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# On Hyperbolic Padovan and Hyperbolic Pell-Padovan Sequences

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***Abstract***

In this article, we extend Padovan and Pell-Padovan numbers to Hyperbolic Padovan and Hyperbolic Pell-Padovan numbers, respectively. Moreover, we obtain Binet-like formulas, generating functions and some identities related to Hyperbolic Padovan and Hyperbolic Pell-Padovan numbers.

**Keywords**: Padovan numbers, Pell-Padovan numbers, Hyperbolic numbers, Hyperbolic Padovan numbers, Hyperbolic Pell-Padovan numbers..

## 1. *Introduction*

Hyperbolic numbers have applications in different areas of mathematics and theoretical physics. In particular, they are related to Lorentz-Minkowski (Space-time) geometry in the plane as well as complex numbers (Catoni 2008). The work on the function theory for hyperbolic numbers can be found in (Aydın 2019, Barreira 2016, Berzsenyi 1977, Deveci 2020, Güncan 2012, Horadam 1963, Khadjiev 2016, Motter 2016, Taş 2021, Taşçı 2018). The set of hyperbolic numbers H can be described in the form

$$\mathbb{H} = \{z = x + hy \mid h \notin \mathbb{R}, h^2 = 1, x, y \in \mathbb{R}\}$$

Addition, subtraction and multiplication of two hyperbolic numbers $z_1$ and $z_2$ are defined by

$$z_1 \pm z_2 = (x_1 + hy_1) \pm (x_2 + hy_2) = (x_1 \pm x_2) + h(y_1 \pm y_2)$$

$$z_1 \times z_2 = (x_1 + hy_1) \times (x_2 + hy_2) = (x_1x_2) + (y_1y_2) + h(x_1y_2 + y_1x_2)$$

On the other hand, the division of two hyperbolic numbers is given by

$$\frac{z_1}{z_2} = \frac{x_1 + hy_1}{x_2 + hy_2}$$

$$\frac{(x_1 + hy_1)(x_2 - hy_2)}{(x_2 + hy_2)(x_2 - hy_2)} = \frac{x_1x_2 + y_1y_2}{x_2^2 - y_2^2} + h\frac{(x_1y_2 + y_1x_2)}{x_2^2 - y_2^2}$$

If $x_2^2 - y_2^2 \neq 0$, then the division $\frac{z_1}{z_2}$ is possible. The hyperbolic conjugation of $z = x + hy$ is defined by $\bar{z} = x - hy$.

## 2. *Materials and Methods*

Padovan sequence is named after Richard Padovan (Voet 2012) and ( Çağman 2021a, Çağman 2021b, Deveci 2015, Deveci 2018, Shannon 2006, Taş 2014) studied Padovan sequence and Pell- Padovan sequence.

The Padovan sequence is the sequence of integers $P_n$ defined by the initial values $P_0 = P_1 = P_2 = 1$ and the recurrence relation

$$P_n = P_{n-2} + P_{n-3}$$

for all $n \geq 3$. The first few values of $P_n$ are $1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37$.

Pell-Padovan sequence is defined by the initial values $R_0 = R_1 = R_2 = 1$ and the recurrence relation

$$R_n = 2R_{n-2} + R_{n-3}$$

for all $n \geq 3$. The first few values of $R_n$ are $1, 1, 1, 3, 3, 3, 7, 9, 17, 25, 43, 67, 111, 177, 289$.

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## 3. *Results*

Firstly, we give the definition of Hyperbolic Padovan sequence.

**Definition 3.1.** The Hyperbolic Padovan sequence is the sequence of hyperbolic numbers $HP_n$ defined by the initial values $HP_0 = 1+h, HP_1 = 1+h, HP_2 = 1+2h$ and the recurrence relation

$$HP_n = P_n + hP_{n+1}$$
$$HP_n = HP_{n-2} + HP_{n-3}$$

for all $n \geq 3$.

The first few values of $HP_n$ are $1+h, 1+h, 1+2h, 2+2h, 2+3h, 3+4h, 4+5h, 5+7h, 7+9h, 9+12h, 12+16h, 16+21h, 21+28h, 28+37h$.

**Theorem 3.1.** The generating function of the Hyperbolic Padovan sequence is

$$g(x) = \frac{1+h+(1+h)x+hx^2}{1-x^2-x^3}$$

**Proof.** Let

$$g(x) = \sum_{n=0}^{\infty} HP_n x^n = HP_0 + HP_1 x + HP_2 x^2 + HP_3 x^3 + \cdots + HP_n x^n + \cdots$$

be generating function of the Hyperbolic Padovan sequence. On the other hand, since

$$x^2 g(x) = HP_0 x^2 + HP_1 x^3 + HP_2 x^4 + HP_3 x^5 + \cdots + HP_{n-2} x^n + \cdots$$

and

$$x^3 g(x) = HP_0 x^3 + HP_1 x^4 + HP_2 x^5 + HP_3 x^6 + \cdots + HP_{n-3} x^n + \cdots$$

we write

$$(1-x^2-x^3)g(x) = HP_0 + HP_1 x + (HP_2 - HP_0)x^2 + (HP_3 - HP_1 - HP_0)x^3 + \cdots + (HP_n - HP_{n-2} - HP_{n-3})x^n + \cdots$$

Now consider $HP_0 = 1+h, HP_1 = 1+h, HP_2 = 1+2h$ and $HP_n = HP_{n-2} + HP_{n-3}$. Thus, we obtain

$$(1-x^2-x^3)g(x) = HP_0 + HP_1 x + (HP_2 - HP_0)x^2$$

$$(1-x^2-x^3)g(x) = 1+h+(1+h)x+hx^2$$

or

$$g(x) = \frac{1+h+(1+h)x+hx^2}{1-x^2-x^3}$$

Hence the proof is completed.

Now we give Binet-like formula for the Hyperbolic Padovan sequence.

**Theorem 3.2.** Binet-like formula for the Hyperbolic Padovan sequence is

$$HP_n = \left(a + h\frac{a}{r_1}\right) r_1^n + \left(b + h\frac{b}{r_2}\right) r_2^n + \left(c + h\frac{c}{r_3}\right) r_3^n$$

where

$$a = \frac{(r_2-1)(r_3-1)}{(r_1-r_2)(r_1-r_3)}, b = \frac{(r_1-1)(r_3-1)}{(r_2-r_1)(r_2-r_3)}, c = \frac{(r_1-1)(r_2-1)}{(r_1-r_3)(r_2-r_3)}$$

and $r_1, r_2, r_3$ are the roots of the equation $x^3 - x^2 - 1 = 0$.

**Proof**. It is easily seen that

$$HP_n = P_n + hP_{n+1}$$

On the other hand, we know that the Binet-like formula for the Padovan sequence is

$$P_n = \frac{(r_2-1)(r_3-1)}{(r_1-r_2)(r_1-r_3)} r_1^n + \frac{(r_1-1)(r_3-1)}{(r_2-r_1)(r_2-r_3)} r_2^n + \frac{(r_1-1)(r_2-1)}{(r_1-r_3)(r_2-r_3)} r_3^n.$$

**Theorem 3.3.**

$$\sum_{k=0}^{n} HP_k = HP_n + HP_{n+1} + HP_{n+2} - (2+3h).$$

**Proof.** By the definition of Hyperbolic Padovan sequence recurrence relation

$$HP_n = HP_{n-2} + HP_{n-3}$$

and

$$HP_0 = HP_2 - HP_{-1}$$
$$HP_1 = HP_3 - HP_0$$
$$HP_2 = HP_4 + HP_1$$
$$\vdots$$
$$HP_{n-2} = HP_n - HP_{n-3}$$
$$HP_{n-1} = HP_{n+1} - HP_{n-2}$$
$$HP_n = HP_{n+2} - HP_{n-1}$$

Then we have

$$\sum_{k=0}^{n} HP_k = HP_n + HP_{n+1} + HP_{n+2} - HP_{-1} - HP_0 - HP_1.$$

Now considering $HP_{-1} = h, HP_0 = 1 + h, HP_1 = 1 + h$ we write

$$\sum_{k=0}^{n} HP_k = HP_n + HP_{n+1} + HP_{n+2} - (2 + 3h)$$

and hence the proof is completed.

Now we investigate the new property of Hyperbolic Padovan numbers in relation to the Padovan matrix formula. We consider the following matrices:

$$Q_3 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, K_3 = \begin{bmatrix} 1+2h & 1+h & 1+h \\ 1+h & 1+h & h \\ 1+h & h & 1 \end{bmatrix}$$

and

$$M_3^n = \begin{bmatrix} HP_{n+2} & HP_{n+1} & HP_n \\ HP_{n+1} & HP_n & HP_{n-1} \\ HP_n & HP_{n-1} & HP_{n-2} \end{bmatrix}$$

**Theorem 3.4.** For all $n \in \mathbb{Z}^+$ we have

$$Q_3^n K_3 = M_3^n.$$

**Proof.** The proof is easily seen that using the induction on $n$.

As well-known Pell-Padovan sequence is defined by the recurrence relation

$$R_n = 2R_{n-2} + R_{n-3}$$

and the initial values $R_0 = R_1 = R_2 = 1$.

Now we define Hyperbolic Pell-Padovan sequence.

**Definition 3.2.** The Hyperbolic Pell-Padovan sequence is defined by the recurrence relation

$$HR_n = R_n + hR_{n-1}$$

$$HR_n = 2HR_{n-2} + HR_{n-3}$$

and the initial values $HR_0 = 1 - h, HR_1 = 1 + h, HR_2 = 1 + h$.

The first few values of $HR_n$ are $1 - h, 1 + h, 1 + h, 3 + h, 3 + 3h, 7 + 3h, 9 + 7h$.

**Theorem 3.5.** Be the generating function of Hyperbolic Pell-Padovan sequence is

$$g(x) = \frac{1 - h + (1 + h)x + (-1 + 3h)x^2}{1 - 2x^2 - x^3}$$

**Proof.** Let

$$g(x) = \sum_{n=0}^{\infty} HR_n x^n = Hr_0 + HR_1 x + HR_2 x^2 + HR_3 x^3 + \cdots + HR_n x^n + \cdots$$

be the generating function of the Hyperbolic Pell-Padovan sequence. On the other hand, since

$$2x^2 g(x) = 2HR_0 x^2 + 2HR_1 x^3 + 2HR_2 x^4 + 2HR_3 x^5 + \cdots + 2HR_{n-2} x^n + \cdots$$

and

$$x^3 g(x) = HR_0 x^3 + HR_1 x^4 + HR_2 x^5 + HR_3 x^6 + \cdots + HR_{n-3} x^n + \cdots$$

we write

$$(1 - 2x^2 - x^3) g(x) = HR_0 + HR_1 x + (HR_2 - 2HP_0)x^2 + (HR_3 - 2HR_1 - HR_0)x^3 + \cdots + (HR_n - 2HR_{n-2} - HR_{n-3})x^n + \cdots$$

Now consider $HR_0 = 1 - h, HR_1 = 1 + h, HR_2 = 1 + h$ and $HR_n = 2HR_{n-2} + HR_{n-3}$. Thus, we obtain

$$(1 - 2x^2 - x^3) g(x) = HR_0 + HR_1 x + (HR_2 - 2HR_0)x^2$$

$$(1 - 2x^2 - x^3) g(x) = 1 - h + (1 + h)x + (-1 + 3h)x^2$$

or

$$g(x) = \frac{1 - h + (1 + h)x + (-1 + 3h)x^2}{1 - 2x^2 - x^3}$$

Hence the proof is completed.

**Theorem 3.6.** The Binet-like formula of Hyperbolic Pell-Padovan sequence is

$$HR_n = \frac{2(\alpha + h)}{\alpha - \beta}\left(1 - \frac{1}{\alpha}\right)\alpha^n - \frac{2(\beta + h)}{\alpha - \beta}\left(1 - \frac{1}{\beta}\right)\beta^n + (1 - h)\gamma^n$$

where

$$\alpha = \frac{1 + \sqrt{5}}{2}, \beta = \frac{1 - \sqrt{5}}{2}, \gamma = -1$$

are roots of the equation $x^3 - 2x - 1 = 0$.

**Proof.** The Binet-like formula of Pell-Padovan sequence is given

$$R_n = 2\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta} - 2\frac{\alpha^n-\beta^n}{\alpha-\beta} + \gamma^{n+1}.$$

Now consider

$$HR_n = R_n + hR_{n-1}$$

so the proof is easily.

**Theorem 3.7.**

$$\sum_{k=0}^{n} HR_k = \frac{1}{2}[(-1-3h) - HR_{n+1} + HR_{n+2} + HR_{n+3}].$$

**Proof.** We know that

$$\sum_{k=0}^{n} R_k = \frac{1}{2}[-1 - R_{n+1} + R_{n+2} + R_{n+3}]$$

and

$$\sum_{k=0}^{n} R_{k-1} = \frac{1}{2}[-3 - 2R_n - R_{n+1} + R_{n+2} + R_{n+3}].$$

Since

$$HR_n = R_n + hR_{n-1}$$

we have

$$\sum_{k=0}^{n} HR_k = \sum_{k=0}^{n} R_k + h\sum_{k=0}^{n} R_{k-1}.$$

So the theorem is proved.

**Theorem 3.8.**

$$\sum_{k=1}^{n} HR_{2k} = R_{2n+1} + hR_{2n} - (n+1) + h(n-1).$$

**Proof.** If we consider the following equalities, then the proof is seen

$$\sum_{k=1}^{n} R_{2k} = R_{2n+1} - (n+1)$$

and

$$\sum_{k=1}^{n} R_{2k-1} = R_{2n} + (n-1)\,.$$

Since

$$HR_n = R_n + hR_{n-1}$$

we have

$$\sum_{k=1}^{n} HR_{2k} = \sum_{k=1}^{n} R_{2k} + h\sum_{k=1}^{n} R_{2k-1}.$$

So the theorem is proved.

Now we investigate the new property of Hyperbolic Pell-Padovan numbers in relation to the Pell-Padovan matrix formula. We consider the following matrices:

$$Q_3 = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$K_3 = \begin{bmatrix} 1+h & 1+h & 1-h \\ 1+h & 1-h & -1+3h \\ 1-h & -1+3h & 3-5h \end{bmatrix}$$

and

$$M_3^n = \begin{bmatrix} HR_{n+2} & HR_{n+1} & HR_n \\ HR_{n+1} & HR_n & HR_{n-1} \\ HR_n & HR_{n-1} & HR_{n-2} \end{bmatrix}$$

**Theorem 3.9.** For all $n \in \mathbb{Z}^+$ we have

$$Q_3^n K_3 = M_3^n.$$

**Proof.** The proof can be obtained easily by induction on $n$.

**Theorem 3.10.** If

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

then we have

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}^n \begin{bmatrix} 1-h \\ 1+h \\ 1+h \end{bmatrix} = \begin{bmatrix} HR_n \\ HR_{n+1} \\ HR_{n+2} \end{bmatrix}.$$

**Proof.** The proof can be seen by mathematical induction on $n$ .

## 4. *Discussion*

We defined Hyperbolic Padovan and Hyperbolic Pell-Padovan numbers and we obtain Binet-like formulas, generating functions and some identities related to Hyperbolic Padovan and Hyperbolic Pell-Padovan numbers.

## *References*

ATANASSOV, K., DIMITROV, D., SHANNON, A., (2009). A remark on ψ-function and pell-padovan's sequence. *Notes on Number Theory and Discrete Mathematics, 15(2), 1-44.*

AYDIN, F.T., (2019). Hyperbolic Fibonacci sequence, *Universal Journal of Mathematics and Applications, 2(2), 59-64.*

BARREİRA, L., POPESCU, L.H., VALLS, C., (2016). Hyperbolic sequences of linear operators and evolution maps, *Milan Journal of Mathematics*, 84(2), 203-216.

BERZSENYİ, G., (1977). Gaussian fibonacci numbers.

CATONİ, F., BOCCALETTİ, D., CANNATA, R., CATONİ, V., NİCHELATTİ, E., ZAMPETTİ, P., (2008). The mathematics of Minkowski space-time: with an introduction to commutative hypercomplex numbers, *Springer Science & Business Media*,

ÇAĞMAN, A., (2021). Repdigits as Product of Fibonacci and Pell numbers, *Turkish Journal of Science , 6 (1) , 31-35* .

ÇAĞMAN, A., (2021). An approach to Pillai's problem with the Pell sequence and the powers of 3, *Miskolc Mathematical Notes 22 (2), 599-610*.

DEVECİ, Ö., KARADUMAN, E., (2015). The pell sequences in finite groups, *Util. Math, 96, 263-276*.

DEVECİ, Ö., SAHANNON, A.G., (2018). The quaternion-pell sequence, *Communications in Algebra, 46(12), 5403-5409*.

DEVECİ, Ö., SAHANNON, A.G., (2020). The complex-type k-fibonacci sequences and their applications, *Communications in Algebra, 46(12), pages 1-16*.

GARGOUBİ, H., KOSSENTİNİ, S., (2016). f-algebra structure on hyperbolic numbers. *Advances in Applied Clifford Algebras, 26(4), 1211-1233*.

GÜNCAN, A., ERBİL, Y., (2012). The q-fibonacci hyperbolic functions, *In AIP Conference Proceedings, American Institute of Physics, volume 1479, pages 946-949*.

HORADAM, A.F., (1963). Complex fibonacci numbers and fibonacci quaternions, *The American Mathematical Monthly, 70(3), 289-291*.

KHADJİEV, D., GÖKSAL, Y. (2016). Applications of hyperbolic numbers to the invariant theory in two-dimensional pseudo-euclidean space. *Advances in Applied Clifford Algebras, 26(2), 645-668.*.

MOTTER, A.E., ROSA, M.A.F. (2016). Hyperbolic calculus. *Advances in Applied Clifford Algebras, 8(1), 109-128*.

SHANNON, A.G., ANDERSON, P.G., HORADAM, A.F. (2006). Properties of cordonnier, perrin and van der laan numbers, *International Journal of Mathematical Education in Science and Technology, 37(7), 825-831*.

TAŞ, S. (2021). The Hyperpolic Quadrapell Sequences. *Eastern Anatolian Journal of Science, Volume VII, Issue I, 25-29*.

TAŞ, S., DEVECİ, Ö., KARADUMAN, E. (2014). The fibonacci-padovan sequences in fnite groups. *Maejo International Journal of Science And Technology, 2014, 8(3), 279-287*.

TAŞÇI, D. (2018). Gaussian padovan and gaussian pell-padovan sequences, *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics*, 67(2), 82-88.

VOET, C. (2012). The poetics of order: Dom hans van der laan's architectonic space", *Architectural Research Quarterly*, 16(2), 137.