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Finite Complete Rewriting Systems for the Monoids M, ρ, and M/ρ

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ABSTRACT

Let M be a monoid and ρ be an equivalence relation on M such that ρ is a congruence. So, ρ is a submonoid of the direct product of monoids $M \times M$, and $M/\rho = \{x\rho: x \in M\}$ is a monoid with the operation $(x\rho)(y\rho) = (xy)\rho$. First, an introductory lemma is proposed, proved and a relevant example is given. Then, it is shown that if ρ can be presented by a finite complete rewriting system, then so can M. As the final part of the main result, it is proved that if ρ can be presented by a finite complete rewriting system, then so can M/ρ.

M, ρ ve M/ρ Monoidleri için Sonlu Tam Yeniden Yazma Sistemleri

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ÖZ

M bir monoid ve ρ, M üzerinde kongrüans olacak biçimde bir denklik bağıntısı olsun. Böylece, ρ, $M \times M$ monoidlerinin direkt çarpımının bir alt monoidi ve $M/\rho = \{x\rho: x \in M\}$ kümesi $(x\rho)(y\rho) = (xy)\rho$ işlemi ile bir monoid olur. Öncelikle, bir giriş lemması ifade ve ispat edilerek konu ile ilgili bir örnek verilmektedir. Daha sonra, eğer ρ bir sonlu tam yeniden yazma sistemi ile takdim edilebilir ise, M'nin de bir sonlu tam yeniden yazma sistemi ile takdim edilebilir olduğu gösterilmektedir. Ana sonucun son kısmında, eğer ρ bir sonlu tam yeniden yazma sistemi ile takdim edilebilir ise, M/ρ monoidinin de bir sonlu tam yeniden yazma sistemi ile takdim edilebilir olduğu gösterilmektedir.

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1. Introduction

String rewriting systems can be regarded as the basis in the development of theoretical computer science and have been widely studied by researchers recently (Squier et al., 1994; Wang, 1998, Wong et al., 2010; Gray, Malheiro, 2011). Max Dehn in (Dehn, 1911) introduced the word problem, whose solvability is one of the fundamental questions in combinatorial semigroup theory, for finitely presented groups. It is well known that there can be no calculus to solve the word problem in finitely presented groups and monoids. So it is important to know which monoids (or groups) have a solvable word problem. A condition for the word problem on a monoid to be solvable is that the monoid can be presented by a *finite complete rewriting system*, or in short an FCRS.

In (Ayık et al., 2005), it is shown that if ρ is finitely presented as a subsemigroup of the direct product $S \times S$, then S is finitely presented where S is a semigroup, and ρ is a congruence on S. It was shown in (Wang, 2007) that if ρ has finite derivation type as a subsemigroup of $S \times S$, then S has finite derivation type and in addition, it is proved that the converse is not true. The author in (Kuyucu, 2011) investigated some relations among ranks of semigroups S, ρ and S/ρ. In (Çalışkan, 2010), some finiteness conditions for the semigroups S, ρ and S/ρ are studied. FCRSs for matrix semigroup presentations are examined in (Özer and Yüksek, 2016). Authors in (Çetinalp and Karpuz, 2021) studied the solvability of the word problem for infinite groups by obtaining a presentation for the crossed product and then finding its complete rewriting system.

2. Materials and Method

2.1. Semigroup presentations and rewriting systems

In this section, we give preliminary material needed in our forthcoming results. The readers who would like further reading on semigroup presentations and rewriting systems may refer to (Book and Otto, 1993; Sims, 1994; Howie, 1995).

Let A be an alphabet, A^+ be the set of all non-empty words over A, and let A^* be the free monoid (i.e. A^+ is determined with the empty word) on A. A semigroup presentation is an ordered pair $\langle A|R\rangle$ where $R \subseteq A^+ \times A^+$ (Gray and Malheiro, 2011). An element a of A is called a generating symbol, while an element (r, s) of R is called a defining relation, and is usually written as $r = s$. If the set of defining symbols $A = \{a_1, a_2, a_n\}$ and the set of defining relations $R = \{r_1 = s_1, r_2 = s_2, \dots, r_n = s_n\}$, then we write $\langle A|R\rangle = \langle a_1, a_2, \dots, a_n | r_1 = s_1, r_2 = s_2, r_n = s_n\rangle$. The semigroup defined by a presentation $\langle A|R\rangle$ is A^+/ρ, where ρ is the least congruence on A^+ containing R. For any two words $u_1, u_2 \in A^+$ we write $u_1 = u_2$ if $u_1\rho$=$u_2\rho$, and $u_1 \equiv u_2$ if they are identical words. We say $u_1 = u_2$ if and only if there is a finite sequence $u_1 \equiv \alpha_1, \alpha_2, \dots, \alpha_k \equiv u_2$ of words from A^+, in which every term α_i $1 < i \leq k$ is obtained from the previous one by applying one relation from R. The semigroup S is finitely generated if the set A is finite; S is finitely presented if the sets A and R are both finite. A monoid presentation is defined similarly (Gray and Malheiro, 2011).

A (string) rewriting system on A is a subset $R \subseteq A^* \times A^*$. A rule of R is an element $(u, v) \in R$, and is also denoted by $u \to v$. A rewriting system is used to replace the words on the left sides of rules with the corresponding right sides. Suppose u and v are in A^*. The single-step reduction relation $\to_R$ is the following relation on A^*,

$u \to_R v$ iff $u \equiv x\alpha y$ and $v \equiv x\beta y$ for some $(\alpha, \beta) \in R$ and $x, y \in A^*$ (Wang, 1998). We say that u is obtained from v using R if there is a sequence of words $u \equiv u_1, u_2, \dots, u_t \equiv v$ with $t \geq 0$ such that u_{i+1} is obtained from u_i in one step. This procedure is called a *multi-step reduction relation* and is denoted by $u \to_R^* v$.

R is called a Noetherian rewriting system if there is no infinite sequence $w \rightarrow_R w_1 \rightarrow_R w_2 \rightarrow_R \cdots$ for any word $w \in A^*$, and confluent if whenever $w \rightarrow_R^* u$ and $w \rightarrow_R^* v$, there is a $z \in A^*$ such that $u \rightarrow_R^* z$ and $v \rightarrow_R^* z$. R is complete if it is both Noetherian and confluent (Pride, 2000). If a complete rewriting system R has a finite generating set, then R is called a *finite complete rewiting system* (FCRS).

If a monoid admits a presentation with respect to some generating set A that forms a finite complete rewriting system R, then the monoid is obtained as an FCRS (Otto, 1984).

3. Results and Discussion

In this section, we state and prove our results on FCRSs for the monoids ρ, M and M/ρ. Let φ_i be the i-th projection from ρ to S for $i = 1,2$. If $\langle X|R\rangle$ and $\langle X_i|R_i\rangle$ are finite semigroup presentations for ρ and M, respectively, then it is obtained that $X_i = \varphi_i(X)$, $R_i = \varphi_i(R) = \{(\varphi_i(r), \varphi_i(s)): (r,s) \in R\}$ $(i = 1,2)$ (Ayık et al., 2005).

Lemma 3.1. Let M be a monoid and ρ be a congruence on M. Let φ_i be defined as above and let X be a finite generating set for ρ. Then, there exist $(u,u) \rightarrow_R^* (v,v)$ for all $(u,u),(v,v) \in X^*$ iff there exist $u, v \in X_i^* = \varphi_i(X)^*$ such that $u \rightarrow_{R_i}^* v$, where $u = \varphi_i(u,u)$, $V = \varphi_i(v,v)$ $(i = 1,2)$.

Proof. Suppose there exist $(u,u) \rightarrow_R^* (v,v)$ for all $(u,u),(v,v) \in X^*$. So, we have

$$(u,u) \equiv (r_1,s_1),(r_2,s_2),\dots,(r_t,s_t) \equiv (v,v)$$

with $t \geq 0$ such that $(r_{j+1},s_{j+1}) \in X^*$ is obtained from $(r_j,s_j) \in X^*$ in one step, (that is, $(r_{j+1},s_{j+1}) \equiv (x,y)(\alpha,\beta)(q,z)$, $(r_j,s_j) \equiv (x,y)(\gamma,\delta)(q,z)$, $(x,y),(q,z) \in X^*$ and $((\alpha,\beta),(\gamma,\delta)) \in R$, $0 \leq j < t$). If we apply φ_i to this squence, then we have

$$\varphi_i(u,u) \equiv \varphi_i(r_1,s_1),\varphi_i(r_2,s_2),\dots,\varphi_i(r_t,s_t) \equiv \varphi_i(v,v)$$

$$u \equiv r_1,r_2,\dots,r_t \equiv v \text{ (or } u \equiv s_1,s_2,\dots,s_t \equiv v)$$

such that $r_{j+1} \in \varphi(X)^* = X_i^*$ is obtained from $r_j \in \varphi(X)^* = X_j^*$ in one step. If we take $u = \varphi_i(u,u)$ and $v = \varphi_i(v,v)$, so we have $u \rightarrow_{R_i}^* v$.

($\Leftarrow$): Assume that there exist $u, v \in X_i^*$ such that $u \rightarrow_{R_i}^* v$. From Lemma 2.1 (Ayık et al., 2005), we have $X_i = \varphi_i(X)$ is a generating set for M. Also, since ρ is a congruence on $M$and from Proposition 2.3 (Ayık et al., 2005) ρ has a finite reflexive generating set, then $(u,u),(v,v) \in \rho$. Since $u \rightarrow_{R_i}^* v$ we have

$$u \equiv q_1,q_2,\dots,q_m \equiv v$$

with $m \geq 0$ such that $q_{k+1} \in X_i^*$ is obtained from $q_k \in X_i^*$ in one step, (that is, $q_{k+1} \equiv apb$, $q_k \equiv azb$, $a,b \in X_i^*$ and $(p,z) \in R_i$, $0 \leq k < m$). By the property of transitivity, we have all relations $(q_1,q_2),(q_2,q_3),\dots,(q_{m-1},q_m) \in \rho = \langle X\rangle$ with $m \geq 0$ such that $(q_{k+1},q_{k+2}) \in X^*$ is obtained from $(q_k,q_{k+1}) \in X^*$ in one step, (that is, $(q_{k+1},q_{k+2}) \equiv (x',y')(\alpha',\beta')(q',z')$,

$(q_k, q_{k+1}) \equiv (x', y')(\gamma', \delta')(q', z')$, $(x', y'), (q', z') \in X^*$ and $\big((\alpha', \beta'), (\gamma', \delta')\big) \in R$, $0 \leq k < m$). Therefore, if we choose $(u, u) = (q_1, q_1)$ and $(v, v) = (q_m, q_m)$, then we have $(u, u) \rightarrow_R^* (v, v)$.

Example 3.2. Let M be a monoid, ρ be a congruence on M, $X = \{(a, a), (b, b)\}$ be a finite generating set for ρ and $\{\big((a^2, a^2), (e, e)\big), \big((b^2, b^2), (e, e)\big), \big((bab, bab), (aba, aba)\big)\}$ be the rewriting rules of R.

If φ_i is applied to R, then $\varphi_i(R) = R_i = \{(a^2, e), (b^2, e), (bab, aba)\}$ is obtained. By choosing $u \equiv ba^5b^3a$ and $v \equiv ab$, we have the following results:

$$u \equiv ba^5b^3a \xrightarrow{(a^2,e)}_{R_i} ba^3b^3a \xrightarrow{(a^2,e)}_{R_i} bab^3a \xrightarrow{(b^2,e)}_{R_i} baba \xrightarrow{(bab,aba)}_{R_i} aba^2 \xrightarrow{(a^2,e)}_{R_i} ab \equiv v$$

and

$$\begin{aligned}(u, u) &\equiv (ba^5b^3a, ba^5b^3a) \equiv (ba^3, ba^3)(a^2, a^2)(b^3a, b^3a) \\ &\xrightarrow{(a^2,a^2)=(e,e)}_R (ba^3b^3a, bab^3a) \equiv (ba, ba)(a^2, a^2)(b^3a, b^3a) \\ &\xrightarrow{(a^2,a^2)=(e,e)}_R (bab^3a, bab^3a) \equiv (bab, bab)(b^2, b^2)(a, a) \\ &\xrightarrow{(b^2,b^2)=(e,e)}_R (baba, baba) \equiv (bab, bab)(a, a) \\ &\xrightarrow{(bab,bab)=(aba,aba)}_R (aba^2, aba^2) \equiv (ab, ab)(a^2, a^2) \\ &\xrightarrow{(a^2,a^2)=(e,e)}_R (ab, ab) \equiv (u, u).\end{aligned}$$

Theorem 3.3. Let M be a monoid and ρ be a congruence on M. If ρ can be presented by an FCRS as a submonoid of $M \times M$, then so can M.

Proof. Let $\langle X|R\rangle$ be a finite monoid presentation for the monoid ρ. Suppose R is an FCRS on X as a submonoid of $M \times M$. Let $\langle X_i|R_i\rangle$ be a finite monoid presentation for M with the above notation. We will prove that R_i is an FCRS on X_i.

Firstly we show that R_i is confluent.

Let $u \rightarrow_{R_i}^* v$ and $u \rightarrow_{R_i}^* w$, where $u, v \in X_i^*$. By Lemma 3.1, there are words $(u, u), (v, v)$ and $(w, w) \in X^*$ such that $(u, u) \rightarrow_R^* (v, v)$ and $(u, u) \rightarrow_R^* (w, w)$, where $u = \varphi_i(u, u)$, $v = \varphi_i(v, v)$ and $w = \varphi_i(w, w)$. Since R is an FCRS on X, there is an irreducible word $(q, q) \in X^*$ such that $(v, v) \rightarrow_R^* (q, q)$ and $(w, w) \rightarrow_R^* (q, q)$. If we take $q = \varphi_i(q, q)$, then, from Lemma 3.1 we have

$$v = \varphi_i(v, v) \rightarrow_{R_i}^* \varphi_i(q, q) = q$$

and

$$w = \varphi_i(w, w) \rightarrow_{R_i}^* \varphi_i(q, q) = q$$

Thus, R_i is confluent.

Now, we want to show that R_i is Noetherian. Suppose there is an infinite sequence such

$$w_1 \rightarrow_{R_i} w_2 \rightarrow_{R_i} w_3 \cdots$$

for all $n \geq 1$, $w_n \in X_i^*$. By Lemma 3.1, for all $n \geq 1$, $w_n \in X_i^*$, there exist $(w_n, w_n) \in X^*$ such that $w_n = \varphi_i(w_n, w_n)$. Therefore, we have the following squence

$$w_1 = \varphi_i(w_1, w_1) \rightarrow_{R_i} w_2 = \varphi_i(w_2, w_2) \rightarrow_{R_i} w_3 = \varphi_i(w_3, w_3) \cdots$$

and

$$(w_1, w_1) \rightarrow_R (w_2, w_2) \rightarrow_R (w_3, w_3) \cdots$$

for all $n \geq 1$, $(w_n, w_n) \in X^*$. Since we suppose R is an FCRS on X, so R is Noetherian, so the squence $(w_1, w_1) \rightarrow_R (w_2, w_2) \rightarrow_R (w_3, w_3) \cdots$ and also $w_1 \rightarrow_{R_i} w_2 \rightarrow_{R_i} w_3 \cdots$ aren't infinite sequences. This a contradiction. As a result, R_i is Noetherian.

Let ρ be a congruence on a monoid M. If ρ is generated by a subset X of $M \times M$, then it is clear that ρ is also generated by X as a congruence on M. Therefore, if ρ is finitely generated as a monoid, then it is also finitely generated as a congruence.

Corollary 3.4. Let M be a monoid and ρ be a congruence on M such that ρ is finitely generated as a congruence on M. If ρ can be presented by an FCRS, then so can M/ρ.

Proof. Suppose ρ can be presented by an FCRS, then it follows from Theorem 3.3 that M can be presented by an FCRS. Also, ρ is finitely presented on any finite generating set such as $Y = \{(\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_k, \beta_k)\}$. Let $\langle X|R \rangle$ and $\langle X_i|R_i \rangle$ be finite monoid presentations for ρ and M, respectively.

By the proof of Lemma 2.5 in (Ayık et al., 2005), M/ρ is defined by the presentation $\langle X_i|R_i \cup \{\alpha_1 = \beta_1, \alpha_2 = \beta_2, \ldots, \alpha_k = \beta_k\} \rangle$, where $Y = \{(\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_k, \beta_k)\}$ is a finite set included by the smallest congruence ρ. As φ_i is the *i* th projection from ρ to M and it is a surjective mapping, we take

$$Y = \{\varphi_i(\alpha_1 = \beta_1), \varphi_i(\alpha_2 = \beta_2), \ldots, \varphi_i(\alpha_k = \beta_k)\}$$

Then $\langle X_i|R_i \cup \{\varphi_i(\alpha_1 = \beta_1), \varphi_i(\alpha_2 = \beta_2), \ldots, \varphi_i(\alpha_k = \beta_k)\} \rangle$ is a finite presentation for M/ρ and by Theorem 3.3, $R_i \cup \{\varphi_i(\alpha_1 = \beta_1), \varphi_i(\alpha_2 = \beta_2), \ldots, \varphi_i(\alpha_k = \beta_k)\}$ is an FCRS on X_i. As a result, M/ρ can be presented by an FCRS.

4. Conclusion

We study a monoid M, a congruence ρ on M, and a rewriting system R on a generating system X in this paper. First, we propose and prove that a necessary and sufficient condition for $(u, u) \rightarrow_R^* (v, v)$ for all $(u, u), (v, v) \in X^*$ is that, there exist $u, v \in X_i^* = \varphi_i(X)^*$ such that $u \rightarrow_{R_i}^* v$. Then we give an example. We show that if ρ can be presented by an FCRS, then so can M. As our final and main result, we prove that if ρ can be presented by an FCRS, then M/ρ can also be presented by an FCRS.

We sincerely hope that this study will make a valuable addition to the literature on semigroups, presentations, and rewriting systems.

Conflict of Interest

The authors declare that there is no conflict of interest.

Authors' contributions

Both authors contributed equally.

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