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New Generalized Hypergeometric Functions

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Abstract — The classical Gauss hypergeometric function ${}_2F_1(\alpha,\beta,\gamma;z)$ and the Kumar confluent hypergeometric function ${}_1F_1(\alpha,\beta;z)$ are defined using a classical Pochammer symbol $(\alpha)_n$ and a factorial function. This research paper will present a two-parameter Pochhammer symbol $(\lambda,\mu)_n$ and discuss some of its properties such as recursive formulae and integral representation. In addition, the generalized Gauss and Kumar confluent hypergeometric functions are defined using the two-parameter Pochhammer symbol and a two-parameter factorial function $(m,\ j)!$ and some of the properties of the new generalized hypergeometric functions were also discussed.

Subject Classification (2020): 33B99, 33C05, 33C15, 33C20.

1. Introduction

The pochhammer symbol is named after the German mathematician Leo Pochhammer, defined as ashifted (rising) factorial [2] and given by

$$(\lambda)_n = \begin{cases} \lambda(\lambda+1)(\lambda+2)(\lambda+3)\cdots(\lambda+(n-1)), & n \in \square \\ 1, & n = 0 \end{cases} \tag{1.1}$$

Rafael and Pariguan in [7] presented the definition of the pochhammer m -symbol as

$$(\lambda)_{n,\mu} = \lambda(\lambda+\mu)(\lambda+2\mu)(\lambda+3\mu)\cdots(\lambda+(n-1)\mu), \tag{1.2}$$

$$(\lambda \in \square,\ \mu \in \square \text{ and } n \in \square)$$

and introduced the m -analogue of the gamma function.

Remark 1.1. When $\mu = 1,$ (the classical Pochhammer symbol).

Srivastava in [11] generalized the Pochhammer symbol using the extended gamma function in [14] as

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$$(\lambda,\mu)_n=\begin{cases}\dfrac{\Gamma_\mu(\lambda+n)}{\Gamma(\lambda)}, & (\Re(\mu)>0;\lambda,n\in\square)\\ (\lambda)_n, & (\mu=0;\lambda,n\in\square)\end{cases} \tag{1.3}$$

A new generalization of the Pochhammer symbol in [11] was proposed by Sahin [14] as

$$(\lambda;r,s;\rho,\eta)_n=\begin{cases}\dfrac{\Gamma_{r,s}^{(\rho,\eta)}(\lambda+n)}{\Gamma(\lambda)}, & (\mathrm{Re}(r)>0,\mathrm{Re}(s)>0,\mathrm{Re}(\rho)>0,\mathrm{Re}(\eta)>0),\\ (\lambda)_n, & (r=1,s=0,\rho=1,\eta=0),\end{cases} \tag{1.4}$$

Where $\Gamma_{r,s}^{(\rho,\eta)}$ is the generalized extended gamma function.

Sahai [18] generalized the Pochhammer symbol using an extended gamma function in [20] as

$$(\lambda;r,\rho,\eta)_n=\frac{\Gamma_r^{(\rho,\eta)}(\lambda+n)}{\Gamma(\lambda)}, \qquad (\mathrm{Re}(\rho)>0,\mathrm{Re}(\eta)>0,\mathrm{Re}(r)>0;\lambda,n\in\square \tag{1.5}$$

Srivastava [16] introduced a generalized Pochhammer symbol from an extended gamma function in [14] as

$$(\lambda;\rho,\{k_v\}_{v\in\square_0})_n=\frac{\Gamma_\rho^{\left(\{k_v\}_{v\in\square_0}\right)}(\lambda+n)}{\Gamma_\rho^{\left(\{k_v\}_{v\in\square_0}\right)}(\lambda)}, \qquad (\lambda,n\in\square). \tag{1.6}$$

Another generalized Pochhammer symbol was defined by Safdar [21] using an extended gamma function in [22] as

$$(\lambda;\rho,\mu)_n=\begin{cases}\dfrac{\Gamma_\rho(\lambda+n;\mu)}{\Gamma(\lambda)}, & (\Re(\rho)>0,\Re(\mu)>0;\lambda,n\in\square),\\ (\lambda;\rho)_n, & (\mu=1;\lambda,n\in\square\setminus\{0\}),\end{cases} \tag{1.7}$$

A factorial function denoted by $(!)$ is given by

$$(\lambda)!=\begin{cases}\lambda(\lambda-1)(\lambda-2)(\lambda-3)\cdots 3\cdot 2\cdot 1, & \lambda\in\square\\ 1, & \lambda=0\end{cases} \tag{1.8}$$

In 2014, Mubeen and Rehman [3] generalized the classical factorial function called (λ,μ)-factorials as

$$(\lambda,\mu)!=\lambda\mu(\lambda\mu-\mu)(\lambda\mu-2\mu)(\lambda\mu-3\mu)\cdots 3\mu\cdot 2\mu\cdot\mu, \qquad \lambda\in\square,\mu>0 \tag{1.9}$$

On simplifying the right-hand side of (1.8), we have

$$(\lambda,\mu)!=\mu^n\lambda!=\mu^n\Gamma(\lambda+1) \tag{1.10}$$

(1.10) is the relationship between the generalized factorial function $(\lambda,\mu)!$ and the gamma function.

Remark 1.2. When $\mu=1$, $(\lambda,\mu)!=(\lambda,1)!=(\lambda)!$ (the classical factorial function).

Other related literatures can be obtained [23, 24, 25 & 26].

Motivated by (1.9), the second part the paper will propose a new generalized Pochhammer symbol $(\lambda,\mu)_n$ and give some of its properties.

2. New Generalised Pochhammer Symbol

The $(\lambda,\mu)_n$ Pochhammer symbol is defined as

$$(\lambda,\mu)_n = \begin{cases} \lambda\mu(\lambda\mu+\mu)(\lambda\mu+2\mu)(\lambda\mu+3\mu)\cdots(\lambda\mu+(n-1)\mu), & \lambda,\mu\in\square,\quad n\in\square \\ 1, & n=0 \end{cases} \tag{2.1}$$

(2.1) can be simplified as

$$(\lambda,\mu)_n = \lambda\mu^n(\lambda+1)(\lambda+2)(\lambda+3)\cdots(\lambda+n-1) = \mu^n(\lambda)_n \tag{2.2}$$

Remark 2.1. When $\mu=1$ in (2.2), $(\lambda,\mu)_n=(\lambda,1)_n=(\lambda)_n$ (i.e. the classical Pochhammer symbol)

Theorem 2.1. The following formulas holds

$$(\lambda\mu,\mu)_n = \mu^n(\lambda\mu)_n \tag{2.3}$$

$$(\lambda+p,\mu)_n = \mu^n(\lambda+p)_n, \qquad p\in\square,\ n\in\square \tag{2.4}$$

$$((\lambda+q)\mu,\mu)_n = \mu^n((\lambda+q)\mu)_n, \qquad q\in\square,\ n\in\square \tag{2.5}$$

From (2.2),

$$(\lambda,\mu)_n = \mu^n\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \tag{2.6}$$

(2.6) is the relationship between the two parameters Pochhammer symbol and the classical gamma function.

Proof To prove equations (2.3), (2.4) and (2.5), put $\lambda=\lambda\mu,\ \lambda=\lambda+p$ and $\lambda=(\lambda+q)\mu$ in (2.2) respectively.

The proof of (2.6) follows from the fact that $(\lambda)_n=\dfrac{\Gamma(\lambda+n)}{\Gamma(\lambda)}$.

In the third part of the paper, we are going to state some recurrence formulae.

3. Main Results

Theorem 3.1. The following formula holds true

$$(\lambda,\mu)_{n+1} = (\lambda+n)(\lambda,\mu)_n \tag{3.1}$$

Proof

$$(\lambda,\mu)_{n+1} = \mu^n\lambda(\lambda+1)(\lambda+2)(\lambda+3)\cdots(\lambda+n-1)(\lambda+n)$$

Using (2.2) in the above equation, we obtained the desired result.

Theorem 3.2. The following formula holds true

$$(\lambda+1,\mu+1)_n - (\lambda,\mu)_n = (\lambda,\mu)_n\left[\frac{\lambda+n}{\lambda}\frac{(\mu+1)^n}{\mu^n}-1\right] \tag{3.2}$$

Proof

$$(\lambda+1,\mu+1)_n = (\mu+1)^n(\lambda+1)(\lambda+2)(\lambda+3)\cdots(\lambda+n-2)(\lambda+n)$$

Multiplying both sides of the equation by $\lambda\mu^n$ and dividing through by $(\mu+1)^n$, we get the required result.

Theorem 3.2.

$$(\lambda,\mu)_n = (\lambda+n-1)(\lambda,\mu)_{n-1} \tag{3.3}$$

Proof

$$(\lambda,\mu)_n = \mu^n \lambda(\lambda+1)(\lambda+2)(\lambda+3)\cdots(\lambda+n-2)(\lambda+n-1)$$

On using (2.2), we obtained the desired result.

Theorem 3.3.

$$(\lambda,\mu)_{m+n} = (\lambda,\mu)_m(\lambda+m,\mu+m)_n \tag{3.4}$$

Proof

$$\begin{aligned}(\lambda,\mu)_{m+n} &= \mu^{m+n}\lambda(\lambda+1)(\lambda+2)(\lambda+3)\cdots(\lambda+m+n-1)\\ &= \mu^{m+n}\lambda(\lambda+1)(\lambda+2)(\lambda+3)\cdots(\lambda+m-1)(\lambda+m)(\lambda+m+1)(\lambda+m+2)\cdots(\lambda+m+n-1)\\ &= (\lambda,\mu)_m(\lambda+m,\mu+m)_n.\end{aligned}$$

Theorem 3.4.

$$(\lambda-1,\mu-1)_{n+1} = \mu(\lambda+n-1)(\lambda-1,\mu-1)_n \tag{3.5}$$

Proof

$$\begin{aligned}(\lambda-1,\mu-1)_{n+1} &= (\mu-1)^{n+1}(\lambda-1)\lambda(\lambda+1)(\lambda+2)(\lambda+3)\cdots(\lambda+n-2)(\lambda+n-1)\\ &= \mu(\lambda+n-1)(\lambda-1,\mu-1)_n.\end{aligned}$$

Theorem 3.5.

$$(\lambda+1,\mu+1)_n = (\lambda+n)(\lambda+1,\mu+1)_{n-1} \tag{3.6}$$

Proof

$$(\lambda+1,\mu+1)_{n-1} = (\mu+1)^{n-1}(\lambda+1)(\lambda+2)(\lambda+3)\cdots(\lambda+n-1) \tag{3.7}$$

Multiplying both sides of (3.7) by $(\mu+1)(\lambda+n)$, we get the desired result.

Theorem 3.6.

$$(\lambda,\mu)_n = \frac{\lambda\mu^n}{(\lambda+n)(\mu+1)^n}(\lambda+1,\mu+1)_n \tag{3.8}$$

Proof

$$(\lambda+1,\mu+1)_n = (\mu+1)^n(\lambda+1)(\lambda+2)(\lambda+3)\cdots(\lambda+n-1)(\lambda+n)$$

Multiplying through by $\lambda\mu^n$ and dividing the result by $(\lambda+n)$, we get (3.8).

Theorem 3.7.

$$(\lambda,\mu)_n = \frac{\mu^n}{(\mu-m)^n}\frac{(\lambda-m,\mu-m)_{n+m}}{(\lambda-m,\mu-m)_m} \tag{3.9}$$

Proof

$$\frac{(\lambda-m,\mu-m)_{n+m}}{(\lambda-m,\mu-m)_m}=\frac{(\mu-m)^{n+m}(\lambda-m)(\lambda-m+1)(\lambda-m+2)\cdots(\lambda+n-1)}{(\mu-m)^m(\lambda-m)(\lambda-m+1)(\lambda-m+2)\cdots(\lambda-1)}$$

$$=(\mu-m)^n\lambda(\lambda+1)(\lambda+2)\cdots(\lambda+n-1)$$

Multiplying both sides by μ^n , we've

$$\frac{\mu^n(\lambda-m,\mu-m)_{n+m}}{(\lambda-m,\mu-m)_m}=(\lambda,\mu)_n(\mu-m)^n$$

on dividing both sides by $(\mu-m)^n$, we get the desired result.

Theorem 3.8.

$$(\lambda,\mu)_n=\frac{\mu^n}{\mu^m(\mu+m)^{n-m}}(\lambda,\mu)_m(\lambda+m,\mu+m)_{n-m} \tag{3.10}$$

Proof

$$(\lambda,\mu)_m(\lambda+m,\mu+m)_{n-m}=\mu^m(\mu+m)^{n-m}\lambda(\lambda+1)(\lambda+2)\cdots(\lambda+m-1)(\lambda+m)(\lambda+m+1)(\lambda+m+2)\cdots(\lambda+n-1)$$

Multiplying both sides by μ^n , we've

$$\mu^n(\lambda,\mu)_m(\lambda+m,\mu+m)_{n-m}=\mu^m(\mu+m)^{n-m}\mu^n\lambda(\lambda+1)(\lambda+2)\cdots(\lambda+n-1)$$

$$=\mu^m(\mu+m)^{n-m}(\lambda,\mu)_n$$

Dividing through by $\mu^m(\mu+m)^{n-m}$, we get the desired result.

Theorem 3.9.

$$(\lambda,\mu)_n=\frac{\mu^n}{(\mu+m)^n}\frac{\Gamma(\lambda+m)\Gamma(\lambda+n)}{\Gamma(\lambda)\Gamma(\lambda+m+n)}(\lambda+m,\mu+m)_n \tag{3.11}$$

Proof Taking the right-hand side of (3.11) and using (2.6), we get

$$\frac{\mu^n}{(\mu+m)^n}\frac{\Gamma(\lambda+m)\Gamma(\lambda+n)}{\Gamma(\lambda)\Gamma(\lambda+m+n)}(\lambda+m,\mu+m)_n=(\lambda,\mu)_n\frac{1}{(\lambda+m,\mu+m)_n}(\lambda+m,\mu+m)_n$$

$$=(\lambda,\mu)_n$$

Theorem 3.10.

$$(\lambda,\mu)_n=\frac{\mu^n}{(\mu-m)^n}\frac{\Gamma(\lambda-m)\Gamma(\lambda+n)}{\Gamma(\lambda)\Gamma(\lambda-m+n)}(\lambda-m,\mu-m)_n \tag{3.12}$$

Proof Taking the right-hand side of (3.12) and using (2.6), we get

$$\frac{\mu^n}{(\mu-m)^n}\frac{\Gamma(\lambda-m)\Gamma(\lambda+n)}{\Gamma(\lambda)\Gamma(\lambda+m+n)}(\lambda-m,\mu-m)_n=(\lambda,\mu)_n\frac{1}{(\lambda-m,\mu-m)_n}(\lambda-m,\mu-m)_n$$

$$.=(\lambda,\mu)_n$$

Corrolary 3.1. The following integral representation holds

$$(\lambda,\mu)_n = \frac{\mu^n}{\Gamma(\lambda)}\int_0^{\infty} t^{\lambda+n-1}e^{-t}dt \tag{3.13}$$

The hypergeometric functions is defined using a factorial function and a Pochhammer symbol. In the fourth part of this paper, we will define new generalized hypergeometric functions using (1.10) and (2.1).

4. Generalized Hypergeometric Functions

The new generalized Gauss and confluent hypergeometric functions are given by

$${}_pF_q\left[\begin{matrix}(\alpha_1,\beta),\alpha_2,\ldots,\alpha_p \\ \delta_1,\delta_2,\ldots,\delta_q\end{matrix};z\right] = \sum_{m=0}^{\infty}\frac{(\alpha_1,\beta)_m(\alpha_2)_m\cdots(\alpha_p)_m}{(\delta_1)_m(\delta_2)_m\cdots(\delta_q)_m}\frac{z^m}{(m,j)!} \tag{4.1}$$

In particular,

$${}_1F_1\left[(\alpha_1,\beta),\delta;z\right] = \sum_{m=0}^{\infty}\frac{(\alpha,\beta)_m}{(\delta)_m}\frac{z^m}{(m,j)!}, \tag{4.2}$$

And

$${}_2F_1\left[(\alpha_1,\beta),\gamma,\delta;z\right] = \sum_{m=0}^{\infty}\frac{(\alpha,\beta)_m(\gamma)_m}{(\delta)_m}\frac{z^m}{(m,j)!} \tag{4.3}$$

Theorem 4.1. The following integral representation holds true

$${}_pF_q\left[\begin{matrix}(\alpha_1,\beta),\alpha_2,\ldots,\alpha_p \\ \delta_1,\delta_2,\ldots,\delta_q\end{matrix};z\right] = \frac{\mu^n}{\Gamma(\lambda)}\int_0^{\infty} t^{\lambda-1}e^{-t}\,{}_{p-1}F_q\left[\begin{matrix}\alpha_2,\alpha_3,\ldots,\alpha_p \\ \delta_1,\delta_2,\ldots,\delta_q\end{matrix};tz\right]dt, \tag{4.4}$$

Proof

$${}_pF_q\left[\begin{matrix}(\alpha_1,\beta),\alpha_2,\ldots,\alpha_p \\ \delta_1,\delta_2,\ldots,\delta_q\end{matrix};z\right] = \sum_{m=0}^{\infty}\frac{(\alpha_1,\beta)_m(\alpha_2)_m\cdots(\alpha_p)_m}{(\delta_1)_m(\delta_2)_m\cdots(\delta_q)_m}\frac{z^m}{(m,j)!}$$

$$= \sum_{m=0}^{\infty}\frac{\mu^n}{\Gamma(\lambda)}\int_0^{\infty} t^{\lambda-1}e^{-t}\frac{(\alpha_2)_m(\alpha_3)_m\cdots(\alpha_p)_m}{(\delta_1)_m(\delta_2)_m\cdots(\delta_q)_m}\frac{(tz)^m}{(m,j)!}dt$$

$$= \frac{\mu^n}{\Gamma(\lambda)}\int_0^{\infty} t^{\lambda-1}e^{-t}\sum_{m=0}^{\infty}\frac{(\alpha_2)_m(\alpha_3)_m\cdots(\alpha_p)_m}{(\delta_1)_m(\delta_2)_m\cdots(\delta_q)_m}\frac{(tz)^m}{(m,j)!}dt$$

$$= \frac{\mu^n}{\Gamma(\lambda)}\int_0^{\infty} t^{\lambda-1}e^{-t}\,{}_{p-1}F_q\left[\begin{matrix}\alpha_2,\alpha_3,\ldots,\alpha_p \\ \delta_1,\delta_2,\ldots,\delta_q\end{matrix};tz\right]dt.$$

Theorem 4.2. The beta-type integral representation holds true

$$ {}_pF_q\left[\begin{matrix}(\alpha_1,\beta),\alpha_2,...,\alpha_p\\ \delta_1,\delta_2,...,\delta_q\end{matrix};z\right]=\frac{1}{B(\alpha_p,\delta_q-\alpha_p)}\int_0^1 t^{\alpha_p-1}(1-t)^{\delta_q-\alpha_p-1}\,{}_{p-1}F_{q-1}\left[\begin{matrix}(\alpha_1,\beta),\alpha_2,...,\alpha_{p-1}\\ \delta_1,\delta_2,...,\delta_{q-1}\end{matrix};tz\right]dt, \tag{4.5}$$

$$(\mathrm{Re}(\alpha_p)>0;\mathrm{Re}(\delta_q)>0;\mathrm{Re}(\beta)\geq 0)$$

Proof

$$ {}_pF_q\left[\begin{matrix}(\alpha_1,\beta),\alpha_2,...,\alpha_p\\ \delta_1,\delta_2,...,\delta_q\end{matrix};z\right]=\sum_{m=0}^{\infty}\frac{(\alpha_1,\beta)_m(\alpha_2)_m\cdots(\alpha_p)_m}{(\delta_1)_m(\delta_2)_m\cdots(\delta_q)_m}\frac{z^m}{(m,j)!}$$

$$=\frac{1}{B(\alpha_p,\delta_q-\alpha_p)}\sum_{m=0}^{\infty}\frac{(\alpha_1,\beta)_m(\alpha_2)_m\cdots(\alpha_{p-1})_m}{(\delta_1)_m(\delta_2)_m\cdots(\delta_{q-1})_m}\int_0^1 t^{\alpha_p+m-1}(1-t)^{\delta_q-\alpha_p-1}\frac{z^m}{(m,j)!}dt$$

$$=\frac{1}{B(\alpha_p,\delta_q-\alpha_p)}\int_0^1 t^{\alpha_p-1}(1-t)^{\delta_q-\alpha_p-1}\sum_{m=0}^{\infty}\frac{(\alpha_1,\beta)_m(\alpha_2)_m\cdots(\alpha_{p-1})_m}{(\delta_1)_m(\delta_2)_m\cdots(\delta_{q-1})_m}\frac{(tz)^m}{(m,j)!}dt$$

$$=\frac{1}{B(\alpha_p,\delta_q-\alpha_p)}\int_0^1 t^{\alpha_p-1}(1-t)^{\delta_q-\alpha_p-1}\,{}_{p-1}F_{q-1}\left[\begin{matrix}(\alpha_1,\beta),\alpha_2,...,\alpha_{p-1}\\ \delta_1,\delta_2,...,\delta_{q-1}\end{matrix};tz\right]dt.$$

Corollary 4.1. The following integral representations hold true

$$ {}_1F_1\left[(\alpha,\beta),\delta;z\right]=\frac{\mu^m}{\Gamma(\lambda)}\int_0^\infty t^{\lambda-1}e^{-t}\,{}_0F_1\left[-,\delta;tz\right]dt, \tag{4.6}$$

$$ {}_2F_1\left[(\alpha,\beta),\delta;z\right]=\frac{\mu^m}{\Gamma(\lambda)}\int_0^\infty t^{\lambda-1}e^{-t}\,{}_1F_1\left[\alpha,\delta;tz\right]dt, \tag{4.7}$$

$$ {}_1F_1\left[(\alpha,\beta),\delta;z\right]=\frac{1}{B(\alpha,\delta-\alpha)}\int_0^1 t^{\alpha-1}(1-t)^{\delta-\alpha-1}\,{}_0F_0\left[-,-;tz\right]dt, \tag{4.8}$$

$$ {}_2F_1\left[(\alpha_1,\beta),\alpha_2,\delta;z\right]=\frac{1}{B(\alpha,\delta-\alpha)}\int_0^1 t^{\alpha-1}(1-t)^{\delta-\alpha-1}\,{}_1F_0\left[(\alpha_1,\beta),-;tz\right]dt. \tag{4.9}$$

Theorem 4.3. The following derivative holds

$$\frac{d}{dz}\,{}_pF_q\left[\begin{matrix}(\alpha_1,\beta),\alpha_2,...,\alpha_p\\ \delta_1,\delta_2,...,\delta_q\end{matrix};z\right]=\frac{1}{j}\frac{(\alpha_1,\beta)_m(\alpha_2)_m\cdots(\alpha_p)_m}{(\delta_1)_m(\delta_2)_m\cdots(\delta_q)_m}\,{}_pF_q\left[\begin{matrix}(\alpha_1+1,\beta+1),\alpha_2,...,\alpha_p\\ \delta_1,\delta_2,...,\delta_q\end{matrix};z\right] \tag{4.10}$$

Proof

$$\frac{d}{dz}\,{}_pF_q\left[\begin{matrix}(\alpha_1,\beta),\alpha_2,...,\alpha_p\\ \delta_1,\delta_2,...,\delta_q\end{matrix};z\right]=\sum_{m=1}^{\infty}\frac{(\alpha_1,\beta)_m(\alpha_2)_m\cdots(\alpha_p)_m}{(\delta_1)_m(\delta_2)_m\cdots(\delta_q)_m}\frac{z^{m-1}}{j^m\Gamma(m)}$$

As $m\to m+1$,

$$\frac{d}{dz}\,{}_pF_q\left[\begin{matrix}(\alpha_1,\beta),\alpha_2,...,\alpha_p\\ \delta_1,\delta_2,...,\delta_q\end{matrix};z\right]=\sum_{m=0}^{\infty}\frac{(\alpha_1,\beta)_{m+1}(\alpha_2)_{m+1}\cdots(\alpha_p)_{m+1}}{(\delta_1)_{m+1}(\delta_2)_{m+1}\cdots(\delta_q)_{m+1}}\frac{z^{m}}{j^{m+1}\Gamma(m+1)}$$

Using (1.6), yields

$$\frac{d}{dz}\,{}_pF_q\left[\begin{matrix}(\alpha_1,\beta),\alpha_2,...,\alpha_p\\ \delta_1,\delta_2,...,\delta_q\end{matrix};z\right]=\frac{1}{j}\frac{(\alpha_1,\beta)(\alpha_2)\cdots(\alpha_p)}{(\delta_1)(\delta_2)\cdots(\delta_q)}\sum_{m=0}^{\infty}\frac{(\alpha_1,\beta)_m(\alpha_2)_m\cdots(\alpha_p)_m}{(\delta_1)_m(\delta_2)_m\cdots(\delta_q)_m}\frac{z^{m}}{(m,j)!}$$

Applying (4.1), we obtain the required result.

Corollary 4.2.

$$\frac{d}{dz}\left\{{}_pF_q\left[(\alpha,\beta),\lambda,\delta;z\right]\right\}=\frac{1}{j}\frac{\lambda(\alpha,\beta)}{\delta}\,{}_2F_1\left[(\alpha+1,\beta+1),\lambda+1,\delta+1;z\right], \tag{4.11}$$

and

$$\frac{d}{dz}\left\{{}_1F_1\left[(\alpha,\beta),\delta;z\right]\right\}=\frac{1}{j}\frac{(\alpha,\beta)}{\delta}\,{}_1F_1\left[(\alpha+1,\beta+1),\delta+1;z\right]. \tag{4.12}$$

Proof.

$${}_2F_1\left[(\alpha,\beta),\lambda,\delta;z\right]=\sum_{m=0}^{\infty}\frac{(\alpha,\beta)_m(\lambda)_m}{(\delta)_m}\frac{z^m}{(m,j)!}$$

$$=\sum_{m=0}^{\infty}\frac{(\alpha,\beta)_m(\lambda)_m}{(\delta)_m}\frac{mz^{m-1}}{(m,j)!}$$

As $m\to m+1$, we have

$$\frac{d}{dz}\,{}_2F_1\left[(\alpha,\beta),\lambda,\delta;z\right]=\sum_{m=0}^{\infty}\frac{(\alpha,\beta)_{m+1}(\lambda)_{m+1}}{(\delta)_{m+1}}\frac{z^m}{j^{m+1}\Gamma(m+1)}$$

$$=\frac{1}{j}\sum_{m=0}^{\infty}\frac{(\alpha,\beta)_{m+1}(\lambda)_{m+1}}{(\delta)_{m+1}}\frac{z^m}{j^{m}\Gamma(m+1)}$$

$$=\frac{1}{j}\frac{\lambda(\alpha,\beta)}{\delta}\sum_{m=0}^{\infty}\frac{(\alpha+1,\beta+1)_m(\lambda+1)_m}{(\delta+1)_m}\frac{z^m}{(m,j)!}$$

$$=\frac{1}{j}\frac{\lambda(\alpha,\beta)}{\delta}\,{}_2F_1[(\alpha+1,\beta+1),\lambda+1,\delta+1;z]$$

5. Families of Generating Functions Relations

In this section, we denote the following array of numbers $\frac{\lambda}{N}, \frac{\lambda+1}{N}, ..., \frac{\lambda+n-1}{N}$ by $\Delta(N,\lambda)$

Theorem 5.1. The following relation holds true

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\,{}_{r+N}F_s\left[\begin{matrix}\Delta(N,\lambda+n),(\alpha_1,\beta),\alpha_2,...,\alpha_r\\ \delta_1,\delta_2,...,\delta_s\end{matrix};zj^N\right]$$
$$={}_{r+N}F_s\left[\begin{matrix}\Delta(N,\lambda),(\alpha_1,\beta),\alpha_2,...,\alpha_r\\ \delta_1,\delta_2,...,\delta_s\end{matrix};z\left(\frac{j}{1-t}\right)^N\right](1-t)^{-\lambda} \tag{5.1}$$

Proof: Given that

$$\left[(1-z)\left(1-\frac{t}{1-z}\right)\right]^{-\lambda}=\left[(1-t)\left(1-\frac{z}{1-t}\right)\right]^{-\lambda}$$

Since

$$(1-z)^{-\lambda}=\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}z^n \tag{5.2}$$

Yields

$$(1-z)^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\left(\frac{t}{1-z}\right)^n=(1-t)^{-\lambda}\sum_{N=0}^{\infty}\frac{(\lambda)_N}{N!}\left(\frac{z}{1-t}\right)^N$$

Applying (5.2) again and (4.1), we get the desired result.

Theorem 5.2

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\,{}_{r+N}F_s\left[\begin{matrix}\Delta(N,-n),(\alpha_1,\beta),\alpha_2,...,\alpha_r\\ \delta_1,\delta_2,...,\delta_s\end{matrix};zj^N\right]$$
$$={}_{r+N}F_s\left[\begin{matrix}\Delta(N,\lambda),(\alpha_1,\beta),\alpha_2,...,\alpha_r\\ \delta_1,\delta_2,...,\delta_s\end{matrix};z\left(\frac{-tj}{1-t}\right)^N\right](1-t)^{-\lambda} \tag{5.3}$$

Proof: The proof of (5.3) is similar to (5.1). This can be obtained from the fact that

$$\left[(1-z)\left(1-\frac{-zt}{1-z}\right)\right]^{-\lambda}=\left[(1+zt)\left(1-\frac{z}{1+zt}\right)\right]^{-\lambda}$$

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