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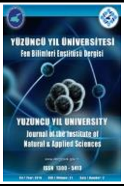

Research Article

A Generalization of G-Nilpotent Units in Commutative Group Rings to Direct Product Groups

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Abstract: Let $V(RG)$ denote the normalized unit group of the group ring RG of a group G over a ring R. The concept of G-nilpotent unit in a commutative group ring has been defined in (Danchev, 2012). In this study, some necessary and sufficient conditions for a normalized unit group in a commutative group ring of a direct product group $G \times H$ to consist only of $G \times H$-nilpotent units have been given and especially some results which are related to groups $G \times C_3$ and $G \times C_4$ have been introduced where C_3 and C_4 are cyclic groups of orders 3 and 4 respectively. In this context, we can say that the paper extends the results in (Danchev, 2012). At the end, an open problem is served as a future work.

Değişmeli Grup Halkalarında G-Nilpotent Birimsel Elemanların Direkt Çarpım Gruplarına Bir Genellemesi

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Öz: $V(RG)$, bir R halkası üzerindeki bir G grubunun RG grup halkasının normalleştirilmiş birim grubunu göstersin. Değişmeli bir grup halkasındaki G-nilpotent birimsel kavramı (Danchev, 2012)'de tanımlanmıştır. Bu çalışmada da, bir $G \times H$ direkt çarpım grubunun değişmeli grup halkasında normallenmiş birimsel elemanlar grubunun sadece $G \times H$-nilpotent birimsel elemanlardan oluşabilmesi için bazı gerek ve yeter şartlar verilmiştir. Ayrıca özel olarak $G \times C_3$ ve $G \times C_4$ gruplarına dair bazı sonuçlar sunulmuştur ki burada C_3 ve C_4 sırasıyla 3 ve 4 mertebeli devirli gruplardır. Bu bağlamda, makale (Danchev, 2012)'deki sonuçları genişletir diyebiliriz. Sonunda, gelecek çalışma için açık problem sunulmuştur.

1. Introduction

Let R be a ring and G be a group. Then the group ring RG is the set of all finite sums $\sum_{g\in G} r(g)g$ where $r(g) \in R$. The operations on the ring structure RG can be seen in (Sehgal, 1978; Karpilovsky, 1982; Milies & Sehgal, 2002; Görentaş, 2020) in detail. The sets of all units that are multiplicative invertible elements and normalized units which have augmentation 1 in RG are shown by $U(RG)$ and $V(RG)$ respectively (Küsmüş, 2020). Augmentation of a unit $u = \sum_{g\in G} r(g)g \in RG$ is defined as follows (Sehgal, 1978; Milies & Sehgal, 2002):

$$\varepsilon(u) = \sum_{g\in G} r(g) \tag{1}$$

Actually, one can see that $\varepsilon: RG \to R$ is a ring homomorphism with the transformation defined as in above equality. The kernel of ε is defined as follows:

$$\Delta(G) = \{\gamma \in RG: \varepsilon(\gamma) = 0\} \tag{2}$$

and it is generated as

$$\Delta(G) = \langle g - 1: g \in G, g \neq 1_G\rangle \tag{3}$$

which is said to be *augmentation ideal* of RG (Sehgal, 1978; Milies & Sehgal, 2002).

The p-primary component of a group G is generally displayed by G_p which consists of elements of order p^k for some $k \in \mathbb{N}$ and so the maximal torsion part G_0 of G is a co-product of primary components as (Danchev, 2010 and 2012).

$$G_0 = \coprod_p G_p \tag{4}$$

All the elements of G are trivial units in $V(RG)$ (Danchev, 2008 and 2009). An element e of a ring R is said to be idempotent if $e^2 = e$ and the set of all idempotent elements is shown by $id(R)$ (Görentaş, 1999). Also, we know that idempotent elements in a group ring RG have been defined as (Danchev, 2010).

$$id(RG) = \langle \sum_{r_g\in id_C(R)} r_g g : g \in G\rangle \tag{5}$$

An element a of R is called by nilpotent if $a^n = 0$ for some $n \in \mathbb{N}$. For a ring R, $N(R)$ is the set of all nilpotent elements in R and is said to be nil-radical of R. For an ideal $S \leq R$, $I(SG; G)$ is a fundamental ideal and $I(RG; H)$ is relative augmentation ideal of RG with respect to $H \leq G$ (Danchev, 2012). As mentioned in (Küsmüş, 2020), Danchev (2012) has defined some sets such as $inv(R) = \{p: p.1_R \in U(R)\}$, $zd(R) = \{p: pr = 0, \exists r \in R\backslash\{0\}\}$ and $supp(G) = \{p: G_p \neq 1\}$. He has also defined the followings:

Definition 1.1. Let $u \in V(RG)$. Then u is said to be G-nilpotent if $u = g(1 + n)$ for some $g \in G$ and $n \in I(N(R)G; G)$.

Definition 1.2. $V(RG)$ is called G-nilpotent if

$$V(RG) = G \times \big(1 + I(N(R)G; G)\big) \tag{6}$$

Under these definitions, Danchev (2012) has formally shown that $V(RG)$ is G-nilpotent if and only if $V(SG) = G$ where $S = R/N(R)$.

By the way, we deal with defining a novel type of units which are lifted from nilpotent elements because nilpotents are also special type elements in a group ring and we have a lot of information and motivation related to nilpotents and nil-radical of a ring in the corresponding literature. We already have some type of units which are well-known such as Bass cyclic units, bicyclic units, etc. By this reason, it is better to generate novel types of units using other type of elements in a group ring.

2. Material and Methods

In this section, we give some motivation and definitions related to the direct products of two commutative groups.

Let G and H be two commutative groups with p-primary and q-primary components G_p and H_q respectively. Utilizing maximal torsion parts of G and H, we show the maximal torsion part of the direct product $D = G \times H$ as follows:

$$D_0 = \coprod_p \coprod_q G_p \times H_q = \coprod_q G_p \times \coprod_q H_q \tag{7}$$

where p and q are prime integers (Küsmüş, 2019).

Due to the fact that $G_p = 1$ means that G has no p-primary component, we indicate by the notation $G_p \times H_q = 1$ that G or H has no p-primary or q-primary components respectively (Küsmüş, 2020).

$$supp_C(G \times H) = \{pq: G_p \times H_q \neq 1\} \tag{8}$$

is said to be the support of $G \times H$ (Küsmüş,2020).

Besides, we use the sets

$$zd_C(R) = \{pq: \exists 0 \neq r \in R, pqr = 0\} \tag{9}$$

and

$$inv_C(R) = \{pq: pq.1 \in U(R)\} \tag{10}$$

are defined in (Küsmüş, 2020).

Throughout the paper, we also need the following propositions and definitions related to the ring R.

Proposition 2.1. Let R be a commutative and unital ring and $N(R)$ be the nil-radical of R. Then (Danchev, 2012).

$$U(R/N(R)) = \{r + N(R): r \in U(R)\} \tag{11}$$

Proposition 2.2. Since R is a commutative and unital ring (Danchev, 2012),

$$inv(R) = inv(R/N(R)) \tag{12}$$

Definition 2.3. Let $\wp$ be the set of all prime integers. Then (Danchev, 2012),

$$np(R) = \{p \in \wp: \exists s \in R/N(R), ps \in N(R)\} \tag{13}$$

Corollary 2.4. $np(R) = zd(R/N(R))$ (Danchev, 2012).

We know that a ring R has nontrivial idempotents if and only if $R/N(R)$ has nontrivial idempotents as well. Actually, we can lift idempotent elements of a ring R from the nil-radical $N(R)$ (Bourbaki, 1989). Hence, if the quotient ring $R/N(R)$ has nontrivial idempotents, we can say R has so as well. Now, we can define $G \times H$-nilpotent units since $G \times H$ is the direct product of groups G and H.

Definition 2.5. Let $u \in V(R(G \times H))$. Then u is said to be $G \times H$-nilpotent if $u = gh(1 + n)$ for some $g \in G, h \in H$ and $n \in I(N(R)G \times H; G \times H)$, we say $V(R(G \times H))$ is $G \times H$-nilpotent if every units in $V(R(G \times H))$ is $G \times H$-nilpotent.

In the next section, we investigate some necessary and sufficient conditions for the normalized unit group $V(R(G \times H))$ to has only $G \times H$-nilpotent units.

3. Results

Firstly, we should note that $C_n = \langle x : x^n = 1 \rangle$ denotes a cyclic group with a generator x of order n throughout the section. Now, recall some definitions in (Küsmüş, 2020) such as

$i)$ $supp_C(G \times H) = \{pq : G_p \times H_q \neq 1\}$

$ii)$ $zd_C(R) = \{pq : \exists 0 \neq r \in R, pqr = 0\}$

$iii)$ $inv_C(R) = \{pq : pq.1_R \in U(R)\}$

Theorem 3.1. $V\big(R(G \times H)\big)$ is $G \times H$-nilpotent $\Leftrightarrow$ R is indecomposable and reduced,

$$V(R/N(R)(G \times H)_0) = (G \times H)_0 \tag{14}$$

and the followings hold:

$i.$ $G \times H$ has only maximal torsion part or
$ii.$ $G \times H \neq (G \times H)_0$ and

$$supp_C(D) \cap [inv_C(R) \cup zd_C(R)] = \emptyset \tag{15}$$

Proof. First, assume that $V\big(R(G \times H)\big)$ is $G \times H$-nilpotent and R is decomposable. Then, there exists a nontrivial $r \in id(R)$. Thus, we can generate a nontrivial unit in the unit group $V(R/N(R)(G \times H))$ such as

$$u = u(r, g, h) = 1_{R/N(R)} - \big(r + N(R)\big) + \big(r + N(R)\big)gh \in V(\frac{R}{N(R)}G \times H) \backslash (G \times H) \tag{16}$$

with the inverse

$$u^{-1} = 1_{R/N(R)} + \big(r + N(R)\big)(-1 + (gh)^{-1}) \tag{17}$$

This contradicts with Prop. 6 in (Danchev, 2012). Similarly, if we assume that R has a nontrivial nilpotent element, then

$$v = 1_{\frac{R}{N(R)}} + \big(f + N(R)\big) - \big(f + N(R)\big)gh \tag{18}$$

is a nontrivial unit where $f \notin N(R)$. This condradiction also shows that R has to be reduced. We know that

$$V\left(\frac{R}{N(R)}D_0\right) \subseteq V\left(\frac{R}{N(R)}D\right) \tag{19}$$

and also $V(R/N(R)D)) = D$ by the assumption. Therefore,

$$V\left(\frac{R}{N(R)}(G \times H)_0\right) = V\left(\frac{R}{N(R)}(G \times H)_0\right) \cap G \times H = (G \times H)_0 \tag{20}$$

and if $G \times H = (G \times H)_0$, we are done. Let us assume that $G \times H \neq (G \times H)_0$ and

$$supp_C(G \times H) \cap inv_C(R) \neq \emptyset \tag{21}$$

In this case, we obtain

$$e = \frac{1}{pq}\left(1 + gh + \cdots + gh^{o(gh)-1}\right) = e^2 \tag{22}$$

which is a nontrivial idempotent where $pq \in supp_C(G \times H) \cap inv_C(R)$. So we can attain a nontrivial unit as above using $e \in id(R)$ which is a contradiction. Hence,

$$supp_C(G \times H) \cap inv_C(R) = \emptyset \tag{23}$$

On the other hand, if

$$supp_C(G \times H) \cap zd_C(R) \neq \emptyset \tag{24}$$

then

$$\big(r + N(R)\big)\big(1 - g_p h_q\big)^{pq} = 0_{R/N(R)} \tag{25}$$

where $pqr = 0$, $g_p \in G_p \leq G$ and $h_q \in H_q \leq H$. Thus

$$u = 1 + \big(r + N(R)\big)(1 - g_p h_q) \tag{26}$$

is a nontrivial unit in $V\big(R/N(R)(G \times H)\big)$ which is another contradiction. So it has to be realized that $supp_C(G \times H) \cap zd_C(R) = \emptyset$. Conversely, let R be an indecomposable and reduced ring and also

$$supp_C(G \times H) \cap [inv_C(R) \cup zd_C(R)] = \emptyset \tag{27}$$

We have

$$V\left(\frac{R}{N(R)}(G \times H)_0\right) \cap (G \times H) = (G \times H)_0 \tag{28}$$

and

$$V\left(\frac{R}{N(R)}G \times H\right) = V\left(\frac{R}{N(R)}(G \times H)_0\right)(G \times H) \tag{29}$$

(May, 1976, p. 491). Extending the group epimorphism $\pi: G \times H \longrightarrow \frac{G \times H}{(G \times H)_0}$ over the quotient ring $R/N(R)$ to

$$\pi: R/N(R)(G \times H) \longrightarrow R/N(R)(\frac{G \times H}{(G \times H)_0}) \tag{30}$$

we get the inclusion

$$V\left(\frac{R}{N(R)} G \times H\right) \subseteq V(R/N(R)(\frac{G \times H}{(G \times H)_0}) \tag{31}$$

Utilizing Lemma 4. in (May, 1976), one can notice that

$$V\left(\frac{R}{N(R)}\left(\frac{G \times H}{(G \times H)_0}\right)\right) = \frac{G \times H}{(G \times H)_0}(1 + N(\frac{R}{N(R)}\left(\frac{G \times H}{(G \times H)_0}\right))^0) \tag{32}$$

Here, we denote the nilpotent elements which have augmentation 0 by $N(\frac{R}{N(R)}\left(\frac{G\times H}{(G\times H)_0}\right))^0$. On the other hand, owing to the fact that

$$1 + N\left(\frac{R}{N(R)}\left(\frac{G \times H}{(G \times H)_0}\right)\right)^0 = \pi\left(1 + N\left(\frac{R}{N(R)} G \times H\right)^0\right) \subseteq \pi(V\left(\frac{R}{N(R)}\left(\frac{G \times H}{(G \times H)_0}\right)\right)) \tag{33}$$

we attain

$$\frac{G \times H}{(G \times H)_0}(1 + N\left(\frac{R}{N(R)}\left(\frac{G \times H}{(G \times H)_0}\right)\right)^0) \subseteq \pi(\frac{G \times H}{(G \times H)_0})\pi(V\left(\frac{R}{N(R)}\left(\frac{G \times H}{(G \times H)_0}\right)\right)) \tag{34}$$

and so

$$\frac{G \times H}{(G \times H)_0}(1 + N\left(\frac{R}{N(R)}\left(\frac{G \times H}{(G \times H)_0}\right)\right)^0) \subseteq \pi(\frac{G \times H}{(G \times H)_0} V\left(\frac{R}{N(R)}\left(\frac{G \times H}{(G \times H)_0}\right)\right)) \tag{35}$$

This means that

$$\pi(V\left(\frac{R}{N(R)}\left(\frac{G \times H}{(G \times H)_0}\right)\right)) \subseteq \pi(\frac{G \times H}{(G \times H)_0} V\left(\frac{R}{N(R)}\left(\frac{G \times H}{(G \times H)_0}\right)\right)) \tag{36}$$

Since the inverse of the above inclusion is clear, one can conclude that

$$\pi\left(V\left(\frac{R}{N(R)}\left(\frac{G \times H}{(G \times H)_0}\right)\right)\right) = \pi(\frac{G \times H}{(G \times H)_0} V\left(\frac{R}{N(R)}\left(\frac{G \times H}{(G \times H)_0}\right)\right)) \tag{37}$$

and thus the image of $V\left(\frac{R}{N(R)} G \times H\right) - (G \times H)V\left(\frac{R}{N(R)}(G \times H)_0\right)$ under π is 0. This shows that

$$V\left(\frac{R}{N(R)} G \times H\right) - (G \times H)V\left(\frac{R}{N(R)}(G \times H)_0\right) \tag{38}$$

is in the kernel of π. We also know that

$$Ker\, \pi \subseteq V\left(\frac{R}{N(R)}(G \times H)_0\right) \tag{39}$$

Then

$$V\left(\frac{R}{N(R)}G\times H\right)\subseteq (G\times H)V\left(\frac{R}{N(R)}(G\times H)_0\right)+V(\frac{R}{N(R)}(G\times H)_0) \quad (40)$$

To sum up, we have the inclusion

$$V\left(\frac{R}{N(R)}G\times H\right)\subseteq (G\times H)V\left(\frac{R}{N(R)}(G\times H)_0\right) \quad (41)$$

As the converse of this inclusion is apparent, the equation

$$V\left(\frac{R}{N(R)}G\times H\right)= (G\times H)V\left(\frac{R}{N(R)}(G\times H)_0\right) \quad (42)$$

hold. Substituting the assumption

$$V\left(\frac{R}{N(R)}(G\times H)_0\right)=(G\times H)_0 \quad (43)$$

into the above equation, we have indicated that

$$V\left(\frac{R}{N(R)}G\times H\right)=(G\times H) \quad (44)$$

as claimed. ■

Theorem 3.2. Let G and H be Abelian groups where $|H|=3$. Then, $V\big(R(G\times H)\big)$ is $G\times H$-nilpotent if and only if

$i)$ $V(R/N(R)G)=G$,

$ii)$ $1+3(a^2+b^2+ab+a+b)\in V(\frac{R}{N(R)})\Leftrightarrow (a,b)\in\{(0,0),(-1,0),(0,-1)\}$.

Proof. ⇒: Assume that $V\big(R(G\times H)\big)$ has only $G\times H$-nilpotent units. In this case, we equivalently have $V\big(R/N(R)(G\times H)\big)=G\times H$. Define a group epimorphism over $G\times H\simeq G\times\langle x: x^3=1\rangle$ as

$$\chi: G\times H\longrightarrow G, \chi(g,h)=g \quad (45)$$

Extending linearly χ over group ring, we attain

$$\bar{\chi}: R/N(R)(G\times H)\longrightarrow R/N(R)G \quad (46)$$

with an element $\gamma=\sum_{gh\in G\times H}\left(r_{gh}+N(R)\right)gh$ which has the image

$$\bar{\chi}(\gamma)=\sum_{gh\in G\times H}\left(r_{gh}+N(R)\right)g \quad (47)$$

Restricting $\bar{\chi}$ to the unit groups yields

$$\chi_V: V(R/N(R)(G\times H))\longrightarrow V(R/N(R)G) \quad (48)$$

with

$$Ker\ \chi_V = V(1+\Delta_{\frac{R}{N(R)G}}(H)) = (1+\langle 1-x, 1-x^2\rangle) \cap V(R/N(R)(G\times H)) \tag{49}$$

Thus,

$$\frac{V(R/N(R)(G\times H))}{V(1+\Delta_{\frac{R}{N(R)}G}(H))} \simeq V(\frac{R}{N(R)}G) \tag{50}$$

and we form a short exact sequence $A \xrightarrow{i} B \xrightarrow{\chi_V} C$ where
$A = V(1+\Delta_{R/N(R)G}(H))$, $B = V(R/N(R)(G\times H))$ and $C = V(R/N(R)G)$. Splitting $A \xrightarrow{i} B \xrightarrow{\chi_V} C$, we obtain a decomposition as $B = A \times C$. One can notice that if

$$V\big(R/N(R)(G\times H)\big) = G\times H \tag{51}$$

then $A = H$ and $C = V(R/N(R)G) = G$. Now, we should also explore necessary and sufficient conditions to be

$$A = V(1+\Delta_{R/N(R)G}(H)) = H \tag{52}$$

Actually, since

$$A = 1+\Delta_{\frac{R}{N(R)}G}(H) \cap V\left(\frac{R}{N(R)}(G\times H)\right) \tag{53}$$

a unit

$$u = 1 + a(1-x) + b(1-x^2) \in \langle x : x^3 = 1\rangle \tag{54}$$

if and only if

$$\begin{aligned} uv = u[1+c(1-x)+d(1-x^2)] = 1 &+ (1-x)(a+c+2ac+bc+ad-bd) \\ &+(1-x^2)(b+d-ac+bc+ad+2bd) = 1 \end{aligned} \tag{55}$$

for some $v = 1 + c(1-x) + d(1-x^2)$ where $a,b,c,d \in R/N(R)G$. Then, we can constitute a system of linear equations as

$$a+c+2ac+bc+ad-bd = 0 \tag{56}$$

$$b+d-ac+bc+ad+2bd = 0 \tag{57}$$

so its matrix equivalent $A\begin{pmatrix} c \\ d\end{pmatrix} = \begin{pmatrix} -a \\ -b\end{pmatrix}$ where

$$A = \begin{pmatrix} 1+2a+b & a-b \\ b-a & 1+a+2b \end{pmatrix} \tag{58}$$

has a unique solution $\begin{pmatrix} c \\ d\end{pmatrix}$ if and only if A is an invertible matrix so we can onclude that

$$detA = 1 + 3(a^2+b^2+ab+a+b) \tag{59}$$

must be a unit in $V(R/N(R)G)$ because of the formula $A^{-1} = \frac{1}{detA}adj(A)$. Hence,

$$1 + a(1-x) + b(1-x^2) \in \langle x : x^3 = 1\rangle \tag{60}$$

and $detA \in V(R/N(R)G)$ yields all of the following possible cases.

Case 1:

$1 + a(1 - x) + b(1 - x^2) = 1$ if and only if $(a, b) = (0,0)$.

Case 2:

$1 + a(1 - x) + b(1 - x^2) = x$ if and only if $(a, b) = (-1,0)$.

Case 3:

$1 + a(1 - x) + b(1 - x^2) = x^2$ if and only if $(a, b) = (0, -1)$.

So we get $ii)$ in the hypothesis. ■

Corollary 3.3. Let G and H be Abelian groups where $|H| = 3$ and char $R = 3$. Then, $V\big(R(G \times H)\big)$ has only $G \times H$-nilpotent units if and only if

$$V(RG) = G \times (1 + I(N(R)G; G)) \tag{61}$$

and $Ker\ \chi = \langle 1 - x, 1 - x^2 \rangle_S$ such that

$$S \times S = \{(0, \mu): \mu \in \mathbb{Z}_3\} \cup \{(\mu, 0): \mu \in \mathbb{Z}_3\} \tag{62}$$

Proof. If char $R = 3$, $\det A$ is

$$1 + 3(a^2 + b^2 + ab + a + b) = 1_{R/N(R)} \tag{63}$$

So, one can clearly deduce that $V(1 + Ker\ \chi_V)$ is $\{1 + a(1 - x) + b(1 - x^2): a, b \in R/N(R)G\}$ and thus $V(1 + Ker\ \chi_V) = H$ if and only if at least one of a and b has to be 0. This requires

$$S \times S = \{(0, \mu): \mu \in \mathbb{Z}_3\} \cup \{(\mu, 0): \mu \in \mathbb{Z}_3\} \tag{64}$$

as claimed. ■

Theorem 3.4. Let G and H be Abelian groups with $|H| = 4$ which is cyclic. Then, $V\big(R(G \times H)\big)$ has not only $G \times H$-nilpotent units if and only if $V\left(\frac{R}{N(R)}G\right) \neq G$ or there exists a unit of the form

$$u(a, b, c) = (1 + 2a + 2c)(1 + 2a^2 + 4b^2 + 2c^2 + 4ab + 4bc + 2a + 4b + 2c) \tag{65}$$

where $a, b, c \in R/N(R)G$.

Proof. Utilizing the epimorphisms in the previous theorem, we can set the same short exact sequence there. In this case, $V\big(R(G \times H)\big)$ has not only $G \times H$-nilpotent units if and only if

$$V(R/N(R)G) \neq G \tag{66}$$

or

$$V(1 + \Delta_{R/N(R)G}(H)) \neq H \tag{67}$$

where $H = \langle x: x^3 = 1 \rangle$. Let

$$u = 1 + a(1 - x) + b(1 - x^2) + c(1 - x^3) \tag{68}$$

be a unit in $V(1+\Delta_{R/N(R)G}(H))$ with the inverse $v = 1 + d(1-x) + e(1-x^2) + f(1-x^3)$. Then $V(1+\Delta_{R/N(R)G}(H)) \neq H$ if and only if u is nontrivial and uv is

$$1 + (1-x)\beta_1 + (1-x^2)\beta_2 + (1-x^3)\beta_3 = 1 \tag{69}$$

where

$$\beta_1 = (a + d + 2ad + bd + cd + ae - ce + af - bf) \tag{70}$$

$$\beta_2 = (b + e - ad + bd + ae + 2be + ce + bf - cf) \tag{71}$$

$$\beta_3 = (c + f - bd + cd - ae + ce + af + bf + 2cf) \tag{72}$$

In this case, $uv = 1$ if and only if $M\begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{pmatrix} -a \\ -b \\ -c \end{pmatrix}$ has a unique solution $\begin{pmatrix} d \\ e \\ f \end{pmatrix}$ where M is

$$\begin{pmatrix} 1+2a+b+c & a-c & a-b \\ b-a & 1+a+2b+c & b-c \\ c-b & c-a & 1+a+b+2c \end{pmatrix} \tag{73}$$

Thus $det\ M$ is invertible in $(R/N(R))G$ and can be stated as

$$(1+2a+2c)(1+2a^2+4b^2+2c^2+4ab+4bc+2a+4b+2c) \tag{74}$$

as claimed in the theorem. ■

Corollary 3.5. Let G and H be Abelian groups with $|H| = 4$ which is cyclic and also $char\ R = 2$. Then, $V\big(R(G \times H)\big)$ has not only $G \times H$-nilpotent units if and only if $V(R/N(R)G) \neq G$ or $Ker\ \chi = \langle 1-x, 1-x^2, 1-x^3 \rangle_T$ such that $T^3 = \{(a,b,c): a,b,c \in R/N(R)G\}$ where at least two of a, b and c is different from $0_{R/N(R)G}$.

Proof. If $V\big(R(G \times H)\big)$ has not only $G \times H$-nilpotent units and $V(R/N(R)G) = G$, then

$$V(1 + Ker\ \chi_V) \tag{75}$$

has to consist nontrivial units. As a unit $u = 1 + a(1-x) + b(1-x^2) + c(1-x^3)$ has to be different from $1, x, x^2$ or x^3. In this case, one can easily check that if only one of a, b or c is 0, u has one of the following forms:

$$u = 1 + a(1-x) + b(1-x^2) \tag{76}$$

$$u = 1 + a(1-x) + c(1-x^3) \tag{77}$$

$$u = 1 + b(1-x^2) + c(1-x^3) \tag{78}$$

Thus u may has a nontrivial form which is a contradiction. Hence, in order to insure that u has to be only $1, x, x^2$ or x^3, we have to choose the parameters a, b, c as claimed. ■

4. Discussion and Conclusion

In this study, we have firstly defined some sets using primes related to a commutative group ring $R(G \times H)$ which is unity of Abelian groups G and H inspring from (Danchev, 2012). Later, we have determined some necessary and sufficient conditions for $V(R(G \times H))$ to be $G \times H$-nilpotent based on our definitions such as $supp_C(G \times H)$, $zd_C(R)$ and $inv_C(R)$ in Theorem 3.1.

Li (1998) has proved that if RG has only trivial units, then $R(G \times C_2)$ has only trivial units as well where $R = \mathbb{Z}$. So, the results on $G \times C_2$-nilpotency of the normalized unit group $V(R(G \times C_2))$ can be similarly obtained using his structure. In this paper, we have acquired some special necessary and sufficient conditions on $G \times H$-nilpotency of $V(R(G \times H))$ for $H = C_3$ and $H = C_4$. As a future work, it may possible to get some results about $G \times C_n$ for a general n. Besides, we should note that the current paper already gives a characterization for $G_1 \times G_2 \times \cdots G_n$ since we can observe that

$$G_1 \times G_2 \times \cdots G_n = \overline{G_1} \times \overline{G_2} \tag{79}$$

where $\overline{G_1} = G_1 \times G_2 \times \cdots G_k$ and $\overline{G_2} = G_{k+1} \times G_{k+2} \times \cdots G_n$ for $1 \leq k < n$. So, it is an easy implementation of this paper and can only be evaluated as an example.

As widely known, units are one of exclusive elements in group rings. In addition, defining a new type of units creates a remarkable area in the theory of group rings. Being able to attract more researchers plays a crucial role by sharing ideas and open problems.
In this context, we think that investigating necessary and sufficient conditions for

$$V\big(R(G \times H)\big) = V(RG) \times (1 + I) \tag{80}$$

where $I = I(N(R)G \times H; G \times H)$ can be appreciated as an open problem and so a future work.

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