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Dynamic Behavior of a Fourth-Order Nonlinear Fuzzy Difference Equation

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Highlights

- This paper focuses on positive solutions of a nonlinear fuzzy difference equation.
- A bounded study of a fuzzy difference equation was carried out.
- Qualitative behavior of the positive solutions of a fuzzy difference equation were investigated.

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Abstract

In this paper, qualitative behavior of the following difference equation

$$w_{n+1}=\frac{Aw_{n-1}}{B+Cw_{n-3}^{p}},\ n\in\mathbb{N}_0,$$

where $A, B, C, w_{-3}, w_{-2}, w_{-1}, w_0$ are positive fuzzy numbers and p is a fixed positive integer, is investigated. It is shown that the equation is a fuzzy difference equation and has solutions converging to zero. The theoretical results obtained are tested with a few simulations. An interesting result is that the equation has some similar dynamics to its ordinary version.

1. INTRODUCTION

Classical difference equations are mathematical equations that delineate the behavior of a variable across successive, interconnected time points. These equations are fundamental models for analyzing how systems evolve and operate over time, relying on precise initial data. The modeling and control of complex systems, particularly, have driven significant research efforts in this domain, spanning numerous disciplines. These equations have many applications both in mathematics and other sciences such as computer engineering, economics, genetics, health sciences, biology, probability, etc. See, e.g., [1,2]. For this reason, for nearly two decades, experts in mathematics and other disciplines have taken a keen interest in difference equations. Conversely, fuzzy difference equations represent an advanced approach, incorporating fuzzy logic to model systems with uncertain information. In contrast to classical equations, these models provide a more realistic and adaptable approach by incorporating uncertainty and imprecise data. This makes them highly effective for tackling real-world problems, where obtaining fully accurate and complete information about input data and system behavior is often unfeasible.

Two studies on two applications of fuzzy difference equations are presented below as examples.

In [3], Deeba et al. discussed the following first-order fuzzy equation

$$w_{n+1}=aw_n+b,\ n\in\mathbb{N}_0, \tag{1}$$

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where $a,b,w_0 \in \mathbb{R}_F$. The equation arises as a mathematical model in population genetics. They also considered the general fuzzy difference equation

$$w_{n+1} = f(w_n, b, a),\ n \in \mathbb{N}_0, \tag{2}$$

where $a,b,w_0 \in \mathbb{R}_F$, $f: \mathbb{R}_c^+ \times \mathbb{R}_c^+ \times \mathbb{R}_c^+ \to \mathbb{R}_c^+$, and $\mathbb{R}_c^+ = \{x : x \ge c\}$, is a continuous and nondecreasing function in its all arguments. In [4], Deeba and Korvin handled the fuzzy difference equation of order two

$$C_{n+1} = C_n - abC_{n-1} + m,\ n \in \mathbb{N}_0, \tag{3}$$

where $a,b,m,C_0,C_{-1} \in \mathbb{R}_F$. Equation (3) is a linearized version of a nonlinear mathematical model that states the concentration of CO_2 in the blood.

Now, let us give a few examples from published studies on the existence, boundedness, and stability of solutions of fuzzy difference equations.

In [5], Zhang et al. studied the following fuzzy equation of Riccati type

$$w_{n+1} = \frac{A + w_n}{B + w_n},\ n \in \mathbb{N}_0, \tag{4}$$

where $A,B,w_0 \in \mathbb{R}_F^+$. In [6], Zhang et al. handled the difference equation

$$w_{n+1} = A + \frac{w_n}{w_{n-1}w_{n-2}},\ n \in \mathbb{N}_0, \tag{5}$$

where $A, w_{-2}, w_{-1}, w_0 \in \mathbb{R}_F^+$.

In [7], Rahman et al. handled the second-order equation

$$w_{n+1} = \frac{w_{n-1}}{A + Bw_{n-1}w_n},\ n \in \mathbb{N}_0, \tag{6}$$

where $A, B, w_{-1}, w_0 \in \mathbb{R}_F^+$.

In [8], Yalçınkaya et al. handled the following third-order equation

$$w_{n+1} = \frac{w_{n-2}}{C + w_{n-2}w_{n-1}w_n},\ n \in \mathbb{N}_0, \tag{7}$$

where $C, w_{-2}, w_{-1}, w_0 \in \mathbb{R}_F^+$. Also, for other works on the fuzzy difference equations, see, e.g. the references [9-13].

In [14], El-Owaidy et al. handled qualitative behavior of nonnegative solutions of the equation

$$t_{n+1} = \frac{\alpha t_{n-1}}{\beta + \gamma t_{n-2}^p},\ n \in \mathbb{N}_0, \tag{8}$$

where $\alpha,\beta,\gamma,t_{-2},t_{-1},t_0 \in \mathbb{R}^+ \cup \{0\}$.

In reference [15], Gümüş and Soykan investigated qualitative behavior of the positive solutions of the following nonlinear system of order three

$$t_{n+1}=\frac{\alpha t_{n-1}}{\beta+\gamma z_{n-2}^{p}},\ z_{n+1}=\frac{\alpha_1 z_{n-1}}{\beta_1+\gamma_1 t_{n-2}^{p}},\ n\in\mathbb{N}_0, \tag{9}$$

where $\alpha,\beta,\gamma,\alpha_1,\beta_1,\gamma_1,t_{-i},z_{-i}\in\mathbb{R}^+$ for $i=0,1,2$ by extending results on the difference equation in (8).

In [16], Yalçınkaya et al. investigated qualitative behavior of the positive solutions of the equation

$$w_{n+1}=\frac{Aw_{n-1}}{1+w_{n-2}^{p}},\ n\in\mathbb{N}_0, \tag{10}$$

where $p\in\mathbb{Z}^+$ and $A,w_{-2},w_{-1},w_0\in\mathbb{R}_F^+$.

In [17], Türk et al. worked qualitative behavior of the positive solutions of the system of order four

$$t_{n+1}=\frac{at_{n-1}}{b+cz_{n-3}^{p}},\ z_{n+1}=\frac{dz_{n-1}}{e+ft_{n-3}^{q}},\ n\in\mathbb{N}_0, \tag{11}$$

where $t_{-i},z_{-i}\in\mathbb{R}^+\cup\{0\}$ for $i=0,1,2,3$ and $p,q,a,b,c,d,e,f\in\mathbb{R}^+$ by reducing system (11) to the system

$$u_{n+1}=\frac{\gamma u_{n-1}}{1+v_{n-3}^{p}},\ v_{n+1}=\frac{\beta v_{n-1}}{1+u_{n-3}^{q}},\ n\in\mathbb{N}_0, \tag{12}$$

where $t_n=\left(\frac{e}{f}\right)^{1/q}u_n$, $z_n=\left(\frac{b}{c}\right)^{1/p}v_n$ with $\gamma=\frac{a}{b}$ and $\beta=\frac{d}{e}$. The authors obtained the next results.

Theorem 1. Assume that $\gamma>1$ and $\beta>1$. Then system (12) has unbounded solutions.

Theorem 2. Assume that $\gamma<1$ and $\beta<1$. Then every positive solution of system (12) is bounded and persists.

Theorem 3. Assume that $\gamma<1$ and $\beta<1$. Then every positive solution (u_n,v_n) of system (12) tends to $(0,0)$ as $n\to\infty$.

In this study, we examine the existence, boundedness, and long-term behavior of positive solutions for the fuzzy equation of order four

$$w_{n+1}=\frac{Aw_{n-1}}{B+Cw_{n-3}^{p}},\ n\in\mathbb{N}_0, \tag{13}$$

where $A,B,C,w_{-3},w_{-2},w_{-1},w_0\in\mathbb{R}_F^+$ and $p\in\mathbb{Z}^+$.

A fuzzy difference equation is a type of difference equation where the solution consists of a sequence of fuzzy numbers. That is to say, its initial conditions consist of fuzzy numbers. In the sequel of our study, we will use some known results on the fuzzy numbers. One can find the aforementioned results in the references [18-20].

In this study we denote by $\mathbb{R}_F$ the space of all fuzzy numbers. For $0<\alpha\le 1$ and $X\in\mathbb{R}_F$, we denote $\alpha-$ cuts of X by $[X]^\alpha=\{x\in(-\infty,+\infty) : X(x)\ge\alpha\}$. Also, if $[X]^0=\overline{\{x\in(-\infty,+\infty) : X(x)\ge 0\}}$, then $[X]^0$ is called the support of X and denoted by $\sup p(X)$.

2. MAIN RESULTS

We here prove the main results of our study on the fuzzy equation (13). We will first show that the positive solutions of (13) exist. We say the sequence (w_n) is a positive solution of the fuzzy difference equation (13) if it satisfies the fuzzy difference equation (13). Fuzzy numbers are intervals of real numbers that satisfy certain conditions. A fuzzy difference equation is one for which such numbers are initial conditions and the fuzzy number rules apply. The property that fuzzy numbers are constructed with real number intervals relates them to a two-dimensional classical difference equation system of the same order. So, a fuzzy difference equation is equivalent to a symmetric system of difference equations. As a result, when studying the solutions of fuzzy difference equations, the solutions of such systems are considered with some conditions. In the following, the solutions of the fuzzy equation (13) will be studied using such an approach.

Theorem 4. If $p\in\mathbb{Z}^+$ and $A,B,C,w_{-3},w_{-2},w_{-1},w_0\in\mathbb{R}_F^+$, then equation (13) has a unique positive fuzzy solution (w_n).

Proof. We suppose that there exists any solution (w_n) of (13) corresponding to the initial conditions $w_{-3},w_{-2},w_{-1},w_0\in\mathbb{R}_F^+$. Then, the α-cuts of $A,B,C,$ and (w_n) are given by

$$\begin{aligned}
[w_n]^\alpha &= [L_n^\alpha, U_n^\alpha],\\
[A]^\alpha &= [A_l^\alpha, A_r^\alpha],\\
[B]^\alpha &= [B_l^\alpha, B_r^\alpha],\\
[C]^\alpha &= [C_l^\alpha, C_r^\alpha]
\end{aligned} \tag{14}$$

for $n=-3,-2,\ldots$ and $\alpha\in(0,1]$. From (13)-(14) and Lemma 1 of [21], we see that

$$\begin{aligned}
[L_{n+1}^\alpha, U_{n+1}^\alpha] = [w_{n+1}]^\alpha
&= \left[\frac{Aw_{n-1}}{B+Cw_{n-3}^p}\right]^\alpha\\
&= \frac{[A]^\alpha[w_{n-1}]^\alpha}{\left[B+Cw_{n-3}^p\right]^\alpha}\\
&= \frac{[A_l^\alpha L_{n-1}^\alpha, A_r^\alpha U_{n-1}^\alpha]}{[B_l^\alpha + C_l^\alpha (L_{n-3}^\alpha)^p, B_r^\alpha + C_r^\alpha (U_{n-3}^\alpha)^p]}\\
&= \left[\frac{A_l^\alpha L_{n-1}^\alpha}{B_r^\alpha + C_r^\alpha (U_{n-3}^\alpha)^p}, \frac{A_r^\alpha U_{n-1}^\alpha}{B_l^\alpha + C_l^\alpha (L_{n-3}^\alpha)^p}\right],
\end{aligned}$$

from which we get

$$L_{n+1}^{\alpha}=\frac{A_l^{\alpha}L_{n-1}^{\alpha}}{B_r^{\alpha}+C_r^{\alpha}(U_{n-3}^{\alpha})^p},\ U_{n+1}^{\alpha}=\frac{A_r^{\alpha}U_{n-1}^{\alpha}}{B_l^{\alpha}+C_l^{\alpha}(L_{n-3}^{\alpha})^p} \tag{15}$$

for $n\in\mathbb{N}_0$ and $\alpha\in(0,1]$. Then, one can see that there exists a unique $(L_n^{\alpha},U_n^{\alpha})$ corresponding to $(L_j^{\alpha},U_j^{\alpha})$, $j=-3,-2,-1,0$ and $\alpha\in(0,1]$.

We will show that $[L_n^{\alpha},U_n^{\alpha}]$ for $\alpha\in(0,1]$, where $(L_n^{\alpha},U_n^{\alpha})$ is any positive solution of (15) corresponding to $(L_j^{\alpha},U_j^{\alpha})$, $j=-3,-2,-1,0$, determines the solution (w_n) of (13) with the positive fuzzy initial conditions w_{-3},w_{-2},w_{-1},w_0 such that

$$[w_n]^{\alpha}=[L_n^{\alpha},U_n^{\alpha}],\ \alpha\in(0,1],\ n=-3,-2,.... \tag{16}$$

Since $A,B,C,w_{-3},w_{-2},w_{-1},w_0\in\mathbb{R}_F^+$, we get

$$\begin{aligned}
&0<A_l^{\alpha_1}\le A_l^{\alpha_2}\le A_r^{\alpha_2}\le A_r^{\alpha_1},\\
&0<B_l^{\alpha_1}\le B_l^{\alpha_2}\le B_r^{\alpha_2}\le B_r^{\alpha_1},\\
&0<C_l^{\alpha_1}\le C_l^{\alpha_2}\le C_r^{\alpha_2}\le C_r^{\alpha_1},\\
&0<L_{-3}^{\alpha_1}\le L_{-3}^{\alpha_2}\le U_{-3}^{\alpha_2}\le U_{-3}^{\alpha_1},\\
&0<L_{-2}^{\alpha_1}\le L_{-2}^{\alpha_2}\le U_{-2}^{\alpha_2}\le U_{-2}^{\alpha_1},\\
&0<L_{-1}^{\alpha_1}\le L_{-1}^{\alpha_2}\le U_{-1}^{\alpha_2}\le U_{-1}^{\alpha_1},\\
&0<L_{0}^{\alpha_1}\le L_{0}^{\alpha_2}\le U_{0}^{\alpha_2}\le U_{0}^{\alpha_1}
\end{aligned} \tag{17}$$

for any $\alpha_1,\alpha_2\in(0,1]$, $\alpha_1\le\alpha_2$. We show by the standard induction method that

$$0<L_n^{\alpha_1}\le L_n^{\alpha_2}\le U_n^{\alpha_2}\le U_n^{\alpha_1},\ n\in\mathbb{N}. \tag{18}$$

From (17) we have that the inequalities in (18) is satisfied for $n=-3,-2,-1,0$. Now, we assume that for $n\le k$, $k\ge1$, (18) is valid. In this case, it follows from (15), (17), (18) for $n\le k$ that

$$L_{k+1}^{\alpha_1}=\frac{A_l^{\alpha_1}L_{k-1}^{\alpha_1}}{B_r^{\alpha_1}+C_r^{\alpha_1}\left(U_{k-3}^{\alpha_1}\right)^p}\le\frac{A_l^{\alpha_2}L_{k-1}^{\alpha_2}}{B_r^{\alpha_2}+C_r^{\alpha_2}\left(U_{k-3}^{\alpha_2}\right)^p}=L_{k+1}^{\alpha_2},$$

$$L_{k+1}^{\alpha_2}=\frac{A_l^{\alpha_2}L_{k-1}^{\alpha_2}}{B_r^{\alpha_2}+C_r^{\alpha_2}\left(U_{k-3}^{\alpha_2}\right)^p}\le\frac{A_r^{\alpha_2}U_{k-1}^{\alpha_2}}{B_l^{\alpha_2}+C_l^{\alpha_2}\left(L_{k-3}^{\alpha_2}\right)^p}=U_{k+1}^{\alpha_2}$$

and

$$U_{k+1}^{\alpha_2}=\frac{A_r^{\alpha_2}U_{k-1}^{\alpha_2}}{B_l^{\alpha_2}+C_l^{\alpha_2}\left(L_{k-3}^{\alpha_2}\right)^p}\le\frac{A_r^{\alpha_1}U_{k-1}^{\alpha_1}}{B_l^{\alpha_1}+C_l^{\alpha_1}\left(L_{k-3}^{\alpha_1}\right)^p}=U_{k+1}^{\alpha_1}.$$

Therefore (18) is satisfied. Also, from (15), we obtain

$$L_1^{\alpha}=\frac{A_l^{\alpha}L_{-1}^{\alpha}}{B_r^{\alpha}+C_r^{\alpha}\left(U_{-3}^{\alpha}\right)^p},\ U_1^{\alpha}=\frac{A_r^{\alpha}U_{-1}^{\alpha}}{B_l^{\alpha}+C_l^{\alpha}\left(L_{-3}^{\alpha}\right)^p},\ \alpha\in(0,1]. \tag{19}$$

Since $A,B,C,w_{-3},w_{-2},w_{-1},w_0\in\mathbb{R}_F^+$, we have that A_l^{α}, A_r^{α}, B_l^{α}, B_r^{α}, C_l^{α}, C_r^{α}, L_{-1}^{α}, U_{-1}^{α}, L_{-2}^{α}, U_{-2}^{α}, L_{-3}^{α} and

U_{-3}^{α} are left continuous. Hence, one can see from (19) that L_1^{α} and U_1^{α} are left continuous, too. Then, by using the standard mathematical induction, one can show for $n \in \mathbb{N}$ that L_n^{α} and U_n^{α} are left continuous.

We need to show that $\overline{\cup_{\alpha\in(0,1]}[L_n^{\alpha},U_n^{\alpha}]}$ is a compact set. We prove that the union $\cup_{\alpha\in(0,1]}[L_n^{\alpha},U_n^{\alpha}]$ is a bounded set in this case. Set $n=1$. Since $A,B,C,w_{-3},w_{-2},w_{-1},w_0 \in \mathbb{R}_F^{+}$, there exist constants P_A, Q_A, P_B, Q_B, P_C, Q_C, P_{-3}, Q_{-3}, P_{-2}, Q_{-2}, P_{-1}, Q_{-1}, P_0, $Q_0 > 0$ such that

$$\begin{aligned}
&[A_l^{\alpha},A_r^{\alpha}] \subset [P_A,Q_A],\\
&[B_l^{\alpha},B_r^{\alpha}] \subset [P_B,Q_B],\\
&[C_l^{\alpha},C_r^{\alpha}] \subset [P_C,Q_C],\\
&[L_{-3}^{\alpha},U_{-3}^{\alpha}] \subset [P_{-3},Q_{-3}],\\
&[L_{-2}^{\alpha},U_{-2}^{\alpha}] \subset [P_{-2},Q_{-2}],\\
&[L_{-1}^{\alpha},U_{-1}^{\alpha}] \subset [P_{-1},Q_{-1}],\\
&[L_0^{\alpha},U_0^{\alpha}] \subset [P_0,Q_0].
\end{aligned} \tag{20}$$

Therefore, from (19)-(20) we can easily see that

$$[L_1^{\alpha},U_1^{\alpha}] \subset \left[\frac{P_A P_{-1}}{Q_B + Q_C\left(Q_{-3}\right)^p}, \frac{Q_A Q_{-1}}{P_B + P_C\left(P_{-3}\right)^p}\right],$$

for $\alpha \in (0,1]$ from which it is clear that

$$\cup_{\alpha\in(0,1]}[L_1^{\alpha},U_1^{\alpha}] \subset \left[\frac{P_A P_{-1}}{Q_B + Q_C\left(Q_{-3}\right)^p}, \frac{Q_A Q_{-1}}{P_B + P_C\left(P_{-3}\right)^p}\right], \tag{21}$$

for $\alpha \in (0,1]$. Also, (21) implies that $\overline{\cup_{\alpha\in(0,1]}[L_1^{\alpha},U_1^{\alpha}]}$ is compact and $\overline{\cup_{\alpha\in(0,1]}[L_1^{\alpha},U_1^{\alpha}]} \subset (0,\infty)$. By the mathematical induction, one may show that

$$\overline{\cup_{\alpha\in(0,1]}[L_n^{\alpha},U_n^{\alpha}]} \text{ is compact, } \overline{\cup_{\alpha\in(0,1]}[L_n^{\alpha},U_n^{\alpha}]} \subset (0,\infty) \text{ for } n \in \mathbb{N}. \tag{22}$$

Therefore, using (18), (22), and the fact that $L_n^{\alpha},U_n^{\alpha}$ are left continuous, and therefore we obtain that $[L_n^{\alpha},U_n^{\alpha}]$ determines a sequence (w_n) compliant with (16).

We now prove that (w_n) is the solution of equation (13) with the initial conditions w_{-3},w_{-2},w_{-1},w_0. Since

$$[w_{n+1}]^{\alpha} = [L_{n+1}^{\alpha},U_{n+1}^{\alpha}] = \left[\frac{A_l^{\alpha}L_{n-1}^{\alpha}}{B_r^{\alpha}+C_r^{\alpha}(U_{n-3}^{\alpha})^p}, \frac{A_r^{\alpha}U_{n-1}^{\alpha}}{B_l^{\alpha}+C_l^{\alpha}(L_{n-3}^{\alpha})^p}\right] = \left[\frac{Aw_{n-1}}{B+Cw_{n-3}^p}\right]^{\alpha},$$

for all $\alpha \in (0,1]$, we see that (w_n) satisfies equation (13), that is it is the solution of equation (13) with w_{-3},w_{-2},w_{-1},w_0.

Let (w_n) be another solution of (13) with w_{-3},w_{-2},w_{-1},w_0. Then arguing as above one can show that

$$[w_n]^\alpha = [L_n^\alpha, U_n^\alpha] \text{ for } \alpha \in (0,1] \text{ and } n \in \mathbb{N}_0. \tag{23}$$

Then from (16) and (23) we have that $[w_n]^\alpha = [w_n]^\alpha$ for $\alpha \in (0,1]$ and $n = -3,-2,...$ from which it holds $w_n = w_n$ for $n = -3,-2,...$. Thus, the proof is completed.

Theorem 5. All positive solutions of (13) are bounded and persist, if $A_r^\alpha < B_l^\alpha$ for all $\alpha \in (0,1]$.

Proof. Assume that (w_n) is a positive solution of (13) compliant with (16). By (15) and Theorem 2, one can obtain

$$[L_n^\alpha, U_n^\alpha] \subset [0, T^\alpha],\ n \in \mathbb{N}, \tag{24}$$

where $T^\alpha = \max\{U_{-1}^\alpha, U_0^\alpha\}$. Since (w_n) is a sequence consisting of positive fuzzy numbers, we can find a constant $T > 0$ satisfying $T^\alpha \le T$ for $\alpha \in (0,1]$. So, $[L_n^\alpha, U_n^\alpha] \subset [0,T]$ for $n \in \mathbb{N}$ from which we get $\cup_{\alpha\in(0,1]}[L_n^\alpha, U_n^\alpha] \subset [0,T]$ for $n \in \mathbb{N}$ so $\overline{\cup_{\alpha\in(0,1]}[L_n^\alpha, U_n^\alpha]} \subseteq [0,T]$. So, the proof is completed.

Theorem 6. If there exists an $\bar{\alpha} \in (0,1]$ such that $B_r^{\bar{\alpha}} < A_l^{\bar{\alpha}}$, then the Equation (13) has unbounded solutions.

Proof. If $\frac{A_l^{\bar{\alpha}}}{B_r^{\bar{\alpha}}} = \gamma$, $\frac{A_r^{\bar{\alpha}}}{B_l^{\bar{\alpha}}} = \beta$, $L_n^{\bar{\alpha}} = u_n$ and $U_n^{\bar{\alpha}} = v_n$ for $n = -3,-2,...,$ then we can apply Theorem 1 to system (15). If there exists an $\bar{\alpha} \in (0,1]$ such that $B_r^{\bar{\alpha}} < A_l^{\bar{\alpha}}$, then there exists a solution (u_n, v_n) of system (15) where $\bar{\alpha} = \alpha$ with (u_{-j}, v_{-j}) for $j = 0,1,2,3$ such that

$$\lim_{n\to\infty} u_n = 0, \quad \lim_{n\to\infty} v_n = \infty. \tag{25}$$

Also, if $u_{-j} < v_{-j}$ for $j = 0,1,2,3,$ we can find positive fuzzy numbers $w_{-3}, w_{-2}, w_{-1}, w_0$ such that

$$[w_j]^\alpha = [L_j^\alpha, U_j^\alpha], \tag{26}$$

for $\alpha \in (0,1]$ and

$$[w_j]^{\bar{\alpha}} = [L_j^{\bar{\alpha}}, U_j^{\bar{\alpha}}] = [u_j, v_j], \tag{27}$$

for $j = -3,-2,-1,0$. Assume that (w_n) is a positive solution of (13) corresponding to $w_{-3}, w_{-2}, w_{-1}, w_0$, and $[w_n]^\alpha = [L_n^\alpha, U_n^\alpha]$ for $\alpha \in (0,1]$. Since (26) and (27) hold and also (L_n^α, U_n^α) satisfies (15) we obtain

$$[w_n]^{\bar{\alpha}} = [L_n^{\bar{\alpha}}, U_n^{\bar{\alpha}}] = [u_n, v_n]. \tag{28}$$

Therefore, from (25), (28) and since

$$\|w_n\| = \sup\max\{|L_n^\alpha|, |U_n^\alpha|\} \ge \max\{|L_n^{\bar{\alpha}}|, |U_n^{\bar{\alpha}}|\} = U_n^{\bar{\alpha}},$$

where sup is taken for all $\alpha \in (0,1]$, it is easily seen that (w_n) is an unbounded sequence. Therefore, the proof is completed.

The following numerical example illustrates the claim of Theorem 6 for some determined values of the parameters A, B, C, and p.

Example 1. We consider (13) with $p = 3$ and the followings:

$$
\begin{aligned}
w_{-3}(x) &= \begin{cases} \frac{10x-3}{2}, & 0.30 \le x \le 0.50, \\ \frac{7-10x}{2}, & 0.50 \le x \le 0.70, \end{cases} \\
w_{-2}(x) &= \begin{cases} 10x-2, & 0.20 \le x \le 0.30, \\ 4-10x, & 0.30 \le x \le 0.40, \end{cases} \\
w_{-1}(x) &= \begin{cases} 4x-0.4, & 0.10 \le x \le 0.35, \\ 2.4-4x, & 0.35 \le x \le 0.60, \end{cases} \\
w_{0}(x) &= \begin{cases} 20x-10, & 0.50 \le x \le 0.55, \\ 12-20x, & 0.55 \le x \le 0.60. \end{cases}
\end{aligned}
\tag{29}
$$

From (29), we get

$$
\begin{aligned}
[w_{-3}]^{\alpha} &= \left[\frac{2\alpha+3}{10}, \frac{7-2\alpha}{10}\right], \\
[w_{-2}]^{\alpha} &= \left[\frac{\alpha+2}{10}, \frac{4-\alpha}{10}\right], \\
[w_{-1}]^{\alpha} &= \left[\frac{\alpha+0.4}{4}, \frac{2.4-\alpha}{4}\right], \\
[w_{0}]^{\alpha} &= \left[\frac{\alpha+10}{20}, \frac{12-\alpha}{20}\right],
\end{aligned}
$$

for all $\alpha \in (0,1]$. Moreover, the parameters A, B, C are

$$
\begin{aligned}
A &= \begin{cases} x-4, & 4 \le x \le 5, \\ 6-x, & 5 \le x \le 6, \end{cases} \\
B &= \begin{cases} 5x-2, & 0.40 \le x \le 0.60, \\ 4-5x, & 0.60 \le x \le 0.80, \end{cases} \\
C &= \begin{cases} x-5, & 5 \le x \le 6, \\ 7-x, & 6 \le x \le 7. \end{cases}
\end{aligned}
\tag{30}
$$

From (30), we get

$$
[A]^{\alpha} = [\alpha+4, 6-\alpha], \; [B]^{\alpha} = \left[\frac{\alpha+2}{5}, \frac{4-\alpha}{5}\right], \; [C]^{\alpha} = [\alpha+5, 7-\alpha],
$$

for all $\alpha \in (0,1]$. Then, we see by Theorem 4 that Equation (13) possesses a positive solution which is unique. Obviously, for $\alpha \in (0,1]$, we have $B_r^{\alpha} < A_l^{\alpha}$. So, by Theorem 6, Equation (13) has unbounded solutions, see Figures 1-4.

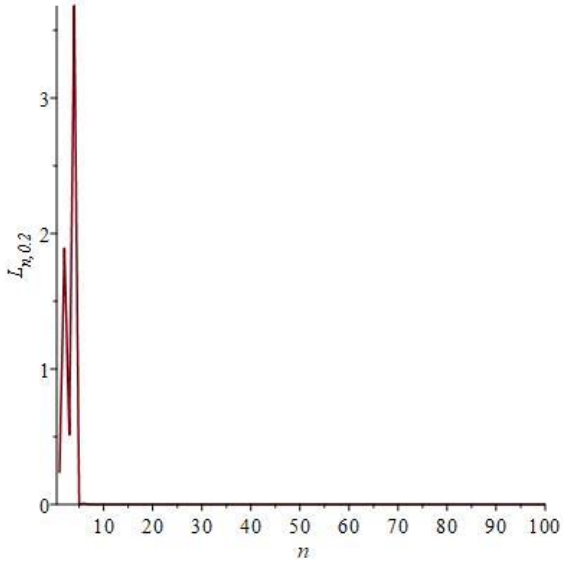

Figure 1. *Plot of* L_n *for* $\alpha = 0.2$

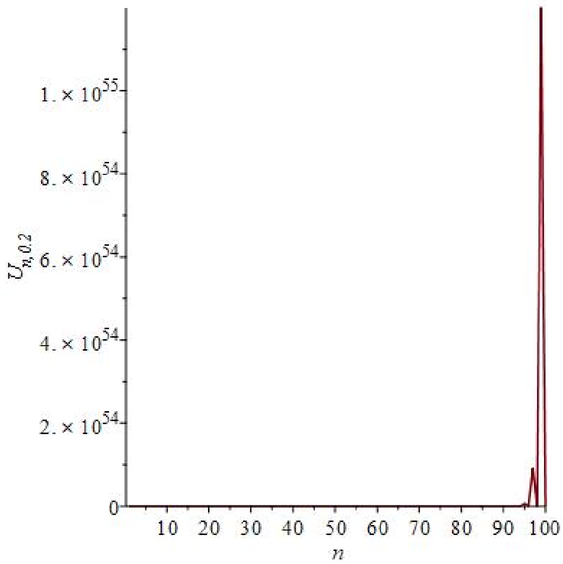

Figure 2. *Plot of* U_n *for* $\alpha = 0.2$

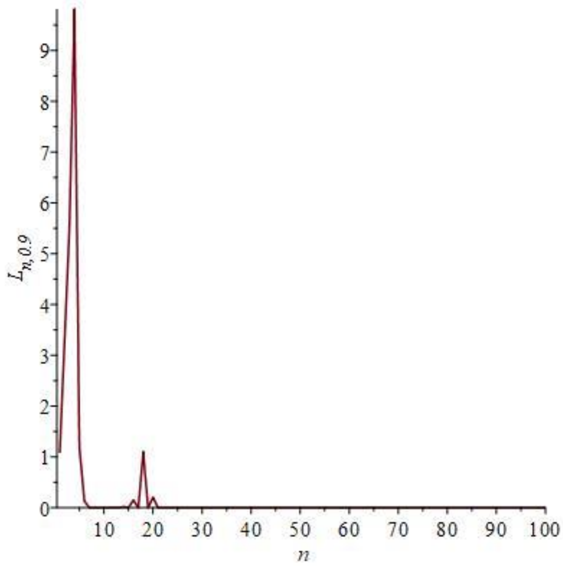

Figure 3. *Plot of* L_n *for* $\alpha = 0.9$

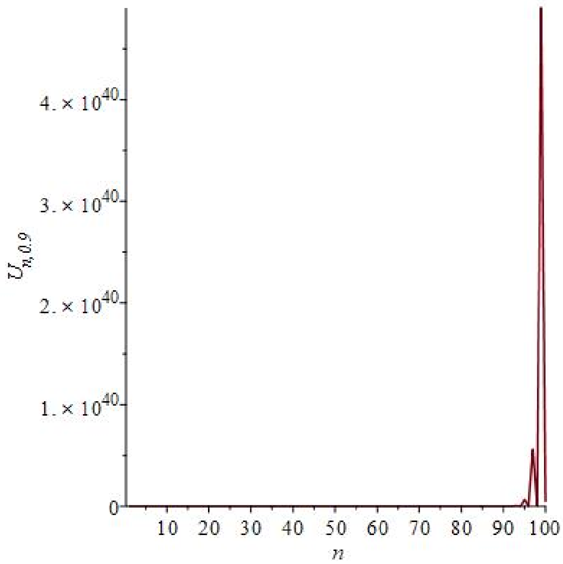

Figure 4. *Plot of* U_n *for* $\alpha = 0.9$

Theorem 7. If $A_r^\alpha < B_l^\alpha$ for all $\alpha \in (0,1]$, then, as $n \to \infty$, every positive solution (w_n) of Equation (13) tends to 0.

Proof. Assume that (w_n) is a solution of (13) such that $A_r^\alpha < B_l^\alpha$ for all $\alpha \in (0,1]$ as given in (14). Then, we can apply Theorem 3 to system (15). So, we get

$$\lim_{n\to\infty} L_n^\alpha = \lim_{n\to\infty} U_n^\alpha = 0. \quad (31)$$

Therefore, from (31) we get $\lim_{n\to\infty} D(w_n, 0) = \lim_{n\to\infty} \sup\left\{\max\left\{\left|L_n^\alpha - 0\right|, \left|U_n^\alpha - 0\right|\right\}\right\} = 0.$

The next numerical example illustrates the claim of Theorem 7 for some determined values of the parameters p, A, B, C.

Example 2. We consider Equation (13) with $p=5$, and the following fuzzy initial conditions

$$
\begin{aligned}
w_{-3}(x) &= \begin{cases} 10x-4, & 0.40 \le x \le 0.50, \\ 6-10x, & 0.50 \le x \le 0.60, \end{cases} \\
w_{-2}(x) &= \begin{cases} 5x-2, & 0.40 \le x \le 0.60, \\ 4-5x, & 0.60 \le x \le 0.80, \end{cases} \\
w_{-1}(x) &= \begin{cases} 10x-3, & 0.30 \le x \le 0.40, \\ 5-10x, & 0.40 \le x \le 0.50, \end{cases} \\
w_{0}(x) &= \begin{cases} 4x-1, & 0.25 \le x \le 0.50, \\ 3-4x, & 0.50 \le x \le 0.75. \end{cases}
\end{aligned} \tag{32}
$$

From (32), we get

$$
\begin{aligned}
[w_{-3}]^{\alpha} &= \left[\frac{\alpha+4}{10}, \frac{6-\alpha}{10}\right], \\
[w_{-2}]^{\alpha} &= \left[\frac{\alpha+2}{5}, \frac{4-\alpha}{5}\right], \\
[w_{-1}]^{\alpha} &= \left[\frac{\alpha+3}{10}, \frac{5-\alpha}{10}\right], \\
[w_{0}]^{\alpha} &= \left[\frac{\alpha+1}{4}, \frac{3-\alpha}{4}\right],
\end{aligned}
$$

for all $\alpha \in (0,1]$. Also, A, B, C are

$$
\begin{aligned}
A &= \begin{cases} 10x-3, & 0.30 \le x \le 0.40, \\ 5-10x, & 0.40 \le x \le 0.50, \end{cases} \\
B &= \begin{cases} x-3, & 3 \le x \le 4, \\ 5-x, & 4 \le x \le 5, \end{cases} \\
C &= \begin{cases} x-1, & 2 \le x \le 3, \\ 4-x, & 3 \le x \le 4. \end{cases}
\end{aligned} \tag{33}
$$

From (33), we get

$$
[A]^{\alpha} = \left[\frac{\alpha+3}{10}, \frac{5-\alpha}{10}\right], \ [B]^{\alpha} = \left[\alpha+3, 5-\alpha\right], \ [C]^{\alpha} = \left[\alpha+1, 4-\alpha\right],
$$

for all $\alpha \in (0,1]$. In this case, Equation (13) possesses a unique solution by Theorem 4. Since $A_r^{\alpha} < B_l^{\alpha}$ for all $\alpha \in [0,1]$, then by Theorem 7, the positive solution (w_n) of Equation (13) converges to 0 as $n \to \infty$, see Figures 5-10.

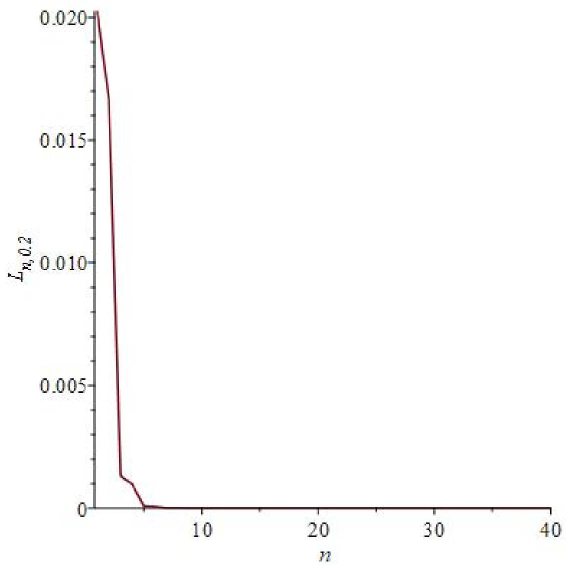

Figure 5. *Plot of* L_n *for* $\alpha = 0.2$

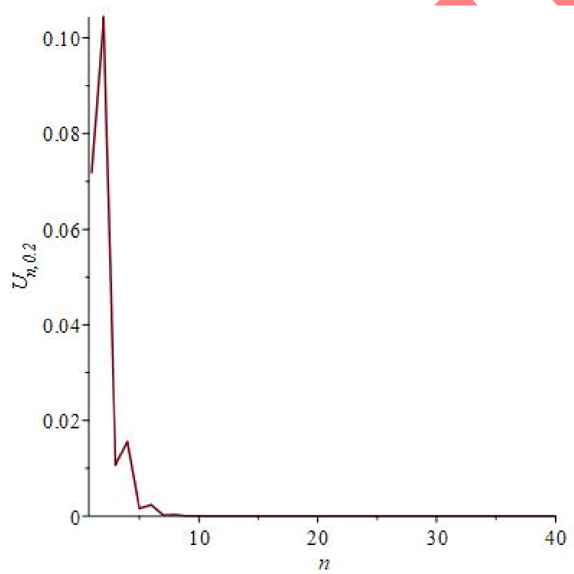

Figure 6. *Plot of* U_n *for* $\alpha = 0.2$

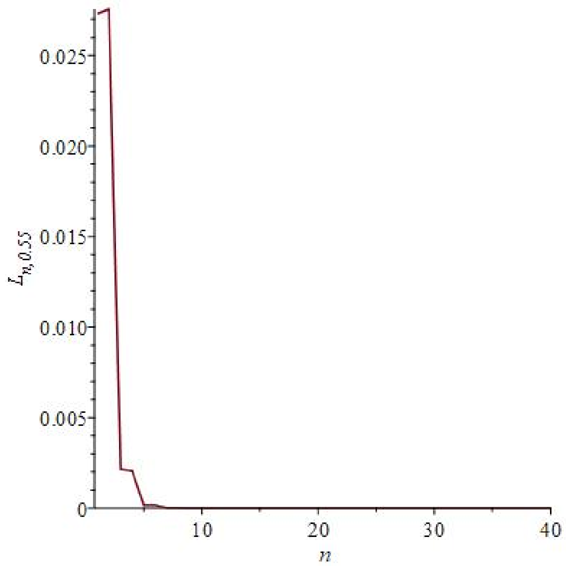

Figure 7. *Plot of* L_n *for* $\alpha = 0.55$

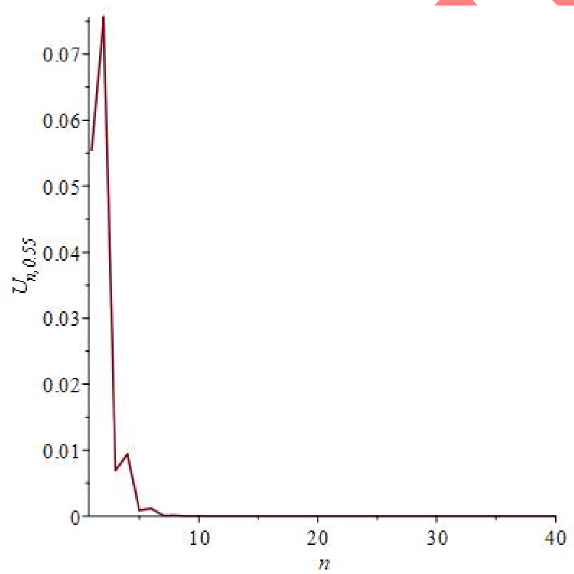

Figure 8. *Plot of* U_n *for* $\alpha = 0.55$

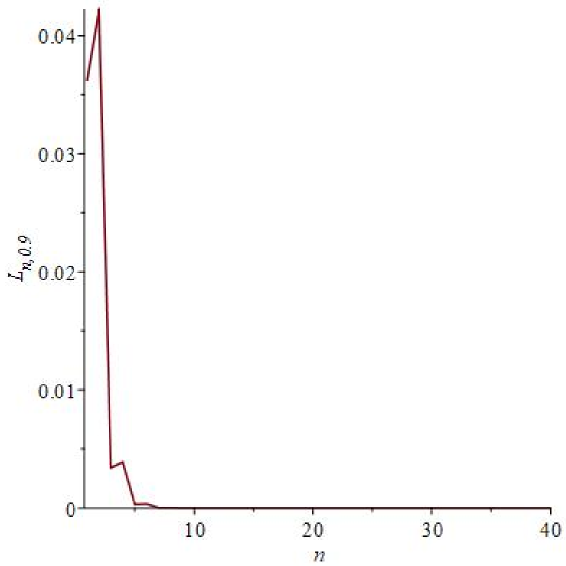

Figure 9. *Plot of* L_n *for* $\alpha = 0.9$

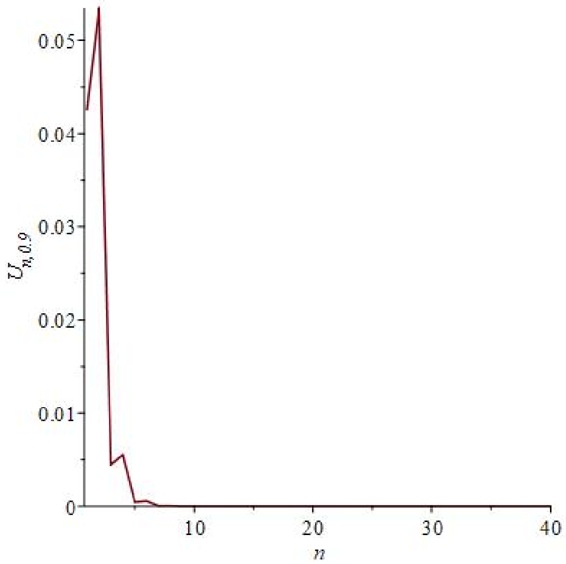

Figure 10. *Plot of* U_n *for* $\alpha = 0.9$

3. CONCLUSION

In this study, the equation

$$z_{n+1} = \frac{Aw_{n-1}}{B + Cw_{n-3}^{p}}, \; n \in \mathbb{N}_0,$$

where $A, B, C, w_{-3}, w_{-2}, w_{-1}, w_0 \in \mathbb{R}_F^+$ and $p \in \mathbb{Z}^+$, which is a fuzzy difference equation, was handled. Here, the equation is discussed in a fuzzy environment and the qualitative behavior of positive solutions, which is one of the main problems of difference equations, is tried to be determined. First, an existence and uniqueness theorem is proved. It is shown that the equation is a fuzzy difference equation. Its positive solutions tend to zero under specific conditions. Additionally, it is established that the equation admits unbounded solutions. In summary, the fuzzy equation exhibits dynamics comparable to its ordinary counterpart.

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

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