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ON $\rho-$ STATISTICAL CONVERGENCE IN TOPOLOGICAL GROUPS

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Abstract. In this study, by using definition of ρ-statistical convergence which was defined by Cakalli [5], we give some inclusion relations between the concepts of ρ-statistical convergence and statistical convergence in topological groups.

1. Introduction

In 1951, Steinhaus [29] and Fast [14] introduced the notion of statistical convergence and later in 1959, Schoenberg [28] reintroduced independently. Caserta et al. [4], Cakalli ([6],[7]), Cinar et al. [8], Colak [9], Connor [10], Et et al. ([11],[12],[13]), Fridy [15], Gadjiev and Orhan [16], Isik and Akbas ([17],[18]), Kolk [19], Mursaleen [20], Salat [21], Sengul et al. ([22]-[27]), Aral et al. ([1],[2],[3]) and many others investigated some arguments related to this notion.

The opinion of statistical convergence depends on the density of subsets of the natural set $\mathbb{N}$. We say that the $\delta(E)$ is the density of a subset E of $\mathbb{N}$ if the following limit exists such that

$$\delta(E)=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\chi_E(k),$$

where χ_E is the characteristic function of E. It is clear that any finite subset of $\mathbb{N}$ has zero natural density and $\delta\left(E^c\right)=1-\delta\left(E\right)$.

We say that the sequence $x=(x_k)$ is statistically convergent to ℓ if for every $\varepsilon>0$,

$$\delta\left(\{k\in\mathbb{N}:|x_k-\ell|\geq\varepsilon\ \}\right)=0.$$

In this case we write $S-\lim x_k=\ell$ or $x_k\to\ell\,(S)$. Equivalently,

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$$\lim_{n\to\infty} \frac{1}{n} \left|\{k \le n : |x_k - \ell| \ge \varepsilon\}\right| = 0.$$

S will denote the set of all statistically convergent sequences.

If x is a sequence such that x_k satisfies property P for all k except a set of natural density zero, then we say that x_k satisfies P for "almost all k ", and we abbreviate this by "$a.a.k.$"

2. Main Results

In this section we give the main results of this article. Now we begin a new definition.

Definition 2.1. *Let X be an abelian topological Hausdorf group. A sequence $(x(k))$ of points in $\mathbb{R}$, the set of real numbers, is called $\rho-$statistically convergent in topological groups to ℓ ($S_\rho(X)-$convergent to ℓ) if there is a real number ℓ for each neighbourhood U of 0 such that*

$$\lim_{n\to\infty} \frac{1}{\rho_n} |\{k \le n : x(k) - \ell \notin U\}| = 0$$

for each $\varepsilon > 0$, where $\rho = (\rho_n)$ is a non-decreasing sequence of positive real numbers tending to ∞ such that $\limsup_n \frac{\rho_n}{n} < \infty$, $\Delta\rho_n = O(1)$, and $\Delta x(n) = x(n+1) - x(n)$ for each positive integer n. In this case we write $S_\rho(X) - \lim x(k) = \ell$ or $x(k) \to \ell\,(S_\rho(X))$. We denote the set of all $\rho-$statistically convergent in topological groups sequences by $S_\rho(X)$. If $\rho = (\rho_n) = n$, $\rho-$statistically convergent in topological groups is coincide statistical convergence in topological groups.

Definition 2.2. *Let X be an abelian topological Hausdorf group. A sequence $x = (x(k))$ of points in $\mathbb{R}$, the set of real numbers, is called $S_\rho(X)$-Cauchy sequence in topological groups if there is a subsequence $(x(k'(n)))$ of x such that $k'(n) \leqslant n$ for each n, $\lim_{n\to\infty} x(k'(n)) = \ell$ and for each neighbourhood U of 0*

$$\lim_{n\to\infty} \frac{1}{\rho_n} |\{k \le n : x(k) - x(k'(n)) \notin U\}| = 0,$$

where $\rho = (\rho_n)$ is a non-decreasing sequence of positive real numbers tending to ∞ such that $\limsup_n \frac{\rho_n}{n} < \infty$, $\Delta\rho_n = O(1)$ and $\Delta x(n) = x(n+1) - x(n)$ for each positive integer n.

Theorem 2.1. *If x is $\rho-$statistically convergent in topological groups, then $S_\rho(X)-\lim x(k) = \ell$ is unique.*

Proof. Suppose that $(x(k))$ has two different $\rho-$statistical in topological groups limits ℓ_1, ℓ_2 say. Since X is a Hausdorff space there exists a neighbourhood U of 0 such that $\ell_1 - \ell_2 \notin U$. Then we may choose a neighbourhood W of 0 such that $W + W \subset U$. Write $z(k) = \ell_1 - \ell_2$ for all $k \in \mathbb{N}$. Therefore for all $n \in \mathbb{N}$,

$$\{k \le n : z(k) \notin U\} \subset \{k \le n : \ell_1 - x(k) \notin W\} \cup \{k \le n : x(k) - \ell_2 \notin W\}.$$

Now it follows from this inclusion that, for all $n \in \mathbb{N}$,

$$|\{k \le n : z(k) \notin U\}| \le |\{k \in I_r : \ell_1 - x(k) \notin W\}| + |\{k \le n : x(k) - \ell_2 \notin W\}|.$$

Since $S_\rho(X)-\lim x(k)=\ell_1$ and $S_\rho(X)-\lim x(k)=\ell_2$ we get

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{\rho_n}\left|\{k\le n: z(k)\notin U\}\right| &\le \lim_{n\to\infty}\frac{1}{\rho_n}\left|\{k\le n:\ell_1-x(k)\notin W\}\right| \\ &\quad+\lim_{n\to\infty}\frac{1}{\rho_n}\left|\{k\le n: x(k)-\ell_2\notin W\}\right|.\end{aligned}$$

This contradiction shows that $\ell_1=\ell_2$. □

Theorem 2.2. *If* $\lim_{k\to\infty}x(k)=\ell$ *and* $S_\rho(X)-\lim y(k)=0$, *then*

$$S_\rho(X)-\lim\left(x(k)+y(k)\right)=\lim_{k\to\infty}x(k).$$

Proof. Let U be any neighborhood of 0. Then we may choose a symetric neighbourhood W of 0 such that $W+W\subset U$. Since $\lim_{k\to\infty}x(k)=\ell$ there exists an integer k_0 such that $k\ge k_0$ implies that $x(k)-\ell\in W$. Hence

$$\lim_{n\to\infty}\frac{1}{\rho_n}\left|\{k\le n:x(k)-\ell\notin W\}\right|\le\lim_{n\to\infty}\frac{k_0}{\rho_n}=0$$

and by the assumption that $S_\rho(X)-\lim y(k)=0$ we have

$$\lim_{n\to\infty}\frac{1}{\rho_n}\left|\{k\le n:y(k)\notin W\}\right|=0.$$

Now we have

$$\{k\le n:(x(k)-\ell)+y(k)\notin U\}\subset\{k\le n:x(k)-\ell\notin W\}\cup\{k\le n:y(k)\notin W\}.$$

Hence

$$\frac{1}{\rho_n}\left|\{k\le n:(x(k)-\ell)+y(k)\notin U\}\right|\le\frac{1}{\rho_n}\left|\{k\le n:x(k)-\ell\notin W\}\right|+\frac{1}{\rho_n}\left|\{k\le n:y(k)\notin W\}\right|$$

It follows from the above inequality that

$$\lim_{n\to\infty}\frac{1}{\rho_n}\left|\{k\le n:(x(k)-\ell)+y(k)\notin U\}\right|=0.$$

Thus $S_\rho(X)-\lim\left(x(k)+y(k)\right)=\lim_{k\to\infty}x(k)$. □

Theorem 2.3. *If a sequence* $x(k)$ *is* ρ*-statistically convergent to* ℓ, *then there are sequences* $y(k)$ *and* $z(k)$ *such that* $\lim_{k\to\infty}y(k)=\ell$, $x=y+z$ *and* $\lim_{n\to\infty}\frac{1}{\rho_n}\left|\{k\le n:x(k)\ne y(k)\}\right|=0$ *and* z *is a* ρ*-statistically null sequence.*

Proof. Let (V_j) be a nested base of neighborhoods of 0. Take $n_0=0$ and choose an increasing sequence (n_j) of positive integers such that

$$\frac{1}{\rho_n}\left|\{k\le n:x(k)-\ell\notin V_j\}\right|<\frac{1}{j}\text{ for }n>n_j.$$

Let us define sequences $y=y(k)$ and $z=z(k)$ in the following way. Write $z(k)=0$ and $y(k)=x(k)$ if $n_0<k\le n_1$ and suppose that $n_j<n_{j+1}$ for $j\ge1$. $z(k)=0$ and $y(k)=x(k)$ if $x(k)-\ell\in V_j$, $y(k)=\ell$ and $z(k)=x(k)-\ell$ if $x(k)-\ell\notin V_j$. Firstly, we prove that $\lim_{k\to\infty}y(k)=\ell$. Let V be any neighborhood of 0. We may choose a positive integer j such that $V_j\subset V$. Then $y(k)-\ell=x(k)-\ell\in V_j$ and so $y(k)-\ell\in V$ for $k>n_j$. If $x(k)-\ell\notin V_j$, then $y(k)-\ell=\ell-\ell=0\in V$. Hence $\lim_{k\to\infty}y(k)=\ell$. Finally we show that $z=z(k)$ is a statistically null sequence. It is enough to show that

$$\lim_{n\to\infty}\frac{1}{\rho_n}\left|\{k\le n:z(k)\ne0\}\right|=0.$$

For any $n \in \mathbb{N}$ any neighborhood V of 0, we have

$$|\{k \leq n : z(k) \notin V\}| \leq |\{k \leq n : z(k) \neq 0\}| .$$

If $j \in \mathbb{N}$ such that $V_j \subset V$ and $\varepsilon > 0$, we are going to show that

$$\frac{1}{\rho_n} |\{k \leq n : z(k) \neq 0\}| < \varepsilon.$$

If $n_p < n \leq n_{p+1}$, then

$$\{k \leq n : z(k) \neq 0\} \subset \{k \leq n : x(k) - \ell \notin V_p\} .$$

If $p > j$ and $n_p < n \leq n_{p+1}$, then

$$\frac{1}{\rho_n} |\{k \leq n : z(k) \neq 0\}| \leq \frac{1}{\rho_n} |\{k \leq n : x(k) - \ell \notin V_p\}| < \frac{1}{p} < \frac{1}{j} < \varepsilon.$$

Thus, the proof is completed. □

Theorem 2.4. *The sequence x is $S_\rho(X)-$convergent if and only if x is $S_\rho(X)-$Cauchy sequence.*

Proof. Assume that x is $S_\rho(X)-$convergent. Since X is a Hausdorff space there exists a neighbourhood U of 0. Then we may choose a neighbourhood Y of 0 such that $Y + Y \subset U$. We can write

$$|\{k \leq n : x(k) - x(k'(n)) \notin U\} \subset \{k \leq n : x(k) - \ell \notin Y\} \cup \{k \leq n : \ell - x(k'(n)) \notin Y\} .$$

Now it follows from this inclusion that, for all $n \in \mathbb{N}$,

$$\frac{1}{\rho_n} |\{k \leq n : x(k) - x(k'(n)) \notin U\}| \leq \frac{1}{\rho_n} |\{k \in I_r : x(k) - \ell \notin Y\}| + \frac{1}{\rho_n} |\{k \leq n : \ell - x(k'(n)) \notin Y\}| .$$

Since $S_\rho(X) - \lim x(k) = \ell$, we get

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{\rho_n} |\{k \leq n : x(k) - x(k'(n)) \notin U\}| &\leq \lim_{n\to\infty} \frac{1}{\rho_n} |\{k \leq n : x(k) - \ell \notin Y\}| \\ &+ \lim_{n\to\infty} \frac{1}{\rho_n} |\{k \leq n : \ell - x(k'(n)) \notin Y\}| . \end{aligned}$$

The proof to the contrary is obvious. □

Theorem 2.5. *Let $\rho = (\rho_n)$ be a non-decreasing sequence of positive real numbers tending to ∞ such that $\limsup_n \frac{\rho_n}{n} < \infty, \Delta\rho_n = O(1)$. If $\frac{\rho_n}{n} \geq 1$ for all $n \in \mathbb{N}$, then $S(X) \subset S_\rho(X)$.*

Proof. If $S(X) - \lim x(k) = \ell$, then for every $\varepsilon > 0$ we have

$$\begin{aligned} \frac{1}{n}|\{k &\leq n : x(k) - \ell \notin U\}| = \frac{\rho_n}{n}\frac{1}{\rho_n}|\{k \leq n : x(k) - \ell \notin U\}| \\ &\geqslant \frac{1}{\rho_n}|\{k \leq n : x(k) - \ell \notin U\}|. \end{aligned}$$

This proves the proof. □

Theorem 2.6. *Let $\rho = (\rho_n)$ and $\tau = (\tau_n)$ be two sequences such that $\rho_n \leqslant \tau_n$ for all $n \in \mathbb{N}$. If $\liminf_{n\to\infty} \frac{\rho_n}{\tau_n} > 0$, then $S_\rho(X) \subset S_\tau(X)$.*

Proof. If $S_\rho(X) - \lim x(k) = \ell$, then for every $\varepsilon > 0$ we can write

$$\frac{1}{\tau_n}|\{k \leq n : x(k) - \ell \notin U\}| \leq \frac{\rho_n}{\tau_n}\frac{1}{\rho_n}|\{k \leq n : x(k) - \ell \notin U\}|.$$

This is enough for proof. □

The following result is obtained from Theorem 2.5 and Theorem 2.6.

Corollary 2.7. *Let $\rho = (\rho_n)$ and $\tau = (\tau_n)$ be two sequences such that $\rho_n \leqslant \tau_n$ and $n < \tau_n$ for all $n \in \mathbb{N}$. If $\liminf_{n\to\infty} \frac{\rho_n}{\tau_n} > 0$, then $S(X) \subset S_\rho(X) \subset S_\tau(X)$.*

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