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High School Students' and Their Teacher's Experiences with Visual Proofs

Lise Öğrencilerinin ve Öğretmenlerinin Görsel İspat ile İlgili Deneyimleri[1]

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Abstract: Proof is one of the main components of both mathematics and mathematics education. However, students and teachers at every level have difficulties in proof and proof teaching. Presenting alternative proof methods to students will not only provide a rich learning environment but also allow the students to have different perspectives by providing alternative ways they can apply for constructing a proof. The visual proofs are one of the alternative proof methods. The visual proofs are seen as valuable tools for mathematics education; it is planned to investigate the views of high school students and their teacher about visual proof. Case study method is used in the study. Participants of the study consisted of four high school students who were nineth-graders and their teacher. The teacher said that after the visual proof activities, the students started to wonder where the formulas came from, in other words, they started questioning, their interest levels increased, and the formulas were kept in their minds more. The high school students stated that they found the visual proof activities enjoyable, and they were happy when they understood the visual proofs and unhappy when they could not understand them.

Keywords: Proof, proof teaching, proof methods, proof without words, visual proof

Öz: İspat, matematiğin ve matematik eğitiminin ana bileşenlerinden biridir. Buna rağmen öğrenciler ve öğretmenler her düzeyde ispat öğreniminde ve öğretiminde güçlük yaşamaktadırlar. Bu bağlamda düşünüldüğünde öğrencilere kullanabilecekleri alternatif ispat yöntemlerinin sunulması ile hem zengin öğrenme ortamları oluşturmuş olacak hem de bir ispatı yapmak için başvurulabilecek alternatif yollar sunulmasıyla öğrenciye farklı bakış açıları kazandırılmış olacaktır. Görsel ispatlar ispat öğretiminde alternatif bir yöntemdir. Bu çalışmada görsel ispat etkinliklerinde yer alan lise öğrencileri ve öğretmenlerinin deneyimlerine dönük görüşleri alınmıştır. Çalışmanın modeli durum çalışmasıdır. Katılımcılar 9. sınıfta öğrenim gören ve uygulama sırasında görsel ispatlar ile deneyim yaşayan dört öğrenci ve onların öğretmenleridir. Öğretmen görsel ispat deneyimi yaşayan öğrencilerin sonraki derslerde formüllerin nereden geldiğini merak ettiklerini başka bir deyişle sorgulamaya başladıklarını, derse ilgilerinin arttığını ve formüllerin akıllarında daha çok kaldığını belirtmiştir. Öğrenciler ise görsel ispat etkinliklerini eğlenceli bulduklarını, ispatı anladıkları zaman mutlu olduklarını buna karşın anlamadıkları zaman mutsuz olduklarını belirtmişlerdir.

Anahtar Kelimeler: İspat, ispat öğretimi, ispat yöntemleri, sözsüz ispat, görsel ispat

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Introduction

Proof is one of the main components of both mathematics and mathematics education (Knuth, 2002). However, students and teachers at every level have difficulties in proof and proof teaching. Proving a theorem in the class shows not only why the theorem is correct but also can enable the student to understand this theorem (Strausova & Hasek, 2012). According to Hanna (2000), a proof is valid only when it is both convincing and legitimized by a mathematician in such a way to provide actual mathematical understanding. In fact, based on the definitions of proof, it is seen that proof has two dimensions. The first dimension is that the correctness of a statement based on theorems and axioms is shown and the other dimension is that the proof has convincing properties (Miller, 2012).

The proof studies have revealed that students have difficulties in proving them (Hawro, 2007; Urhan & Bülbül, 2016; Weber, 2001). Numerous studies have still been conducted on those difficulties and how those difficulties can be eliminated. According to Almeida (1996), most students use empirical methods in mathematical proof. In this case, some students are convinced about the correctness of the theorem by examples in particular, individuals who do have sufficient level of the conceptual knowledge concerning the definitions do not prefer deductive methods (Doruk, 2016). But if it is aimed to enable students to gain competence about proof, explanatory proofs should be placed in proof teaching (Gierdien, 2007). Visual presentations and diagrams are important components used in explanatory proofs.

It is widely understood that visual arguments help discover new results and produce more formal proofs (Bardelle, 2010). Just algebra formulas or just diagrams cannot have the power of directly giving all the information. If both are used together, this facilitates embody the proof (Rinvold & Lorange, 2013). According to Brown (2008), mathematics is a diverse field and there are many ways to prove something. It is therefore not right to use only verbal and symbolic proofs in mathematics. Since information can be presented both verbally and non-verbally, only using sentences for reasoning can be restrictive (Hanna & Sidoli, 2007). For this reason, it is wrong to say that mathematical information can be transferred only through formal arguments if elements such as visualization are present (Rodd, 2000). As a matter of fact, visual presentations concretize abstract concepts (Flores, 2000). Thus, it is necessary to benefit from visualization in mathematics

[1] Bu çalışma ilk yazarın doktora tezinden üretilmiştir.

education when considering that our ability to perceive visual concepts is strong enough (Miller, 2012).

In studies examining the use of geometric representations in algebra, the importance of using geometric models to understand the relationships between algebra and arithmetic has been mentioned (Flores, 2000). The presentation of the proof as a part of the problem-solving process such as correcting ideas, switching between different presentations and working on the diagrams is important for the proof teaching process (Ball et al., 2002). Tymoczko notes that informal proofs are more convincing and lead to new discoveries (cited in Hanna, 1989).

According to the high school mathematics curriculum in Turkey, the skill of "selecting the most appropriate proof method in the process of proving a mathematical proposition" along with many behaviors through the mathematical reasoning among mathematical process skills desired for students to acquire is one of the desired skills (Ministry of Education [MoNE], 2013, p.8). However, many teachers do not integrate reasoning and proving in their mathematics teaching (Herbert et al., 2015). For example, it was determined that the teachers presented the proof through the transfer of information, they did not involve the students in the cognitive processes of constructing the proof and expressed the meaning and steps of making proof in a limited way (Uğurel & Moralı, 2010).

Although most students have distinct characteristics from each other, the proofs are usually presented in an unchanging and definite format (Lockhart, 2009). However, for the learning and teaching process to be more meaningful, it is necessary to determine individual differences such as students' interests, learning needs, readiness level, and learning style and to consider these differences in determining teaching methods and techniques. Hence, it is suggested to include different proof methods in the proofs of theorems (MoNE, 2018). Presenting alternative proof methods to students in this context will not only provide rich learning settings but also allow the student to have different perspectives by providing alternative ways he/she can apply to construct a proof.

It may not be very correct to expect a high level of proof skills for students who are recently learning proofs because the students prove first a special situation through manipulations such as drawing, then visualize the actions in their minds using more special cases and eventually make generalization (Semadeni, 1980, cited by. Harel & Sowder, 1998). As a matter of fact, most students do not use general mathematical reasoning and algebraic symbols simultaneously. They need special examples of this. Using visual representations makes algebraic terms concrete.

It is important especially for the students who have just started the symbolic use of variables to concretize the meaning of the manipulation of symbols. Visual presentations guide students to understand the steps of algebraic manipulation. Of course, this should not signify that visual presentations are not important for those who have already gained this ability. Then, these presentations are also a source of inspiration for new discoveries (Flores, 2000).

The contribution of visual presentations to mathematical proofs has been investigated particularly for years. The studies have tried to answer the question how visual presentations can be used to justify a mathematical statement even though they provide ideas or evidence for the mathematical statement (Hanna & Sidoli, 2007). According to Flores (2000), geometric representations of numerical relationships are an important tool for explaining algebraic ideas. These presentations are a guideline for the students learning to use algebra as a tool for generalization and justification.

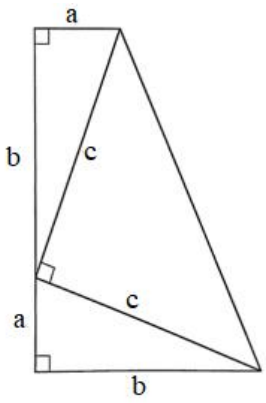
The theorems are usually proved formally. However, people can be ensured to use a different proof form by performing geometrical operations on diagrams. Diagrams have advantages such as storing information and supporting perceptual inferences and the proofs with diagrams bring an intuitive view that will facilitate the understanding of the proof. Moreover, such proofs are understood more easily than algebraic proofs (Jamnik et. al, 1997).

Visual Proofs

There is not a common view in the literature on expressions which are used for "visual proofs". For example, proof without words, proofs by picture, diagrammatic proofs, and pictorial proofs are among the expressions used for visual proofs (Polat & Demircioğlu, 2016). Visual proofs are not the recently discovered innovations (Alsina & Nelsen, 2010). However, visual proofs have been paid increasingly attention in mathematical research and mathematics education applications in the last years. Visual proofs are in the form of deductive steps based on figures, diagrams, and plots. This means that it is possible to understand the proof by reading pictures. In the literature, visual proofs are described as non-verbal proofs which are based only on diagrams and numbers, letters, arrows, dots, and symbolic expressions (Bardelle, 2010). They will help us to see why a specific mathematical expression is correct and even how it is addressed while proving the accuracy of a mathematical statement (Gierdien, 2007). In other words, visual proof is the mathematical drawings that illustrate the proof of a mathematical statement without words (Bell, 2011). Visual proofs can consist of a single shape or multiple shapes. Figure 1 shows examples of visual proof.

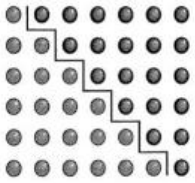
It is thought that visual proofs play important roles in mathematics from primary school to university (Alsina & Nelsen, 2010). The visual proofs are powerful tools because they can make proof construction accessible to students at any level (Reid & Vargas, 2018). Discussions and explanations made for understanding visual proofs provide opportunities for finding the connections between different mathematical ideas. In the visual proof process, visual representation is converted to textual representation. With this, the student performs mathematical actions. (Marco et al., 2022). In this way, visual proofs improve student comprehension (Gierdien, 2007) and they are valuable tools for mathematics education (Miller, 2012).

The students having appropriate mathematical knowledge will see the drawing and meaning in visual proofs, understand the proof, and develop their visual proof ability by confirming each stage in the proof (Sigler et al., 2016). Truly, visual proofs are like a jigsaw puzzle given to the reader to discover the mathematical truth underlying the picture. So visual proofs are the tools stimulating communication and mathematical thinking (Doyle et.al, 2014).

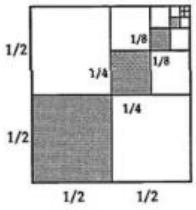


$$A = 2.\frac{1}{2}ab + \frac{1}{2}c^2 = \frac{1}{2}$$

$$c^2 = a^2 + b^2$$



$$1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$$



$$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \cdots = \frac{1}{3}$$

Figure 1. Examples of visual proofs (Nelsen, 1993)

Generally, proving is not an easy task for students. Especially for students who are beginner on proof, completing a proof can be possible only with certain guidance (Mudaly, 2013). The students who are trying to gain proof skills can lose control in the proof process. Therefore, the guidance of the teacher as well as presenting the proof into small pieces are important in proof teaching (Heinze & Reiss, 2004).

Although visual proofs are considered as useful tools in mathematics education (Borwein & Jörgenson, 2002), there are very few studies about their application especially in the high school curriculum (Uğurel, et. al, 2016). However, there is not much empirical evidence on effect of visual proof activities and their pedagogical potentiality in high school mathematics classrooms (Marco, et. al, 2022). When both the role of visualization in mathematics education and the difficulties experienced by the students in proving are taken into consideration, visual proofs can be seen as an alternative method in proof teaching. Thus, in recent decades, visual proof has been suggested to be implemented in high school mathematics classrooms to develop students' competencies in proving (Reid & Vallejo Vargas, 2016). Generally, the students do not have experience with proof at high school level, and when they are asked to prove, they just give example. Therefore, it is important to reveal the students' and teacher's experiences related to an educational process that includes activities for a different proof method. With the results of this research, different proof methods can be given more place in proof teaching. This study is important to reveal what are the views of high school students and the teacher who experienced with visual proof. For this purpose, the study was conducted to find the answers to the question of "What are the views of high school students and their teacher about visual proofs?"

Method

Research Model

Case study method as a form of qualitative research approach was used in the study. Case study was preferred because it is a detailed and in-depth examination of a situation (Creswell, 2013). The data were collected by taking the opinions of the students and the teacher about the visual proof activities.

Participants

This study was conducted in a public school in the central Anatolian region of Turkey. For the second semester, sixteen activities of visual proofs were applied. After the instruction period, one-on-one interviews were carried out with four students and their teacher about their views on the visual proofs related to the instruction period. The students live in rural area whose income levels are not very high. The parents of the students are farmers. Their mathematics achievements are not very high.

The content of the subject is the reason for conducting the study with nineth-grade students. Also, the study was conducted with nineth-grade students to ensure the students who have just started the proof subject to encounter the visual proofs as an alternative proof method and to gain skills about the structure of the proof by achieving the proof step by step through the activities of visual proofs. Four students and the mathematics teacher were interviewed about the visual proofs process. The students were observed throughout the activities then they were selected for interview. Four students who were selected according to purposive sampling were included in the study to conduct individual interviews, considering the student's ability to express herself/himself, her/his score and her/his interest in the lesson. Below is the information about the participants.

According to the exam score, Yunus is the most successful student in the class. He participates in the lesson effectively and can answer questions. Can does not participate effectively in the activities. He is reluctant. However, in the observations, it was noticed that he made correct determinations for the questions in the activities. Therefore, he was included in the study. Mehtap expressed her negative feelings about the lesson at the beginning of the activities, but after a few activities, she participated effectively in the activities, and she stated that she enjoyed these activities. Also, it was noted that there was a noticeable improvement in her score. Although Ferit was the most unsuccessful student in the class, he was good at the activities. So, he was included in the study.

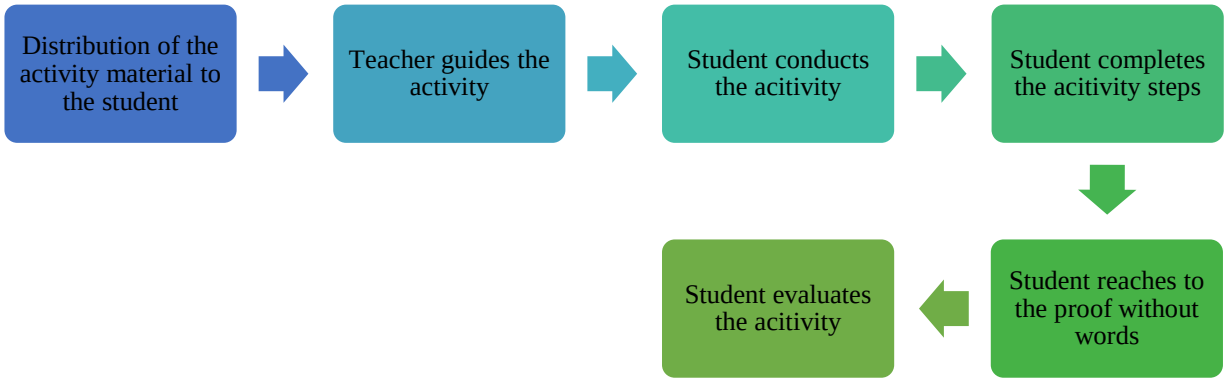
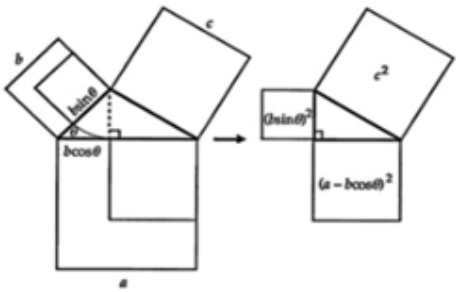


Figure 2. The teaching scheme of the course

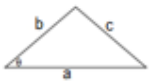
Data Collection

In the study, the activities of visual proofs were applied for a semester. Each of the 16 activities used in the study was designed by the researchers in such a way that the students would go through the proofs step by step. The activities were finalized by taking the opinions of three experts. A sample activity is presented in Figure 3. Firstly, the activity material was handed out to the students, the students were guided by the teacher depending on the instructions and the students reached the visual proofs after completing the activity stages. Lastly, the activity was evaluated by the students. In this evaluation, students were asked about the points they had difficulty with and the points they like or dislike. Figure 2 shows the teaching scheme for the course.

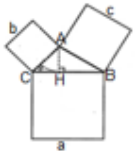
As can be seen from Figure 3, the students were first asked to discuss the visual proof. Then, what would be the other theorem to be used in proof was discussed. After these discussions, the proof was made step by step. For example, the first student was asked to draw a triangle. Then squares are drawn on the sides of this triangle. Then, the sides are found with trigonometric relations. In the last visual, equality is reached by using the Pythagorean theorem. Then, the obtuse triangle state has been discussed.



- What does figure means to you? Discuss.
- Discuss which theorem is used in this theorem.
- Let's draw acute triangle with sides a, b, c and let's call θ one of the angles of the triangle.



- Let's draw squares with sides a, b, c on sides a, b, c of this triangle.



- Find the side lengths IBHI, IAHI and ICHI using trigonometric relations.

- We will use one of the important mathematics theorem as follows to complete the proof. For you what is the theorem?

Figure 3. A sample activity

16 visual proof activities were performed for a semester to classroom. Afterwards, the interviews were conducted by the researchers with four students and the teacher. They were interviewed about their opinions, the process and experiences and the issues they had faced or did not have difficulties in the visual proofs as well as their positive or negative thoughts about the process. These interviews were recorded using a voice recorder.

The Role of the Researchers

The researchers were only observers of some activities in the study but some of them carried out the activities together with the mathematics teacher. It has been observed that the students are more active in the lessons which were conducted by the teacher and the researchers. This is because the researchers can guide the students better in the proof process, since they have more information about the stages of the proof process; also, the teacher can guide the students better with questions since he is aware of knowledge funds of his students. Besides, the teacher is qualified to carry out the visual proof activities. Because he studied visual proof within the scope of the course at a state university. After the activities, interviews with teacher and students were made by the researchers. So, the researchers were present at every stage of the process in this study and took an active role in the process. Therefore, the researchers are in a participatory-observer position. In addition, during the data collection stage, the researchers took care to play an impartial role in revealing the views of the teachers and students.

Data Collection Tool

To reveal the positive and negative effects of teaching proof with visual proofs and to put the effects of the lived experience semi-structed interview form was created by researchers. With this form, the researchers got the opinions from teacher and the students about positive and negative views on visual proofs, the effects of visual proofs and their use in the course. Four volunteer students were interviewed about the process. Interviews were conducted by one of the researchers at the end of the second semester.

Data Analysis

Content analysis technique is used to examine the cases in depth. The main purpose of the content analysis technique is to reach the concepts and relationships that can explain the collected data. Merriam (2009/2013, p. 195), states that all qualitative data analysis is content analysis because it states that the analysis of the content of the interviews, field notes and documents. Interviews were analyzed with content analysis by two experts. Categories were created as a result of the analysis of the interview with the teacher are "Necessary, Unnecessary, Lack of basic concepts related to the proof, Not to know how to start the proof, Visuality, Starting the proof, Interesting, Enjoyable, Permanent, Preparation of events, Interest, Permanence, Curiosity, Questioning boring, Failing to understand, Failing to remember". Categories were created as a result of the analysis of the interview with the students are "Conceptual understanding, Convincing, Enjoyable, Reminding the subjects, Seeing what is asked in the figure, Explaining what is asked, Doing the operation, Visualization, Connection, Positive feeling, Negative feeling, Guidance".

Reliability-Validity

There are strategies that can be used to ensure validity and reliability in qualitative research (Merriam, 2009/2013, p. 202). Validity and reliability are intertwined in qualitative research. For this reason, what has been done regarding the reliability and validity of the study is given in Table 1.

Table 1. Reliability and validity strategies of the study

Reliability-Validity	Strategy	Explanation
	Detailed description	Details of the study were described and direct quotations from the interviews were included.
	Triangulation	Analyzes were made by different researchers, and the consensus opinions were presented as a conclusion (researcher triangulation)
	Adequate participation	At the beginning of the study, it was not determined how many students would be interviewed. When the researchers thought that the data had reached a certain satisfaction, ended the interviews.
	Purposive sampling	Qualitative research tries to reach deep knowledge. For this reason, purposive sampling technique was used to get deeper information about the process.
	Longtime participation	Before starting the interviews, the researcher communicated with the students. Thus, it ensured that the students were comfortable during the interview.
	Transparency	Transparency was tried to be provided by explaining the research strategy, how the data were collected and how the analyzes were carried out.

Findings

Results and Interpretations of the Teacher's Opinions

The presence of a limited number of studies on visual proofs for application has led us to have little knowledge about the applicability of visual proofs. Therefore, the opinions of the teacher, who is the practitioner of the lesson and plays a major role in guiding students, about the application of visual proofs in the class were obtained.

Table 2 shows the answers and categories of the teacher given to the question of "What do you think about including proofs in the curriculum for high school students?"

Table 2. Teacher's opinions about including proofs in the curriculum

Categories	Subcategories	Teacher's Opinion
Necessary Unnecessary	Basic proofs Permanence Exam system	"We have included the proof in a few subjects. For example, we proved that there are infinite reel numbers between two real numbers and $\sqrt{2}$ is irrational. We could not give detailed information since the students did not understand a lot. Therefore, it would be better if there are simpler proofs that can be understood by the students. Besides, it seems unnecessary if we think on the basis of exams. It would be important and useful in terms of understanding what comes from where. At least, the permanence of information is ensured."

When the teacher's opinions about including the proofs in the curriculum for the high school students were examined, it was observed that the students did not understand much about the proofs. Therefore, it was believed that the teachers could not give the details about proofs, it would be more appropriate to use simple proofs the students can understand during the class. The teacher also stated that since they are not involved in the national wide exam system, they should not be included in the curriculum.

Table 3 shows the answer and categories given to the question of "How are the proof skills of the high school students?".

Table 3. The teacher's opinions about the proof skills of the students

Categories	Teacher's opinion
Lack of basic concepts related to the proof Not to know how to start the proof	"As the students from university, they do not know where to start. Therefore, it would be better if they have such basis before. They can at least derive something rather than memorizing the formula."

When examining the teacher's opinions about the high school students' proof skills, the teacher stated that the high school students did not know how to start the proof and they did not know basic concepts related to the proof. Therefore, it can be asserted that the appropriate proofs suggested by curriculums should be introduced to the students so that they gain the idea of proof at an early age.

Table 4 shows the answer and categories given to the question of "What do you think about how the proof skill of the students can be improved?". When examining the teacher's opinions about what to do to improve proof skill of the students, the teacher stated that something should be done about benefiting from visuality and how the students will start the proof they have difficulty in.

Table 4. The teacher's opinions about the improvement of proof skill

Categories	Teacher's Opinion
Visuality Starting the Proof	"Visual things can appeal more. I have not thought about this issue before. We actually start with acceptance. Something can be made in order for us to realize these acceptances."

Table 5 shows the answers and categories given to the question of "What do you think about including visual proofs in the high school curriculum?". When examining the teacher's opinions about including the visual proofs in the high school, the teacher considered that the visual proofs should be included in the high school since these visual proofs increased the students' interest in the proof, they were enjoyable and provided the permanence of the information. However, the teacher stated that it was not easy to prepare these activities. Thus, the experts can design the visual proof activities. So, the teachers can apply the ready activities more efficiently.

Table 5. The teacher's opinions about including visual proofs in high school curriculum

Categories	Subcategories	Teacher's Opinion
Positive	Interesting Enjoyable Permanent	"I think they are good proofs. These proofs have increased the interest of students in the class. Even sometimes they said that "Let us prove". Since they are visual, the students use their different aspects, so they become more permanent. It increases the student engagement, and it is more enjoyable.
Negative	Preparation of events	"Yes, it would be nice to apply. But it is not so easy to prepare such events. But we can apply ready to use activities."

Table 6 shows the answers and categories given to the question of "Are there any positive aspects in students after these activities in your opinion? If any, can you give an example?". According to Table 6, the positive aspects the teacher found in the students after the activities were that the students wondered where the formulas came from, in other words, they started questioning, their interest levels increased, and the formulas were kept in their minds more.

Table 6. The teacher's opinions about the positive aspects of the students

Categories	Teacher's Opinion
Interest Permanence Curiosity Questioning	"Their interest increased, and they asked different questions; "Are we using this formula or that formula?". I mean, the formulas themselves began to stick in their minds more. A few students started to wonder what came from where. They started asking me these questions in the class."

Table 7 shows the answers and categories given to the question of "Are there any negative aspects of students after these activities in your opinion? If any, can you give an example?". According to Table 7, the teacher stated that some students got bored since they did not understand the activities, some students were very active during the activity but later they did not remember anything about the activity.

Table 7. The teacher's opinions about negative aspects of students

Categories	Teacher's Opinion
Boring Failing to understand Failing to remember	Since it attracted interest of some students, the activities developed, and they were efficient but did not make any contribution to some students. So, they were bored with it. They were bored more in some activities. Sometimes, the students like what they do during the activity but then they gain nothing about the activity. For example, one student was very effective in cosine, but I realized that the student remembered the formula. But I saw this was not like such in some students.

Results and Interpretations of the Students' Opinions

The students gave positive answers to the question of "What are your thoughts about including visual proofs in high school curriculum?". When the students' answers were examined, the categories of "conceptual understanding, persuasion, enjoyable, reminding the subjects" were created concerning the visual proofs. Table 8 shows the categories prepared for this question and the examples from the answers given by the students. When Table 8 was examined, it was seen that the students stated that they wanted to know the origin of the formulas and of mathematical subject. They stated that knowing the origin of a formula convinced them. They also considered these activities enjoyable. Although they expressed that they enjoyed these proofs, they emphasized that they had difficulties in seeing what was wanted from the image.

When examining the answers given to the question "Were there points you had difficulties in doing the activities of visual proofs? If any, explain with examples", the challenging points for them were determined as "seeing what is wanted from the figure, explaining the wanted and doing the operation". Table 9 shows the categories concerning the points the students had difficulties in activities of visual proofs and the examples from answers of the students.

Table 8. The students' opinions about including visual proofs in high school curriculum.

Categories	Students' answers
Conceptual understanding	• I think it should be included. We find areas, but we do not know where they come from. • It should be included. I think what you do is very logical. Because when we do, we don't know why we are doing. But through these proofs, we can understand where they came from. • Proof teaches us where the formula comes from. This eases us. • I was doing something, but I did not understand why I was doing it. Now we are using these proofs. This comes from here and that comes from there.
Convincing	• We should learn those so we can be convinced. Constructing these proofs is convincing. • Mathematics convinced me. I want to examine the proofs. • The written proof convinces me. Since it is convincing, it becomes easier for me.
Enjoyable	• They found reasonable things. Nice things are found by applying formulas. It is also enjoyable. • I do not like mathematics very much. The class became fun in this way.
Reminding the subjects	• There are students who cannot remember the subject. Such things are important for exams. For those who forgot some things, for example I remembered most of the things I had forgotten with these proofs.

When Table 9 was examined, the students stated that they had difficulty in seeing what they were supposed to see in the figures. However, in general, they expressed that they had more difficulties in understanding the figure when they just started the activities. This became easier afterwards, and they could understand the figure more easily later. It was also found that the students were not good at operation during the activities, and they stated that they had difficulties with this subject.

To the question of "What did visual proof give you? Explain with examples", the students generally gave examples from positive effects of visual proofs. When their answers were analyzed, "visualization, connection" categories were formed as the effects of visual proofs. Table 10 shows the categories created for this question and the examples from the answers of the students.

Table 9. The students' opinions about the challenging points in visual proofs

Categories	Students' Responses
Seeing what is wanted in the figure	• I did not see what I supposed to see. I saw some matters easily. • I could not see everything in the figures. For example, those questions with squares are easy. I studied for one hour, but I couldn't figure out. Actually, it can be understood when you think and look. You just need to look longer. If you think based on formula, you cannot find it. • I could not see many things about triangles. Now I can see saying that this angle is like this and that angle is like that.
Explaining what is wanted	• I think properly in my mind. When I say that I got it and try to explain, I mess it up.
Doing the operation	• I could not remember the formula, and this compelled me. • I had difficulties in finding areas.

Table 10. The student's opinions about the effect of the visual proofs

Categories	Students' Answers
Visualization	• Formulas should not contain only mathematics. It had an effect on thinking with figure. • For example, when I say a. b the area of a rectangle with sides a and b comes to my mind. • For example, we cannot visualise it. But when we have visuals, we do it easier. So, it becomes clearer.
Connection	• Geometry was figural in the past, now the algebra has become figural. I mean, mathematics and algebra are associated. • I think there is not much difference between math and geometry. I mean I started to think that way. I have realised that I use the same things in both. For example, it is about mathematics but also used in geometry. We also learn geometry while learning math. • I could not make a connection between the written things before. Now I can.

When Table 10 was examined, it was observed that the students talked about the positive effects of visual proofs. They mentioned that the visual proofs had an effect on their visualization skills. In addition, they also expressed that showing an algebraic statement with figures was effective in seeing the connection between mathematics and geometry.

To the question "Can you describe your feelings when you understood and did not understand the visual proofs?", the students expressed that they were happy when they understood, and they were unhappy when they did not. Table 11 shows the expressions of the students about their feelings.

Table 11. The students' opinions about their feelings when they understood and did not understand the visual proofs

Categories	Students' Answers
Positive feeling	• I became happy. I know now that this came from here. • Joy, of course, I say I can. This also makes me very happy. There is joy of knowing how it is done.
Negative feeling	• I'm angry at myself when I can't see. • I am afraid of saying the wrong things, not be able to talk.

As seen in Table 11, the students expressed that they felt positive when they understood the visual proofs and they felt negative when they did not understand the visual proofs.

When the students were asked the question "Can you construct the visual proofs by yourself?", all of them stated that they could not construct the visual proofs alone and they needed someone to guide. In this sense, the visual proof activities are useful for learning proof process to the students who have just started the proof. Table 12 shows the dialogues about the students' requests to be guided by someone in activities of the visual proofs.

Table 12. The students' opinions about whether or not they can construct visual proofs alone.

Categories	Students' Opinions
Guidance	• I get bored when I am alone. I quit. When I get stuck, I say enough. If someone helps me to the points I have stuck, I can continue. • If somebody helps me, then I can do it. For example, remind me of the formula that I did not. • I cannot do it alone. If you remind me, I can.

When Table 12 was examined, it was observed that the students thought that they cannot do visual proofs without guidance. In general, it was concluded that the students considered that somebody should remind them the properties of formulas that they cannot remember.

Discussion and Results

This study suggests that visual proof can be used as an alternative proof method in the high school. According to teacher's opinion, the high school students did not know how to start the proof and they did not know basic concepts related to the proof. So, the students could not understand the proofs. Teacher said that it would be more appropriate to use simple proofs. When studies on proof are examined, it is known that students have difficulty in starting the proof (Bell, 2011; Morash, 1987, p. 146). Students fail to understand that proof includes successive logical steps and cannot transfer this to the proof process (Uğurel & Moralı, 2010; Urhan & Bülbül, 2016). In this respect, if student proves step by step like in visual proof activities, it can be eliminating these difficulties related to proof. The teacher also stated that visualization can be used to make easier for students to do proof. In the

literature, it is stated that visualization can be used in proof (Flores, 2000; Hanna & Sidoli, 2007). In this sense, it can be considered that the use of visual proofs would be appropriate if these proofs can be adjusted to the level of the students. Therefore, it can be asserted that the appropriate proofs suggested by curriculums should be introduced to the students so that they gain the idea of proof at an early age. According to the teacher, after the visual proof activities the students started to wonder where the formulas came from, they started questioning, their interest increased, and formulas became more permanent in some student's minds. Therefore, as stated in other studies, visual proof activities are effective for students' learning (Demircioğlu & Polat, 2015; İpek, 2010).

When examining the teachers' opinions about the negative aspects in students after the activities, he stated that some students got bored since they did not understand the activities. Like Mudaly's (2013) study, it is not always an easy task to understand visual proofs. Sometimes visual proofs can be complicated for students, so the teacher can provide a flowchart arrangement and use bold print and color in the diagram, to help students focus on important parts (Nirode, 2017). Also, the teacher stated that some students were very active during the activity but later they did not remember anything about the activity. Further studies are needed to evaluate this situation. Studies can shed light on this situation by revealing the students' approaches to proof after performing visual proof activities.

Consequently, it can be asserted that visual proofs gain importance in order for students to gain proof process skills. Therefore, the activities of visual proofs among the informal proofs that will allow the student to play an active role in the proof process can be presented as an alternative method in proof teaching. For this, it is necessary especially for teachers to know visual proofs and use them in their lessons.

The students who participated the application said that visual proofs are effective in visualization and connecting mathematical subjects. The students stated that they wanted to know the origin of a formula and of a mathematical subject. Knowing the origin of a property convinces the students. This result supports the result of Tekin and Konyalıoğlu (2010) indicating that proofs based on visual figures informed the students about how the formulas were found and where they came from and helped them learn permanently by avoiding memorization. In addition, according to Karras (2012), visual reasoning can ensure to understand mathematical concepts. Similarly, Doruk et al., (2014) expressed in their study that the information was more persistent since the pre-service teacher had concrete experiences in symbolic proof activities. The proofs were seen to be useful in terms of understanding what comes from where and the permanence of the information. The students can understand what comes from where in the formulas with visual proofs. So, it can be said that visual proof supports discovering the formulas. This result is similar to the diagrams supports discovering the proof's key idea (Marco, et. al, 2022).

The high school students stated that they found the visual proof activities enjoyable, and they felt positive feelings when they understood the visual proofs and felt negative feelings when they could not understand them. According to some studies, high school students, pre-service teachers and teachers find visual proofs enjoyable and amusing (Demircioğlu & Polat; Doruk et al., 2014; İpek, 2010; Uğurel et al., 1996). Therefore, visual proofs arouse curiosity of students and increased their motivation to construct proof. In addition, according to results, it was seen that the students mostly talked about the positive effects of the visual proofs. This is compatible with the results of the studies in the literature (Demircioğlu & Polat, 2015; Kristiyajati & Wiyaja, 2018; Uğurel et al., 2016). Also, according to Nirode (2017), arranging a visual proof is like solving a puzzle, and most people enjoy working on puzzles. Indeed, in many visual proofs, diagrams are rotated and flipped, just like puzzles.

Although the students stated that they enjoyed the visual proofs, they had difficulties. The challenging points for the students while constructing the visual proofs were determined as "seeing the wanted in the figure, explaining the wanted, and doing operation". However, in general, the students expressed that they had more difficulties in understanding the things shown when they just started the activities, but this became easier afterwards. This result is similar to the results of studies (Demircioğlu & Polat, 2016; Uğurel et al., 2016).

The students mentioned that visual proofs affected their visualization skills. This situation is like the results of some studies **(**Doruk et al., 2014; Uğurel et al., 2016). In addition, the students stated that showing an algebraic expression with figures was effective in seeing the connection between mathematics and geometry. Nirode (2017) stated that he used visual proofs as a bridge between informal reasoning and writing the complete proof. But in the study of Demircioğlu and Polat (2016) the preservice teachers mentioned that the visual proofs were appropriate for the geometry class, their application areas were limited. This shows that even the preservice teacher could not gain this connection skill (Polat & Demircioğlu, 2021). However, making connection between fields is important for mathematics education.

The students thought that they cannot do visual proofs without guidance. According to Marco, et. al (2022) study, this is due to the need to reduce students' doubts when producing their proofs like. So, we can say that step by step proof teaching approach can be used in proof teaching.

Limitations

The most important limitation of this study was that it was not possible to find visual proofs for all nineth-grade subjects. So, the results were restricted with these proofs. In addition, the cosine theorem was used in the proof of the Pythagorean theorem, although it was not included in the nineth-grade.

Suggestions

According to high school students, visual proofs made concrete some mathematical concepts. So visual proof can be used to provide students with concrete awareness of both mathematical concepts and proofs.

The students are irrelevance and unsuccessful in proof. However, in this study, students found visual proof activities effective and interesting. Therefore, it can be recommended to use visual proofs in proof teaching.

In the teacher's view, it can be difficult to prepare visual proof activities. So, these activities can be prepared by experts and presented as an alternative method to proof teaching.

It has been observed that students have difficulty in seeing what is required in visual proof. Therefore, by using various methods such as coloring or textual clues to understand the visual in these proofs this difficulty can be eliminated.

Author Contribution Rate

The first author carried out the planning, literature review and implementation process of the study. All authors contributed to the data collection process. The first and second authors analyzed the data. All authors contributed to the writing of the article and read and approved the final version of the study.

Ethical Declaration

The purposes and procedure of the current study were granted approval from the Ethics Committee of the Sivas Cumhuriyet University (Ethics Committee's Decision Date: 13.12.2021, Ethics Committee Approval Issue Numbers E-30568564850-2100343018).

Conflict of Interest

The authors declare that there is no conflict of interest with any institution or person within this study.

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