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On Third Order Bronze Fibonacci Quaternions

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Abstract. In this study, we define third order bronze Fibonacci quaternions. We obtain the generating functions, the Binet's formula and some properties of these quaternions. We give d'Ocagne's-like and Cassini's-like identity and we use q-determinants for quaternionic matrices to give the Cassini's identity for third order bronze Fibonacci quaternions.

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1. Introduction

The real quaternions were first defined by Sir William Rowan Hamilton in 1843, [13]. Let $\mathbb{Q}$ be a four dimensional vector space over $\mathbb{R}$ with an ordered base $1, i, j, k$. A real quaternion is a vector quaternion

$$Q_n = a + bi + cj + dk \in \mathbb{Q},$$

where $a, b, c, d \in \mathbb{R}$. The product of two quaternions in this space is defined by

$$i^2 = j^2 = k^2 = -1,$$

$$ij = -ji = k, jk = -kj = i, ki = -ik = j.$$

Horodam in [15] defined Fibonacci and Lucas quaternions as follows:

$$Q_n = F_n + F_{n+1} + F_{n+2}i + F_{n+3}k$$

and

$$K_n = L_n + L_{n+1} + L_{n+2}i + L_{n+3}k,$$

where F_n and L_n are the nth Fibonacci and Lucas numbers [14]. Fibonacci quaternions and the properties of these sequences are also studied by Halici in [8, 9]. They also studied Gaussian Fibonacci, complex Fibonacci and dual Fibonacci quaternions, see [10–12]. Generalized Fibonacci quaternions are studied by Swammy in [17]. The properties of bicomplex Fibonacci and generalized dual Fibonacci quaternions are studied in [2] and [18], respectively. k-Pell and k-Pell Lucas quaternions are studied in [7] and k-Fibonacci and k-Lucas quaternions over Z_p in [19]

In [4] Cerda-Morales introduced third order Jacobsthal quaternions and Jacobsthal-Lucas quaternions. The author also studied bicomplex third order Jacobsthal quaternions in [3] and dual third-order Jacobsthal quaternions in [5, 6]. They obtained the generating function, Binet Formula, d'Ocagne-like identity and Cassini-like identity for these quaternions.

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However, the Cassini Identity for generalized third order Pell numbers and third order bronze Fibonacci numbers is obtained in [16] and [1], respectively.

2. Preliminaries

In this study, we will define and study third order bronze Fibonacci quaternions by using third order bronze Fibonacci numbers which are studied in [1]. The authors investigated the generalized third order bronze Fibonacci sequence and three specific sequences which are derived from its De-Moivre Type Identities. The sequence $\{\mathfrak{B}_n^G\}$ with the recurrence relation $\mathfrak{B}_n^G = 3\mathfrak{B}_{n-1}^G + \mathfrak{B}_{n-2}^G + \mathfrak{B}_{n-3}^G$ for $n \geq 3$, where $\mathfrak{B}_0^G, \mathfrak{B}_1^G, \mathfrak{B}_2^G$ are any arbitrary numbers not all being zero, is called a generalized third order bronze Fibonacci sequence [1]. The sequences derived from the De-Moivre Type Identities of this sequence are: third order bronze Lucas sequence $\{\mathfrak{B}_n^L\}$, modified third order bronze Fibonacci sequence $\{\mathfrak{B}_n^M\}$ and third order bronze Fibonacci sequence $\{\mathfrak{B}_n^F\}$ [1].

The first ten terms of above sequences are presented in the following table.

Table 1. The Third Order Bronze Fibonacci Numbers

n	0	1	2	3	4	5	6	7	8	9	10
$\mathfrak{B}_n^{\mathfrak{L}}$	3	3	11	39	131	443	1499	5071	17155	58035	196331
$\mathfrak{B}_n^{\mathfrak{M}}$	1	2	7	24	81	274	927	3136	10609	35890	121415
$\mathfrak{B}_n^{\mathfrak{F}}$	1	3	10	34	115	389	1316	4452	15061	50951	172366

Some of identities of third order bronze Fibonacci sequences that will be used in this study are:

– The generating function for generalized third order bronze Fibonacci numbers

$$\mathfrak{B}^G(x) = \frac{\mathfrak{B}_0^G + (\mathfrak{B}_1^G - 3\mathfrak{B}_0^G)x + (\mathfrak{B}_2^G - 3\mathfrak{B}_1^G - \mathfrak{B}_0^G)x^2}{1 - 3x - x^2 - x^3}, \tag{2.1}$$

where $\mathfrak{B}^G(x) = \sum_{n=0}^{\infty} \mathfrak{B}_n^G x^n$ [1].

– The Binet's Formula for third order bronze Fibonacci numbers

$$\mathfrak{B}_n^G = d_1\alpha_1^n + d_2\alpha_2^n + d_3\alpha_3^n, \tag{2.2}$$

where

$$\begin{aligned} d_1 &= \frac{\mathfrak{B}_0^G\alpha_2\alpha_3 - \mathfrak{B}_1^G(\alpha_2+\alpha_3) + \mathfrak{B}_2^G}{(\alpha_2-\alpha_1)(\alpha_3-\alpha_1)}, \\ d_2 &= \frac{-\mathfrak{B}_0^G\alpha_1\alpha_3 + \mathfrak{B}_1^G(\alpha_1+\alpha_3) - \mathfrak{B}_2^G}{(\alpha_3-\alpha_2)(\alpha_2-\alpha_1)}, \\ d_3 &= \frac{\mathfrak{B}_0^G\alpha_1\alpha_2 - \mathfrak{B}_1^G(\alpha_1+\alpha_2) + \mathfrak{B}_2^G}{(\alpha_3-\alpha_2)(\alpha_3-\alpha_1)} \end{aligned} \tag{2.3}$$

and $\alpha_1, \alpha_2, \alpha_3$ are roots of the equation $x^3 - 3x^2 - x - 1 = 0$, i.e,

$$\begin{aligned} \alpha_1 &= 1 + U + V, \\ \alpha_2 &= 1 - \frac{1}{2}(U+V) + i\frac{\sqrt{3}}{2}(U-V), \\ \alpha_3 &= 1 - \frac{1}{2}(U+V) - i\frac{\sqrt{3}}{2}(U-V), \end{aligned} \tag{2.4}$$

where $U = \sqrt[3]{2 + \sqrt{4 - \frac{64}{27}}}$, $V = \sqrt[3]{2 - \sqrt{4 - \frac{64}{27}}}$, $UV = \frac{4}{3}$, and $U^3 + V^3 = 4$ [1].

– The linear sum for generalized third order bronze Fibonacci numbers [1]

$$\sum_{k=0}^{n} \mathfrak{B}_k^G = \frac{1}{4}(\mathfrak{B}_{n+3}^G - 2\mathfrak{B}_{n+2}^G - 3\mathfrak{B}_{n+1}^G - \mathfrak{B}_2^G + 2\mathfrak{B}_1^G + 3\mathfrak{B}_0^G). \tag{2.5}$$

Now, we define quaternionic matrices which will be used to obtain Cassini's Identity. Let $M_{m\times n}(\mathbb{Q})$ be the set of all $m \times n$ matrices with quaternion entries and $M_n(\mathbb{Q})$ the set of all square matrices with quaternion entries. The properties of these matrices are studied in [20]. For $A = A_1 + A_2 j \in M_n(\mathbb{Q})$, the $2n \times 2n$ complex matrix

$$\begin{bmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{bmatrix}$$

is called the *complex adjoint of the quaternion matrix* A and is denoted by χ_A. This matrix is uniquely determined by A and some of the properties of this matrix given in [20] are
a) $\chi_{I_n} = I_{2n}$,
b) $\chi_{AB} = \chi_A \chi_B$,
c) $\chi_{A+B} = \chi_A + \chi_B$,
d) χ_A is unitary, Hermitian or normal$\Leftrightarrow$ A is unitary, Hermitian or normal, respectively.

The author in [20] also defined the q-determinant of A to be det(χ_A), i.e. $|A|_q = |\chi_A|$ and obtained the following results which will be used in this study
a) A is invertible $\Leftrightarrow$ $|A|_q \neq 0$,
b) $|AB|_q = |A|_q|B|_q$, consequently $|A^{-1}|_q = |A|_q^{-1}$,
c) $|PAQ|_q = |A|_q$, for any elementary matrices P and Q.

3. Third Order Bronze Fibonacci Quaternions and Some Properties

We define *third order generalized bronze Fibonacci quaternions* by

$$\mathfrak{B}Q_n^G = \mathfrak{B}_n^G + \mathfrak{B}_{n+1}^G i + \mathfrak{B}_{n+2}^G j + \mathfrak{B}_{n+3}^G k,$$

where $\mathfrak{B}_n^G$ is the nth generalized third order bronze Fibonacci number. Then, it can be shown that for $n \geq 3$

$$\mathfrak{B}Q_n^G = 3\mathfrak{B}Q_{n-1}^G + \mathfrak{B}Q_{n-2}^G + \mathfrak{B}Q_{n-3}^G.$$

By using recurrence relation and initial terms for bronze Fibonacci numbers, we also define

$$\begin{aligned} \mathfrak{B}_{-1}^L &= -1, \mathfrak{B}_{-2}^L = -5, \\ \mathfrak{B}_{-1}^M &= 0, \mathfrak{B}_{-2}^M = -1, \\ \mathfrak{B}_{-1}^F &= 0, \mathfrak{B}_{-2}^L = 0. \end{aligned} \tag{3.1}$$

The sum of the first n terms of the generalized third order bronze Fibonacci quaternions sequence can be given by the following theorem.

Theorem 3.1. *The linear sum of the first n terms of the generalized third order bronze Fibonacci quaternion sequence is*

$$\sum_{k=0}^{n} \mathfrak{B}Q_k^G = \frac{1}{4}\{\mathfrak{B}Q_{n+3}^G - 2\mathfrak{B}Q_{n+2}^G - 3\mathfrak{B}Q_{n+1}^G + \mathfrak{B}_0^G(3 - i - j - k) + \mathfrak{B}_1^G(2 + 2i - 2j - 2k) - \mathfrak{B}_2^G(1 + i + j + 5k)\}. \tag{3.2}$$

Proof.

$$\begin{aligned} \sum_{k=0}^{n} \mathfrak{B}Q_k^G &= \sum_{k=0}^{n} \mathfrak{B}_k^G + \sum_{k=0}^{n} \mathfrak{B}_{k+1}^G i + \sum_{k=0}^{n} \mathfrak{B}_{k+2}^G j + \sum_{k=0}^{n} \mathfrak{B}_{k+3}^G \\ &= \sum_{k=0}^{n} \mathfrak{B}_k^G + (\sum_{k=0}^{n+1} \mathfrak{B}_k^G - \mathfrak{B}_0^G)i + (\sum_{k=0}^{n+2} \mathfrak{B}_k^G - \mathfrak{B}_0^G - \mathfrak{B}_1^G)j + (\sum_{k=0}^{n+3} \mathfrak{B}_k^G - \mathfrak{B}_0^G - \mathfrak{B}_1^G - \mathfrak{B}_2^G)k. \end{aligned}$$

By using (2.5), we find

$$\sum_{k=0}^{n}\mathfrak{B}Q_k^G = \frac{1}{4}\{\mathfrak{B}_{n+3}^G - 2\mathfrak{B}_{n+2}^G - 3\mathfrak{B}_{n+1}^G - \mathfrak{B}_2^G + 2\mathfrak{B}_1^G + 3\mathfrak{B}_0^G\} + \frac{1}{4}(\mathfrak{B}_{n+4}^G - 2\mathfrak{B}_{n+3}^G - 3\mathfrak{B}_{n+2}^G - \mathfrak{B}_2^G + 2\mathfrak{B}_1^G - \mathfrak{B}_0^G)i$$
$$+\frac{1}{4}(\mathfrak{B}_{n+5}^G - 2\mathfrak{B}_{n+4}^G - 3\mathfrak{B}_{n+3}^G - \mathfrak{B}_2^G - 2\mathfrak{B}_1^G - \mathfrak{B}_0^G)j + \frac{1}{4}(\mathfrak{B}_{n+6}^G - 2\mathfrak{B}_{n+5}^G - 3\mathfrak{B}_{n+4}^G - 5\mathfrak{B}_2^G - 2\mathfrak{B}_1^G - \mathfrak{B}_0^G)k$$

which implies (3.2). □

Corollary 3.2. *Linear sums for the sequences* $\{\mathfrak{B}Q_n^L\}$, $\{\mathfrak{B}Q_n^M\}$, *and* $\{\mathfrak{B}Q_n^F\}$ *can be calculated as:*

$$\begin{aligned}
\sum_{k=0}^{n}\mathfrak{B}Q_k^L &= \frac{1}{4}\{\mathfrak{B}Q_{n+3}^L - 2\mathfrak{B}Q_{n+2}^L - 3\mathfrak{B}Q_{n+1}^L + 4 - 8i - 20j - 64k\},\\
\sum_{k=0}^{n}\mathfrak{B}Q_k^M &= \frac{1}{4}\{\mathfrak{B}Q_{n+3}^M - 2\mathfrak{B}Q_{n+2}^M - 3\mathfrak{B}Q_{n+1}^M - 4i - 12j - 40k\},\\
\sum_{k=0}^{n}\mathfrak{B}Q_k^F &= \frac{1}{4}\{\mathfrak{B}Q_{n+3}^F - 2\mathfrak{B}Q_{n+2}^F - 3\mathfrak{B}Q_{n+1}^F - 1 - 5i - 17j - 57k\}.
\end{aligned}$$

4. Generating Functions and Binet's Formula

In this section, we obtain the generating function and Binet's formula for generalized third order bronze Fibonacci quaternion sequence and its three specific sequences.

Theorem 4.1. *The generating function for generalized third order bronze Fibonacci quaternion sequence* $\{\mathfrak{B}Q_n^G\}$ *is given by*

$$\mathfrak{B}Q^G(x) = \frac{1}{1-3x-x^2-x^3}\{\mathfrak{B}Q_0^G + (\mathfrak{B}Q_{-2}^G + \mathfrak{B}Q_{-1}^G)x + \mathfrak{B}Q_{-1}^G x^2\}, \tag{4.1}$$

where $\mathfrak{B}Q^G(x) = \sum_{n=0}^{\infty}\mathfrak{B}Q_n^G x^n$.

Proof. Let $\mathfrak{B}Q^G(x) = \sum_{n=0}^{\infty}\mathfrak{B}Q_n^G x^n$. Then,

$$\begin{aligned}
\mathfrak{B}Q^G(x) &= \sum_{n=0}^{\infty}\mathfrak{B}_n^G x^n + \sum_{n=0}^{\infty}\mathfrak{B}_{n+1}^G x^n i + \sum_{n=0}^{\infty}\mathfrak{B}_{n+2}^G x^n j + \sum_{n=0}^{\infty}\mathfrak{B}_{n+3}^G x^n k,\\
&= \mathfrak{B}^G(x) + \frac{1}{x}(\mathfrak{B}^G(x) - \mathfrak{B}_0^G)i + \frac{1}{x^2}(\mathfrak{B}^G(x) - \mathfrak{B}_0^G - \mathfrak{B}_1^G x) + \frac{1}{x^3}(\mathfrak{B}^G(x) - \mathfrak{B}_0^G - \mathfrak{B}_1^G x - \mathfrak{B}_2^G x^2)k.
\end{aligned}$$

By using equation (2.1) and the fact that $\mathfrak{B}_{n+1} - 3\mathfrak{B}_n = \mathfrak{B}_{n-1} + \mathfrak{B}_{n-2}$ and $\mathfrak{B}_{n+2} - 3\mathfrak{B}_{n+1} - \mathfrak{B}_n = \mathfrak{B}_{n-1}$ we find

$$\mathfrak{B}Q^G(x) = \frac{1}{1-3x-x^2-x^3}\{\mathfrak{B}_0^G + (\mathfrak{B}_{-2}^G + \mathfrak{B}_{-1}^G)x + \mathfrak{B}_{-1}^G x^2 + (\mathfrak{B}_1^G + (\mathfrak{B}_{-1}^G + \mathfrak{B}_0^G)x + \mathfrak{B}_0^G x^2)i$$
$$+(\mathfrak{B}_2^G + (\mathfrak{B}_0^G + \mathfrak{B}_1^G)x + \mathfrak{B}_1^G x^2)j + (\mathfrak{B}_3^G + (\mathfrak{B}_1^G + \mathfrak{B}_2^G)x + \mathfrak{B}_2^G x^2)k\},$$

which implies equation (4.1). □

By using (3.1) we can give the following corollary.

Corollary 4.2. *The generating functions for the sequences* $\{\mathfrak{B}Q_n^L\}$, $\{\mathfrak{B}Q_n^M\}$ *and* $\{\mathfrak{B}Q_n^F\}$ *can be calculated as follows;*

$$\begin{aligned}
\mathfrak{B}Q^L(x) &= \frac{1}{1-3x-x^2-x^3}\{3 - 6x - x^2 + (3 + 2x + 3x^2)i + (11 + 6x + 3x^2)j + (39 + 14x + 11x^2)k\},\\
\mathfrak{B}Q^M(x) &= \frac{1}{1-3x-x^2-x^3}\{1 - x + (2 + x + x^2)i + (7 + 3x + 2x^2)j + (24 + 9x + 10x^2)k\},\\
\mathfrak{B}Q^F(x) &= \frac{1}{1-3x-x^2-x^3}\{1 + (3 + x + x^2)i + (10 + 4x + 3x^2)j + (34 + 13x + 10x^2)k\}.
\end{aligned}$$

Theorem 4.3. *The Binet's formula for generalized third order bronze Fibonacci quaternion sequence* $\{\mathfrak{B}Q_n^G\}$ *is given by*

$$\mathfrak{B}Q_n^G = d_1\omega_1\alpha_1^n + d_2\omega_2\alpha_2^n + d_3\omega_3\alpha_3^n,$$

where d_1, d_2, d_3 *are coefficients given in* (2.3), $\alpha_1, \alpha_2, \alpha_3$ *are roots of the equation* $x^3 - 3x^2 - x - 1 = 0$ *given in* (2.4) *and*

$$\omega_i = 1 + \alpha_i i + \alpha_i^2 j + \alpha_i^3 k, i = 1, 2, 3.$$

Proof. Let $\mathfrak{B}Q_n^G = \mathfrak{B}_n^G + \mathfrak{B}_{n+1}^G i + \mathfrak{B}_{n+2}^G j + \mathfrak{B}_{n+3}^G k$ be the n-th generalized third order bronze Fibonacci Quaternion, then by using the Binet Formula for generalized bronze Fibonacci numbers (2.2) we find

$$\begin{aligned}\mathfrak{B}Q_n^G &= d_1\alpha_1^n + d_2\alpha_2^n + d_3\alpha_3^n + (d_1\alpha_1^{n+1} + d_2\alpha_2^{n+1} + d_3\alpha_3^{n+1})i \\ &\quad + (d_1\alpha_1^{n+2} + d_2\alpha_2^{n+2} + d_3\alpha_3^{n+2})j + (d_1\alpha_1^{n+3} + d_2\alpha_2^{n+3} + d_3\alpha_3^{n+3})k \\ &= d_1(1 + \alpha_1 i + \alpha_1^2 j + \alpha_1^3 k)\alpha_1^n + d_2(1 + \alpha_2 i + \alpha_2^2 j + \alpha_2^3 k)\alpha_2^n + d_3(1 + \alpha_3 i + \alpha_3^2 j + \alpha_3^3 k)\alpha_3^n\end{aligned}$$

which proves the theorem. □

Corollary 4.4. *Binet's Formulas for the sequences* $\{\mathfrak{B}Q_n^L\}$, $\{\mathfrak{B}Q_n^M\}$, *and* $\{\mathfrak{B}Q_n^F\}$ *can be calculated as:*

$$\mathfrak{B}Q_n^L = \frac{3\alpha_2\alpha_3 - 3(\alpha_2 + \alpha_3) + 11}{(\alpha_2 - \alpha_1)(\alpha_3 - \alpha_1)}\omega_1\alpha_1^n + \frac{-3\alpha_1\alpha_3 + 3(\alpha_1 + \alpha_3) - 11}{(\alpha_3 - \alpha_2)(\alpha_2 - \alpha_1)}\omega_2\alpha_2^n + \frac{3\alpha_1\alpha_2 - 3(\alpha_1 + \alpha_2) + 11}{(\alpha_3 - \alpha_2)(\alpha_3 - \alpha_1)}\omega_3\alpha_3^n$$

or

$$\mathfrak{B}Q_n^L = \omega_1\alpha_1^n + \omega_2\alpha_2^n + \omega_3\alpha_3^n,$$

$$\mathfrak{B}Q_n^M = \frac{\alpha_2\alpha_3 - 2(\alpha_2 + \alpha_3) + 7}{(\alpha_2 - \alpha_1)(\alpha_3 - \alpha_1)}\omega_1\alpha_1^n + \frac{-\alpha_1\alpha_3 + 2(\alpha_1 + \alpha_3) - 7}{(\alpha_3 - \alpha_2)(\alpha_2 - \alpha_1)}\omega_2\alpha_2^n + \frac{\alpha_1\alpha_2 - 2(\alpha_1 + \alpha_2) + 7}{(\alpha_3 - \alpha_2)(\alpha_3 - \alpha_1)}\omega_3\alpha_3^n$$

and

$$\mathfrak{B}_n^F = \frac{\alpha_2\alpha_3 - 3(\alpha_2 + \alpha_3) + 10}{(\alpha_2 - \alpha_1)(\alpha_3 - \alpha_1)}\omega_1\alpha_1^n + \frac{-\alpha_1\alpha_3 + 3(\alpha_1 + \alpha_3) - 10}{(\alpha_3 - \alpha_2)(\alpha_2 - \alpha_1)}\omega_2\alpha_2^n + \frac{\alpha_1\alpha_2 - 3(\alpha_1 + \alpha_2) + 10}{(\alpha_3 - \alpha_2)(\alpha_3 - \alpha_1)}\omega_3\alpha_3^n.$$

5. Matrix Representation of Generalized Third Order Bronze Fibonacci Quaternions

In this section, we will use the matrix representation of quaternions to find the expression of $\mathfrak{B}Q_{n+m}^G$. Define the matrix

$$\mathbf{B} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Then, in [1] it is shown that

$$\mathbf{B}^n = \begin{bmatrix} \mathfrak{B}_n^F & \mathfrak{B}_{n-1}^F + \mathfrak{B}_{n-2}^F & \mathfrak{B}_{n-1}^F \\ \mathfrak{B}_{n-1}^F & \mathfrak{B}_{n-2}^F + \mathfrak{B}_{n-3}^F & \mathfrak{B}_{n-2}^F \\ \mathfrak{B}_{n-2}^F & \mathfrak{B}_{n-3}^F + \mathfrak{B}_{n-4}^F & \mathfrak{B}_{n-3}^F \end{bmatrix} \tag{5.1}$$

and $\det \mathbf{B}^n = 1$. We define the matrix

$$\mathfrak{B}Q = \begin{bmatrix} \mathfrak{B}Q_2^G & \mathfrak{B}Q_1^G + \mathfrak{B}Q_0^G & \mathfrak{B}Q_1^G \\ \mathfrak{B}Q_1^G & \mathfrak{B}Q_0^G + \mathfrak{B}Q_{-1}^G & \mathfrak{B}Q_0^G \\ \mathfrak{B}Q_0^G & \mathfrak{B}Q_{-1}^G + \mathfrak{B}Q_{-2}^G & \mathfrak{B}Q_{-1}^G \end{bmatrix} \tag{5.2}$$

and give the next theorem.

Theorem 5.1. *If* $\mathfrak{B}Q_n^G$ *is the n-th generalized third order bronze Fibonacci quaternion then,*

$$\mathfrak{B}Q \cdot \mathbf{B}^n = \begin{bmatrix} \mathfrak{B}Q_{n+2}^G & \mathfrak{B}Q_{n+1}^G + \mathfrak{B}Q_n^G & \mathfrak{B}Q_{n+1}^G \\ \mathfrak{B}Q_{n+1}^G & \mathfrak{B}Q_n^G + \mathfrak{B}Q_{n-1}^G & \mathfrak{B}Q_n^G \\ \mathfrak{B}Q_n^G & \mathfrak{B}Q_{n-1}^G + \mathfrak{B}Q_{n-2}^G & \mathfrak{B}Q_{n-1}^G \end{bmatrix}$$

Proof. We will use induction by n to prove this theorem. For $n = 0$ the result is obvious. Suppose it is true for $n := n$ and let us show that it is true for $n := n + 1$, too.

$$\mathfrak{B}Q \cdot \mathbf{B}^{n+1} = (\mathfrak{B}Q \cdot \mathbf{B}^n) \cdot \mathbf{B} = \begin{bmatrix} \mathfrak{B}Q^G_{n+2} & \mathfrak{B}Q^G_{n+1} + \mathfrak{B}Q^G_n & \mathfrak{B}Q^G_{n+1} \\ \mathfrak{B}Q^G_{n+1} & \mathfrak{B}Q^G_n + \mathfrak{B}Q^G_{n-1} & \mathfrak{B}Q^G_n \\ \mathfrak{B}Q^G_n & \mathfrak{B}Q^G_{n-1} + \mathfrak{B}Q^G_{n-2} & \mathfrak{B}Q^G_{n-1} \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3\mathfrak{B}Q^G_{n+2} + \mathfrak{B}Q^G_{n+1} + \mathfrak{B}Q^G_n & \mathfrak{B}Q^G_{n+2} + \mathfrak{B}Q^G_{n+1} & \mathfrak{B}Q^G_{n+2} \\ 3\mathfrak{B}Q^G_{n+1} + \mathfrak{B}Q^G_n + \mathfrak{B}Q^G_{n-1} & \mathfrak{B}Q^G_{n+1} + \mathfrak{B}Q^G_n & \mathfrak{B}Q^G_{n+1} \\ 3\mathfrak{B}Q^G_n + \mathfrak{B}Q^G_{n-1} + \mathfrak{B}Q^G_{n-2} & \mathfrak{B}Q^G_n + \mathfrak{B}Q^G_{n-1} & \mathfrak{B}Q^G_n \end{bmatrix} = \begin{bmatrix} \mathfrak{B}Q^G_{n+3} & \mathfrak{B}Q^G_{n+2} + \mathfrak{B}Q^G_{n+1} & \mathfrak{B}Q^G_{n+2} \\ \mathfrak{B}Q^G_{n+2} & \mathfrak{B}Q^G_{n+1} + \mathfrak{B}Q^G_n & \mathfrak{B}Q^G_{n+1} \\ \mathfrak{B}Q^G_{n+1} & \mathfrak{B}Q^G_n + \mathfrak{B}Q^G_{n-1} & \mathfrak{B}Q^G_n \end{bmatrix}.$$

□

Corollary 5.2. *For $n \geq 0$ we have*

$$\mathfrak{B}Q^G_{n+1} = \mathfrak{B}Q^G_1 \mathfrak{B}^F_n + (\mathfrak{B}Q^G_0 + \mathfrak{B}Q^G_{-1})\mathfrak{B}^F_{n-1} + \mathfrak{B}Q^G_0 \mathfrak{B}^F_{n-2}.$$

Proof. $\mathfrak{B}Q^G_{n+1}$ is the (2,1) entry of the matrix $\mathfrak{B}Q \cdot \mathbf{B}^n$ which is the product of the third row of the matrix $\mathfrak{B}Q$ given in (5.2) and the first column of the matrix $\mathbf{B}^n$ given in (5.1). □

Now, for $n \geq 0$ let us define

$$\mathbf{Y}_n = \begin{bmatrix} \mathfrak{B}Q^G_{n+2} & \mathfrak{B}Q^G_{n+1} + \mathfrak{B}Q^G_n & \mathfrak{B}Q^G_{n+1} \\ \mathfrak{B}Q^G_{n+1} & \mathfrak{B}Q^G_n + \mathfrak{B}Q^G_{n-1} & \mathfrak{B}Q^G_n \\ \mathfrak{B}Q^G_n & \mathfrak{B}Q^G_{n-1} + \mathfrak{B}Q^G_{n-2} & \mathfrak{B}Q^G_{n-1} \end{bmatrix}.$$

It can be easily shown that $\mathbf{Y}_{n+1} = \mathbf{B} \cdot \mathbf{Y}_n$ and the bellow theorem holds.

Theorem 5.3. *For $n, m \geq 0$ we have*

(1) $\mathbf{Y}_n = \mathbf{B}^n \mathbf{Y}_0$,
(2) $\mathbf{Y}_0 \mathbf{B}^n = \mathbf{B}^n \mathbf{Y}_0$,
(3) $\mathbf{Y}_{n+m} = \mathbf{Y}_n \mathbf{B}^m$.

Proof. (1) We can show this by induction. For $n = 0$ it is obvious, Let's suppose that this equality is satisfied for $n := n$ then,

$$\mathbf{Y}_{n+1} = \mathbf{B} \cdot \mathbf{Y}_n = \mathbf{B} \cdot \mathbf{B}^n \cdot \mathbf{Y}_0 = \mathbf{B}^{n+1} \cdot \mathbf{Y}_0.$$

(2) We will use induction over n. It can be easily shown by straightforward calculation that

$$\mathbf{Y}_0 \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{Y}_0.$$

Suppose that this equality is true for $n := n$, then

$$\mathbf{Y}_0 \cdot \mathbf{B}^{n+1} = \mathbf{Y}_0 \cdot \mathbf{B}^n \cdot \mathbf{B} = \mathbf{B}^n \cdot \mathbf{Y}_0 \cdot \mathbf{B} = \mathbf{B}^n \cdot \mathbf{B} \cdot \mathbf{Y}_0 = \mathbf{B}^{n+1} \cdot \mathbf{Y}_0.$$

(3) From 1 and 2 it follows that

$$\mathbf{Y}_{n+m} = \mathbf{B}^{n+m} \cdot \mathbf{Y}_0 = \mathbf{B}^n \cdot \mathbf{B}^m \cdot \mathbf{Y}_0 = \mathbf{B}^n \cdot \mathbf{Y}_0 \cdot \mathbf{B}^m = \mathbf{Y}_n \cdot \mathbf{B}^m.$$

□

Theorem 5.4. *For $n, m \geq 0$ we have*

$$\mathfrak{B}Q^G_{n+m} = \mathfrak{B}Q^G_n \mathfrak{B}^F_m + (\mathfrak{B}Q^G_{n-1} + \mathfrak{B}Q^G_{n-2})\mathfrak{B}^F_{m-1} + \mathfrak{B}Q^G_{n-1}\mathfrak{B}^F_{m-2}.$$

Proof. From the above theorem we have $\mathbf{Y}_{n+m} = \mathbf{Y}_n \mathbf{B}^m$ or

$$\mathbf{Y}_{n+m} = \begin{bmatrix} \mathfrak{B}Q^G_{n+m+2} & \mathfrak{B}Q^G_{n+m+1} + \mathfrak{B}Q^G_{n+m} & \mathfrak{B}Q^G_{n+m+1} \\ \mathfrak{B}Q^G_{n+m+1} & \mathfrak{B}Q^G_{n+m} + \mathfrak{B}Q^G_{n+m-1} & \mathfrak{B}Q^G_{n+m} \\ \mathfrak{B}Q^G_{n+m} & \mathfrak{B}Q^G_{n+m-1} + \mathfrak{B}Q^G_{n+m-2} & \mathfrak{B}Q^G_{n+m-1} \end{bmatrix}$$

$$= \begin{bmatrix} \mathfrak{B}Q^G_{n+2} & \mathfrak{B}Q^G_{n+1} + \mathfrak{B}Q^G_n & \mathfrak{B}Q^G_{n+1} \\ \mathfrak{B}Q^G_{n+1} & \mathfrak{B}Q^G_n + \mathfrak{B}Q^G_{n-1} & \mathfrak{B}Q^G_n \\ \mathfrak{B}Q^G_n & \mathfrak{B}Q^G_{n-1} + \mathfrak{B}Q^G_{n-2} & \mathfrak{B}Q^G_{n-1} \end{bmatrix} \cdot \begin{bmatrix} \mathfrak{B}^F_m & \mathfrak{B}^F_{m-1} + \mathfrak{B}^F_{m-2} & \mathfrak{B}^F_{m-1} \\ \mathfrak{B}^F_{m-1} & \mathfrak{B}^F_{m-2} + \mathfrak{B}^F_{m-3} & \mathfrak{B}^F_{m-2} \\ \mathfrak{B}^F_{m-2} & \mathfrak{B}^F_{m-3} + \mathfrak{B}^F_{m-4} & \mathfrak{B}^F_{m-3} \end{bmatrix}.$$

Since $\mathfrak{B}Q^G_{n+m}$ is the (3,1) entry of the matrix $\mathbf{Y}_{n+m}$ it is equal to the product of the third row of the matrix $\mathbf{Y}_n$ and the first row of the matrix $\mathbf{B}^m$. □

6. Some Identities for Third Order Bronze Fibonacci Quaternions

In this section, we first give the d'Ocagne's-like identity and Cassini's-like identity and then we obtain Cassini's identity for third order bronze Fibonacci quaternions by using q-determinants for matrices with quaternionic entries.

Theorem 6.1. *(d'Ocagne's-like identity). Let $\mathfrak{B}Q_m^G$ be the nth bronze Fibonacci quaternion and $n \geq m \geq 0$, then the d'Ocagne's-like identity for third order bronze Fibonacci quaternions is given by*

$$\mathfrak{B}Q_m^G \cdot \mathfrak{B}Q_{n+1}^G - \mathfrak{B}Q_{m+1}^G \cdot \mathfrak{B}Q_n^G = \sum_{i<j} d_i d_j(\alpha_j - \alpha_i)\alpha_i^m \alpha_j^m(\omega_i\omega_j\alpha_j^{n-m} - \omega_j\omega_i\alpha_i^{n-m}),$$

where d_1, d_2, d_3 are given in (2.3), $\alpha_1, \alpha_2, \alpha_3$ in (2.4) and $\omega_1, \omega_2, \omega_3$ in (4.3).

Proof. The proof can be shown by using the Binet's formula (4.3)

$$\begin{aligned}
&(d_1\omega_1\alpha_1^m + d_2\omega_2\alpha_2^m + d_3\omega_3\alpha_3^m)\cdot(d_1\omega_1\alpha_1^{n+1} + d_2\omega_2\alpha_2^{n+1} + d_3\omega_3\alpha_3^{n+1})\\
&\qquad - (d_1\omega_1\alpha_1^{m+1} + d_2\omega_2\alpha_2^{m+1} + d_3\omega_3\alpha_3^{m+1})\cdot(d_1\omega_1\alpha_1^{n} + d_2\omega_2\alpha_2^{n} + d_3\omega_3\alpha_3^{n})\\
&= \sum_{i<j} d_i d_j(\omega_i\omega_j\alpha_i^m\alpha_j^n(\alpha_j - \alpha_i) + \omega_j\omega_i\alpha_j^m\alpha_i^n(\alpha_i - \alpha_j))\\
&= \sum_{i<j} d_i d_j(\alpha_j - \alpha_i)\alpha_i^m\alpha_j^m(\omega_i\omega_j\alpha_j^{n-m} - \omega_j\omega_i\alpha_i^{n-m}).
\end{aligned}$$

□

For $m := n+1$ in d'Ocagne's we get the Cassini's-like identity for third order bronze Fibonacci quaternions in the next result.

Corollary 6.2. *(Cassini's-like identity). For a natural number n we have*

$${\mathfrak{B}Q_{n+1}^G}^2 - \mathfrak{B}Q_{n+2}^G \cdot \mathfrak{B}Q_n^G = \sum_{i<j} d_i d_j(\alpha_j - \alpha_i)\alpha_i^n\alpha_j^n(\omega_i\omega_j\alpha_i - \omega_j\omega_i\alpha_j).$$

Now, we give Cassini's identity for generalized third order bronze Fibonacci quaternions by using q-determinants.

Theorem 6.3. *Cassini's Identity for generalized third order bronze Fibonacci quaternions is given by*

$$\begin{vmatrix} \mathfrak{B}Q_{n+2}^G & \mathfrak{B}Q_{n+1}^G & \mathfrak{B}Q_n^G \\ \mathfrak{B}Q_{n+1}^G & \mathfrak{B}Q_n^G & \mathfrak{B}Q_{n-1}^G \\ \mathfrak{B}Q_n^G & \mathfrak{B}Q_{n-1}^G & \mathfrak{B}Q_{n-2}^G \end{vmatrix}_q = \begin{vmatrix} \mathfrak{B}Q_2^G & \mathfrak{B}Q_1^G & \mathfrak{B}Q_0^G \\ \mathfrak{B}Q_1^G & \mathfrak{B}Q_0^G & \mathfrak{B}Q_{-1}^G \\ \mathfrak{B}Q_0^G & \mathfrak{B}Q_{-1}^G & \mathfrak{B}Q_{-2}^G \end{vmatrix}_q.$$

Proof. The proof is given by mathematical induction, using recurrence relations and the properties of q-determinant of matrices with quaternionic entries. For $n = 0$ the result is obvious. Let us assume that identity is satisfied for $n := n$ then by using recurrence relation it can be easily proved that it is satisfied for $n := n+1$

$$\begin{aligned}
\begin{vmatrix} \mathfrak{B}Q_{n+3}^G & \mathfrak{B}Q_{n+2}^G & \mathfrak{B}Q_{n+1}^G \\ \mathfrak{B}Q_{n+2}^G & \mathfrak{B}Q_{n+1}^G & \mathfrak{B}Q_n^G \\ \mathfrak{B}Q_{n+1}^G & \mathfrak{B}Q_n^G & \mathfrak{B}Q_{n-1}^G \end{vmatrix}_q &= \begin{vmatrix} 3\mathfrak{B}Q_{n+2}^G + \mathfrak{B}Q_{n+1}^G + \mathfrak{B}Q_n^G & \mathfrak{B}Q_{n+2}^G & \mathfrak{B}Q_{n+1}^G \\ 3\mathfrak{B}Q_{n+1}^G + \mathfrak{B}Q_n^G + \mathfrak{B}Q_{n-1}^G & \mathfrak{B}Q_{n+1}^G & \mathfrak{B}Q_n^G \\ 3\mathfrak{B}Q_n^G + \mathfrak{B}Q_{n-1}^G + \mathfrak{B}Q_{n-2}^G & \mathfrak{B}Q_n^G & \mathfrak{B}Q_{n-1}^G \end{vmatrix}_q \\
&= \begin{vmatrix} \mathfrak{B}Q_n^G & \mathfrak{B}Q_{n+2}^G & \mathfrak{B}Q_{n+1}^G \\ \mathfrak{B}Q_{n-1}^G & \mathfrak{B}Q_{n+1}^G & \mathfrak{B}Q_n^G \\ \mathfrak{B}Q_{n-2}^G & \mathfrak{B}Q_n^G & \mathfrak{B}Q_{n-1}^G \end{vmatrix}_q = \begin{vmatrix} \mathfrak{B}Q_{n+2}^G & \mathfrak{B}Q_{n+1}^G & \mathfrak{B}Q_n^G \\ \mathfrak{B}Q_{n+1}^G & \mathfrak{B}Q_n^G & \mathfrak{B}Q_{n+1}^G \\ \mathfrak{B}Q_n^G & \mathfrak{B}Q_{n-1}^G & \mathfrak{B}Q_{n-2}^G \end{vmatrix}_q.
\end{aligned}$$

□

Corollary 6.4. *Cassini's Identity for $\{\mathfrak{B}Q_n^F\}$, $\{\mathfrak{B}Q_n^L\}$, $\{\mathfrak{B}Q_n^M\}$ sequences is given by*

$$\begin{vmatrix} \mathfrak{B}Q_{n+2}^F & \mathfrak{B}Q_{n+1}^F & \mathfrak{B}Q_n^F \\ \mathfrak{B}Q_{n+1}^F & \mathfrak{B}Q_n^F & \mathfrak{B}Q_{n-1}^F \\ \mathfrak{B}Q_n^F & \mathfrak{B}Q_{n-1}^F & \mathfrak{B}Q_{n-2}^F \end{vmatrix}_q = 1088, \tag{6.1}$$

$$\begin{vmatrix} \mathfrak{B}Q^L_{n+2} & \mathfrak{B}Q^L_{n+1} & \mathfrak{B}Q^L_{n} \\ \mathfrak{B}Q^L_{n+1} & \mathfrak{B}Q^L_{n} & \mathfrak{B}Q^L_{n-1} \\ \mathfrak{B}Q^L_{n} & \mathfrak{B}Q^L_{n-1} & \mathfrak{B}Q^L_{n-2} \end{vmatrix}_q = 31234981888 + 31426560000i, \tag{6.2}$$

$$\begin{vmatrix} \mathfrak{B}Q^M_{n+2} & \mathfrak{B}Q^M_{n+1} & \mathfrak{B}Q^M_{n} \\ \mathfrak{B}Q^M_{n+1} & \mathfrak{B}Q^M_{n} & \mathfrak{B}Q^M_{n-1} \\ \mathfrak{B}Q^M_{n} & \mathfrak{B}Q^M_{n-1} & \mathfrak{B}Q^M_{n-2} \end{vmatrix}_q = 349660 - 501220i. \tag{6.3}$$

Proof. Since $\mathfrak{B}Q^F_n = \mathfrak{B}^F_n + \mathfrak{B}^F_{n+1}i + \mathfrak{B}^F_{n+2}j + \mathfrak{B}^F_{n+3}k$ we can write

$$\mathfrak{B}Q^F_n = (\mathfrak{B}^F_n + \mathfrak{B}^F_{n+1}i) + (\mathfrak{B}^F_{n+2} + \mathfrak{B}^F_{n+3}i)j.$$

Now, let

$$A = \begin{bmatrix} \mathfrak{B}Q^F_2 & \mathfrak{B}Q^F_1 & \mathfrak{B}Q^F_0 \\ \mathfrak{B}Q^F_1 & \mathfrak{B}Q^F_0 & \mathfrak{B}Q^F_{-1} \\ BQ^F_0 & \mathfrak{B}Q^F_{-1} & \mathfrak{B}Q^F_{-2} \end{bmatrix}$$

be a matrix with quaternion entries, and

$$\begin{aligned} A_1 &= \begin{bmatrix} \mathfrak{B}^F_2 + \mathfrak{B}^F_3 i & \mathfrak{B}^F_1 + \mathfrak{B}^F_2 i & \mathfrak{B}^F_0 + \mathfrak{B}^F_1 i \\ \mathfrak{B}^F_1 + \mathfrak{B}^F_2 i & \mathfrak{B}^F_0 + \mathfrak{B}^F_1 i & \mathfrak{B}^F_{-1} + \mathfrak{B}^F_0 i \\ \mathfrak{B}^F_0 + \mathfrak{B}^F_1 i & \mathfrak{B}^F_{-1} + \mathfrak{B}^F_0 i & \mathfrak{B}^F_{-2} + \mathfrak{B}^F_{-1} i \end{bmatrix} \\ &= \begin{bmatrix} 10 + 34i & 3 + 10i & 1 + 3i \\ 3 + 10i & 1 + 3i & 0 + 1i \\ 1 + 3i & 0 + 1i & 0 + 0i \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} A_2 &= \begin{bmatrix} \mathfrak{B}^F_4 + \mathfrak{B}^F_5 i & \mathfrak{B}^F_3 + \mathfrak{B}^F_4 i & \mathfrak{B}^F_2 + \mathfrak{B}^F_3 i \\ \mathfrak{B}^F_3 + \mathfrak{B}^F_4 i & \mathfrak{B}^F_2 + \mathfrak{B}^F_3 & \mathfrak{B}^F_1 + \mathfrak{B}^F_2 i \\ \mathfrak{B}^F_2 + \mathfrak{B}^F_3 i & \mathfrak{B}^F_1 + \mathfrak{B}^F_2 i & \mathfrak{B}^F_0 + \mathfrak{B}^F_1 i \end{bmatrix} \\ &= \begin{bmatrix} 115 + 89i & 34 + 115i & 10 + 34i \\ 34 + 115i & 10 + 34i & 3 + 10i \\ 10 + 34i & 3 + 10i & 1 + 3i \end{bmatrix} \end{aligned}$$

matrices with complex entries, then $A = A_1 + A_2 j$. We define the complex adjoint of the quaternion matrix A by

$$\chi_A = \begin{bmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{bmatrix}$$

and define the q-determinant of A as $|A|_q = |\chi_A|$, [20]. Then by using the properties of the determinant of block matrices we have

$$\begin{aligned} |\chi_A| &= \begin{vmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{vmatrix} = det A_1 det(\overline{A_1} + \overline{A_2} A_1^{-1} A_2), \\ A_1^{-1} &= \frac{1}{4} \begin{bmatrix} -1 + i & 2 - 4i & 5 + i \\ 2 - 4i & -2 + 14i & -1 - 2i \\ 5 + i & -1 - 2i & -3 - 5i \end{bmatrix}, \\ \overline{A_2} A_1^{-1} A_2 &= \begin{bmatrix} 1316 - 4452i & 389 - 1316i & 115 - 389i \\ 389 - 1316i & 115 - 389i & 34 - 115i \\ 115 - 389i & 34 - 115i & 10 - 34i \end{bmatrix}, \\ \overline{A_1} + \overline{A_2} A_1^{-1} A_2 &= \begin{bmatrix} 1326 - 4486i & 392 - 1326i & 116 - 392 \\ 392 - 1326i & 116 - 392i & -116i + 34 \\ 116 - 392i & -116i + 34 & 10 - 34i \end{bmatrix} \end{aligned}$$

and

$$det(\overline{A_1} + \overline{A_2} A_1^{-1} A_2) = -272 + 272i,$$

$$detA_1 = -2 - 2i.$$

Then,

$$|A|_q = |\chi_A| = (-2 - 2i)(-272 + 272) = 1088.$$

By the above theorem, we get equation (6.1).

Equations (6.2) and (6.3) are obtained similarly. □

7. Conclusion

There have been many studies about Fibonacci and other second order quaternions. As for third order quaternions, there are studies about Jacobsthal, dual Jacobsthal and bicomplex Jacobsthal third order quaternions. We introduce generalized third order bronze Fibonacci quaternions and its three specific sequences. We present Binet's formula, generating functions, matrix representation for these sequences, d'Ocagne's-like and Cassini's-like formulas. Furthermore, we give a new method to calculate the Cassini's Identity for generalized third order bronze Fibonacci quaternions and its three specific sequences.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The author has read and agreed to the published version of the manuscript.

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