$. Therefore $(\alpha - \alpha^n)(\beta - \beta^n) \in J(R[A, B])$.

(2) ⇒ (1) Let $x, y \in A$ and write $\alpha = (x, 0, 0, \cdots), \beta = (y, 0, 0, \cdots) \in R[A, B]$. By assumption, we have $(\alpha - \alpha^n)(\beta - \beta^n) = \big((x - x^n)(y - y^n), 0, 0, \cdots\big) \in J(R[A, B])$, and so $(x - x^n)(y - y^n) \in J(A)$. Hence A is a J-n-like ring. Similarly, we show that B is a J-n-like ring. We complete the proof. □

Example 2.19. *Let R be a J-n-like ring. Then the ring $T = \{(x, y) \mid x - y \in J(R), x, y \in R\}$ is also a J-n-like ring.*

Proof. We consider the ring homomorphism $\varphi : T \to R$, $(x, y) \mapsto x$ where $(x, y) \in T$. Obviously, for every element x in R, we may find an element (x, x) in T corresponding to x, therefore φ is an epimorphism. Similarly, consider the ring homomorphism $\psi : T \to R$, $(x, y) \mapsto x$ where $(x, y) \in T$. In a similar way, ψ is an epimorphism. But $ker(\varphi) \cap ker(\psi) = 0$, thus T is isomorphic to the subdirect product of R and R. By Proposition 2.12, we ultimately have T is J-n-like. □

Lemma 2.20. *For every prime integer p, $\mathbb{Z}_p$ is a generalized p-like ring.*

Proof. For any $a \in \mathbb{Z}_p$, by Fermat's Theorem, $a^{p-1} = 1$. Then $a - a^p = 0$. Hence $\mathbb{Z}_p$ is a generalized p-like ring. $\square$

Example 2.21. *$\mathbb{Z}_5$ is a generalized 5-like ring, $\mathbb{Z}_{13}$ is a generalized 13-like ring.*

Lemma 2.22. *If $p_1, p_2, \cdots, p_s$ are distinct prime integers, then $\mathbb{Z}_{p_1} \oplus \mathbb{Z}_{p_2} \oplus \cdots \oplus \mathbb{Z}_{p_s}$ is a generalized $\big(\prod_{i=1}^{s}(p_i - 1) + 1\big)$-like ring.*

Proof. For any $(a_i), (b_i) \in \mathbb{Z}_{p_1} \oplus \mathbb{Z}_{p_2} \oplus \cdots \oplus \mathbb{Z}_{p_s}$, since $a_i^{p_i-1} (i = 1, 2, \cdots, s)$ is an idempotent, we have $a_i^{p_i-1} = a_i^{\prod_{i=1}^{s}(p_i-1)}$, thus $a_i^{\prod_{i=1}^{s}(p_i-1)+1} = a_i^{(p_i-1)+1} = a_i^{p_i}$. As for each p_i is a prime, we have $a_i^{p_i} = a_i$, hence $a_i^{\prod_{i=1}^{s}(p_i-1)+1} = a_i$, at last $(a_i - a_i^{\prod_{i=1}^{s}(p_i-1)+1})(b_i - b_i^{\prod_{i=1}^{s}(p_i-1)+1}) = 0$. Therefore, $\mathbb{Z}_{p_1} \oplus \mathbb{Z}_{p_2} \oplus \cdots \oplus \mathbb{Z}_{p_s}$ is a generalized $\big(\prod_{i=1}^{s}(p_i - 1) + 1\big)$-like ring. $\square$

Theorem 2.23. *If $n = p_1^{r_1} p_2^{r_2} \cdots p_s^{r_s}$ where $p_1, p_2, \cdots, p_s$ are distinct primes, and $r_1, r_2, \cdots, r_s \in \mathbb{N}$, then $\mathbb{Z}_n$ is a J-$\big(\prod_{i=1}^{s}(p_i - 1) + 1\big)$-like ring.*

Proof. By Lemma 2.22, $\mathbb{Z}_{p_1} \oplus \mathbb{Z}_{p_2} \oplus \cdots \oplus \mathbb{Z}_{p_s}$is a generalized $\big(\prod_{i=1}^{s}(p_i - 1) + 1\big)$-like ring. Thus,

$$\begin{aligned}\mathbb{Z}_n/J(\mathbb{Z}_n) &\cong \mathbb{Z}_{{p_1}^{r_1}} \oplus \mathbb{Z}_{{p_2}^{r_2}} \oplus \cdots \oplus \mathbb{Z}_{{p_s}^{r_s}}/J(\mathbb{Z}_{{p_1}^{r_1}}) \oplus J(\mathbb{Z}_{{p_2}^{r_2}}) \oplus \cdots \oplus J(\mathbb{Z}_{{p_s}^{r_s}})\\ &\cong \mathbb{Z}_{{p_1}^{r_1}}/J(\mathbb{Z}_{{p_1}^{r_1}}) \oplus \mathbb{Z}_{{p_2}^{r_2}}/J(\mathbb{Z}_{{p_2}^{r_2}}) \oplus \cdots \oplus \mathbb{Z}_{{p_s}^{r_s}}/J(\mathbb{Z}_{{p_s}^{r_s}})\\ &\cong \mathbb{Z}_{p_1} \oplus \mathbb{Z}_{p_2} \oplus \cdots \oplus \mathbb{Z}_{p_s}\end{aligned}$$

is a generalized $\big(\prod_{i=1}^{s}(p_i - 1) + 1\big)$-like ring, as

$$\mathbb{Z}_n \cong \mathbb{Z}_{{p_1}^{r_1}} \oplus \mathbb{Z}_{{p_2}^{r_2}} \oplus \cdots \oplus \mathbb{Z}_{{p_s}^{r_s}}, \qquad \text{(see [5])}$$

therefore, $\mathbb{Z}_n$ is a J-$\big(\prod_{i=1}^{s}(p_i - 1) + 1\big)$-like ring. $\square$

Lemma 2.24. *Let p, q be distinct prime integers, and let $R = \mathbb{Z}_{(p)} \cap \mathbb{Z}_{(q)} = \big\{\frac{m}{n} \in \mathbb{Q} \mid m, n \in \mathbb{Z}, p \nmid n, q \nmid n\big\}$. Then $J(R) = pqR$.*

Proof. Straightforward. $\square$

Theorem 2.25. *Let $R = \mathbb{Z}_{(p)} \cap \mathbb{Z}_{(q)}$ where p, q are distinct primes. Then R is a J-$\big((p-1)(q-1) + 1\big)$-like ring.*

Proof. Let $\frac{m}{n}, \frac{m'}{n'} \in R = \mathbb{Z}_{(p)} \cap Z_{(q)}$, and $(m,n) = 1, (m',n') = 1$. Where $\frac{m}{n}$ can regard as the element in $\mathbb{Z}_{(p)}$, since $\mathbb{Z}_{(p)}/J(\mathbb{Z}_{(p)}) \cong \mathbb{Z}_p$, $\overline{(\frac{m}{n})} = \overline{(\frac{m}{n})}^p$. Then, we can get $\overline{(\frac{m}{n})}^{p-1}$ is an idempotent. Similarly, $\frac{m'}{n'}$ can regard as the element in $\mathbb{Z}_{(q)}$. Since $\mathbb{Z}_{(q)}/J(\mathbb{Z}_{(q)}) \cong \mathbb{Z}_q$, $\overline{(\frac{m'}{n'})} = \overline{(\frac{m'}{n'})}^q$. So $\overline{(\frac{m'}{n'})}^{q-1}$ is an idempotent, too. Let $k = (p-1)(q-1)+1$. We have $\overline{(\frac{m}{n})}^k = \big(\overline{(\frac{m}{n})}^{p-1}\big)^{(q-1)}\overline{(\frac{m}{n})} = \overline{(\frac{m}{n})}^{p-1}\overline{(\frac{m}{n})} = \overline{(\frac{m}{n})}$, that is $\overline{(\frac{m}{n})} - \overline{(\frac{m}{n})}^k = \bar{0}$. Similarly, $\overline{(\frac{m'}{n'})} - \overline{(\frac{m'}{n'})}^k = \bar{0}$, therefore,

$$(\tfrac{m}{n}) - (\tfrac{m}{n})^k \in J(\mathbb{Z}_{(p)}),\ (\tfrac{m'}{n'}) - (\tfrac{m'}{n'})^k \in J(\mathbb{Z}_{(q)}).$$

Let $(\frac{m}{n}) - (\frac{m}{n})^k = p\frac{m_1}{n_1}$, where $\frac{m_1}{n_1} \in \mathbb{Z}_{(p)}$, hence $pm_1n^k = n_1m(n^{k-1} - m^{k-1})$. If $q|n_1$, then $q|pm_1n^k$, furthermore $q \nmid p$, $q \nmid n$, so $q \mid m_1$, this is a contradiction with $(n_1,m_1) = 1$, then we get $q \nmid n_1$, at last $\frac{m_1}{n_1} \in Z_{(q)}$. Hence, $\frac{m_1}{n_1} \in R$. Let $(\frac{m'}{n'}) - (\frac{m'}{n'})^k = q\frac{m_2}{n_2}$, similarly, $p \mid n_2$, then $\frac{m_2}{n_2} \in R$. Therefore, by Lemma 2.24, we have $\big((\frac{m}{n}) - (\frac{m}{n})^k\big)\big((\frac{m'}{n'}) - (\frac{m'}{n'})^k\big) = pq\frac{m_1m_2}{n_1n_2} \in pqR = J(R)$. Consequently, R is a J-$\big((p-1)(q-1)+1\big)$-like ring. $\square$

3. P-n-like rings

Definition 3.1. The *prime radical* $P(R)$ is the intersection of all the prime ideals in R, equivalently, $P(R) = \{x \in R \mid RxR \text{ is nilpotent}\}$.

As is well known, $P(R)$ is a semiprime ideal; that is, if for any ideal I of R with $I^2 \subseteq P(R)$, then $I \subseteq P(R)$ and it is also a nil ideal and $P(R) \subseteq J(R)$.

Definition 3.2. Let n be a fixed integer. A ring R is called a *P-n-like ring* provided that $(a - a^n)(b - b^n) \in P(R)$ for every element a, b in R.

Recall that an element $a \in R$ is called *strongly nilpotent* if every sequence $a_1, a_2, a_3, \ldots$ such that $a_1 = a$ and $a_{i+1} \in a_iRa_i$ is eventually zero.

Lemma 3.3. *Let R be a ring. Then $P(R) = \{x \in R \mid x$ is a strongly nilpotent element$\}$.*

Proof. See [6, Exercise 10.17]. $\square$

Theorem 3.4. *A ring R is a P-n-like ring if and only if*

(1) *R is a J-n-like ring,*
(2) *$J(R) = P(R)$.*

Proof. $\Rightarrow$: Since $P(R) \subseteq J(R)$, (1) holds. Let $x \in J(R)$, $r \in R$. Then, we get $(x - x^n)(rx - (rx)^n) \in P(R)$, that is

$$(1 - x^{n-1})xrx(1 - (rx)^{n-1}) \in P(R).$$

As $1 - x^{n-1} \in U(R)$, $1 - (rx)^{n-1} \in U(R)$, we have $xRx \in P(R)$, then $x \in P(R)$ because $P(R)$ is a semiprime ideal. Therefore, $J(R) \subseteq P(R) \subseteq J(R)$, as desired.

$\Leftarrow$: As $J(R) = P(R)$, we easily obtain the result. $\square$

Recall that a ring R is called to be 2-*primal* if $P(R) = Nil(R)$.

Corollary 3.5. *If a ring R is P-n-like, then R is 2-primal.*

Proof. Assume that R is a P-n-like ring. Then, by Theorem 3.4, R is J-n-like and $J(R) = P(R)$. According to Corollary 2.5, we get $Nil(R) \subseteq J(R)$, and so $Nil(R) \subseteq J(R) = P(R) \subseteq Nil(R)$, as asserted. $\square$

There are J-n-like rings which are not P-n-like.

Example 3.6. *Let $R = \mathbb{Z}_{(2)} \cap \mathbb{Z}_{(3)}$ is a J-3-like ring but not P-3-like.*

Proof. By Theorem 3.4, if R is a P-n-like ring, then $P(R) = J(R)$, and so $P(R) = \{\frac{m}{n} \mid m, n \in \mathbb{Z}, 2 \nmid n, 3 \nmid n\}$, there exists $s \in R$ such that $(\frac{m}{n})^s = 0$, then $m^s = 0$; that is, $m = 0$. So $P(R) = 0$. But for any $\frac{1}{5}, \frac{1}{7} \in R$, $(\frac{1}{5} - (\frac{1}{5})^3)(\frac{1}{7} - (\frac{1}{7})^3) \neq 0$, hence R is not a P-3-like ring. $\square$

Recall that a ring R is *periodic* if for any $a \in R$, there exist distinct $m, n \in \mathbb{N}$ such that $a^m = a^n$.

Theorem 3.7. *Every P-n-like ring is a periodic ring.*

Proof. Suppose that R is a P-n-like ring and let $a \in R$. Then $(a - a^n)(a - a^n) \in P(R)$. As $P(R)$ is a nil ideal, there exists an integer m such that $(a - a^n)^{2m} = 0$. Hence $a - a^n \in Nil(R)$, by [1], R is a periodic ring. The proof is completed. $\square$

Lemma 3.8. *Suppose R_1, R_2 are two P-n-like rings. Then $R = R_1 \oplus R_2$ is a P-n-like ring.*

Proof. It is suffice to show that $(P(R_1), P(R_2)) \subseteq P(R_1 \oplus R_2)$. For any $(x_1, x_2) \in (P(R_1), P(R_2))$, there exists a sequence $(x_{01}, x_{02}), (x_{11}, x_{12}), \cdots, (x_{n1}, x_{n2})$, where $(x_{i1}, x_{i2}) = (x_{i-1,1}, x_{i-2,2})(R_1 \oplus R_2)(x_{i-1,1}, x_{i-2,2}), (i = 1, 2, 3, \cdots)$ is eventually zero; that is, $x_{i1} = x_{i-1,1}R_1x_{i-1,1}, (i = 1, 2, 3, \cdots)$, this implies that$x_{n1} = 0$, $x_{i2} = x_{i-1,2}R_2x_{i-1,2}, (i = 1, 2, 3, \cdots)$, and this yields that $x_{m2} = 0$. Let $k = max\,\{n, m\}$. Then $x_{k1} = 0, x_{k2} = 0$. Hence, $(x_{k1}, x_{k2}) = 0$. Thus, $(x_1, x_2) \in (P(R_1) \oplus P(R_2))$, the conclusion is obvious. $\square$

Proposition 3.9. *Every finite direct product of P-n-like rings is also P-n-like.*

Proof. It is obvious by Lemma 3.8. $\square$

Proposition 3.10. *Every subring of a P-n-like ring is also a P-n-like ring.*

Proof. Let R be a P-n-like ring and let $x, y \in S \subseteq R$. Then $(x - x^n)(y - y^n) \in P(R) \cap S$. For any element $x \in P(R) \cap S$, and a sequence $x_0, x_1, \cdots, x_n, \cdots$ in R where $x = x_0, x_n \in x_{n-1}Rx_{n-1}$ for each n, hence $x_m = 0$ for some m, so $x \in P(S)$; that is, $P(R) \cap S \subseteq P(S)$. Hence $(x - x^n)(y - y^n) \in P(S)$. □

Proposition 3.11. *Every finite subdirect product of P-n-like rings is also a P-n-like ring.*

Proof. It is obvious by Proposition 3.9 and 3.10. □

Lemma 3.12. *Let K be an ideal of a ring R with $K^2 = 0$. If R/K is a P-n-like ring, then R is a P-n-like ring.*

Proof. Let $x, y \in R$, then $\overline{x}, \overline{y} \in R/K$, as R/K is a P-n-like ring, we have

$$(\overline{x} - \overline{x}^n)(\overline{y} - \overline{y}^n) \in P(R/K).$$

For any sequence $x_0, x_1, \cdots, x_n, \cdots \in R$, where $x_0 = (x - x^n)(y - y^n), x_i \in x_{i-1}Rx_{i-1}$, then $\overline{x_0}, \overline{x_1}, \cdots, \overline{x_n} \in R/K$, as R/K is a P-n-like ring. Thus, we have $\overline{x_m} = 0$ for some $m \in N$. As $K^2 = 0$, then $x_{n+1} \in x_nKx_n = 0$, hence x_0 is a strongly nilpotent, that is $x_0 \in P(R)$, thus $(x - x^n)(y - y^n) \in P(R)$. Therefore, R is a P-n-like ring. □

Theorem 3.13. *Let $R/I, R/J$ be P-n-like rings where I, J are the ideals of R. Then $R/(IJ)$ is a P-n-like ring.*

Proof. We construct a ring homomorphism $\varphi : R/(I \cap J) \to R/J$, where $x + I \cap J \mapsto x + J$. $\psi : R/(I \cap J) \to R/I$, where $x + I \cap J \mapsto x + I$. Obviously, φ and ψ are two epimorphisms. $ker(\varphi) \cap ker(\psi) = 0$, then $R/(I \cap J)$ is the subdirect products of R/J and R/I, by Proposition 3.9, we have $R/(I \cap J)$ is a P-n-like ring by Lemma 3.12. Clearly, we see

$$R/(I \cap J) \cong R/(IJ)\big/(I \cap J)/(IJ),$$

as $IJ \subseteq I \cap J$, thus $\big((I \cap J)/(IJ)\big)^2 = 0$, then we have $R/(IJ)$ is a P-n-like ring. □

Corollary 3.14. *Let I be an ideal of a ring R. If R is a P-n-like ring, then so is R/I^m for all $m \in \mathbb{N}$.*

Example 3.15. *$\mathbb{Z}_{p^r}$ is a P-p-like ring, since for every element $a \in \mathbb{Z}_{p^r}$, we have $a - a^p = 0$, and $(J(\mathbb{Z}_{p^r})) = (p\mathbb{Z}_{p^r})^r = 0$.*

Theorem 3.16. *Let $n = p_1^{r_1} p_2^{r_2} \cdots p_s^{r_s}$ where $p_1, p_2, \cdots, p_s$ are distinct prime integers, and $r_1, r_2, \cdots, r_s \in \mathbb{N}$. Then $\mathbb{Z}_n$ is a P-$\big(\prod_{i=1}^{s}(p_i - 1) + 1\big)$-like ring.*

Proof. Since $\mathbb{Z}_{p^r}$ is commutative, we can get $J(\mathbb{Z}_{p^r}) = p\mathbb{Z}_{p^r} = Nil(\mathbb{Z}_{p^r}) = P(\mathbb{Z}_{p^r})$, by Theorem 2.23, $\mathbb{Z}_n$ is a P-$\big(\prod_{i=1}^{s}(p_i - 1) + 1\big)$-like ring. □

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