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Research Article

Semitopological δ-groups

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Abstract

The aim of this paper is to introduce semitopological δ-group and topological δ-group with the concept of δ-group which arise from approximately algebraic structures. Furthermore, it is shown that product space determined with δ-topological subspaces is a δ-topological space. Fundamental system of open δ-neighborhoods and related properties were investigated.

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1. Introduction

A topological group is an algebraic group endowed with a topology so that the multiplication and inversion operations are continuous in all variables together. A semitopological group is an algebraic group endowed with a topology such that for each variable, only the multiplication operation is continuous [4].

Efremovič developed the theory of proximity spaces in the early 1950s when he axiomatically defined the proximity relation "A is near B" for subsets A and B of any set X [2,3]. The set X together with this relation was called an proximity space and is a natural generalization of a topological group. He demonstrated that a topology can be introduced in a proximity space by defining the closure of a subset A of X as the collection of all points of X near A. One can discover compact introduction to the theory of proximity spaces and their generalizations in [12]. Furthermore, for descriptive proximities one can see [1].

In 2017 and 2018, approximately semigroups, approximately ideals, approximately groups, approximately subgroups and approximately rings were introduced by İnan [5–8]. Approximately Γ-semigroups and approximately Γ-rings were also defined [9, 14]. Some instances of these approximately algebraic structures in digital images equipped with proximity relations were described in these articles. Approximately algebraic structures provide a foundation for more applicable fields like image analysis and classification difficulties.

The aim of this paper is to introduce semitopological δ-group and topological δ-group with the concept of δ-group which arise from approximately algebraic structures. Furthermore, it is shown that product space determined with δ-topological subspaces is a

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δ-topological space. Fundamental system of open δ-neighborhoods and related properties were investigated.

2. Preliminaries

Definition 2.1 ([2,3])**.** Let X be a non-empty set and δ be a relation on $P(X)$. δ is called an Efremovič proximity that satisfy following axioms:
(A_1) $A\ \delta\ B$ implies $B\ \delta\ A$,
(A_2) $A\ \delta\ B$ implies $A \neq \emptyset$ and $B \neq \emptyset$,
(A_3) $A \cap B \neq \emptyset$ implies $A\ \delta\ B$,
(A_4) $A\ \delta\ (B \cup C)$ iff $A\ \delta\ B$ or $A\ \delta\ C$,
(A_5) $\{x\}\ \delta\ \{y\}$ iff $x = y$,
(A_6) $A\ \underline{\delta}\ B$ implies $\exists E \subseteq X$ such that $A\ \underline{\delta}\ E$ and $E^c\ \underline{\delta}\ B$
for all $A, B, C \in P(X)$ and all $x, y \in X$. Also, (X, δ) is called an Efremovič proximity space.

In a discrete space, a non-abstract point has a location and has features that can be measured [10]. Let X be a non-empty set of non-abstract points.

Probe functions $\varphi_i : X \to \mathbb{R}$ represent a feature of a sample point in a picture. Let $\Phi(x) = (\varphi_1(x), \cdots, \varphi_n(x))$ $(n \in \mathbb{N})$ be an object description, which is a feature vector of x, which provides a description of each $x \in X$. After the choosing a set of probe functions, one obtain a descriptive proximity relation.

Definition 2.2 ([11])**.** Let X be a non-empty set of non-abstract points, Φ be an object description and A be a subset of X. Then the set description of A is defined as

$$\mathcal{Q}(A) = \{\Phi(a) \mid a \in A\}.$$

Definition 2.3 ([11,13])**.** Let X be a non-empty set of non-abstract points and A, B be two subsets of X. Then the descriptive (set) intersection of A and B is defined as

$$A \underset{\Phi}{\cap} B = \{x \in A \cup B \mid \Phi(x) \in \mathcal{Q}(A) \ \ and\ \Phi(x) \in \mathcal{Q}(B)\}.$$

Definition 2.4 ([11])**.** Let X be a non-empty set of non-abstract points and A, B be any two subsets of X. If $\mathcal{Q}(A) \cap \mathcal{Q}(B) \neq \emptyset$, then A is called descriptively near B and denoted by $A\delta_\Phi B$. If $\mathcal{Q}(A) \cap \mathcal{Q}(B) = \emptyset$, then $A\ \underline{\delta}_\Phi\ B$ reads A is descriptively far from B.

Throught this manuscript it considered Efremovič proximity relation with the notation δ and descriptive Efremovič proximity relation with the notation δ_Φ [11].

Lemma 2.5 ([11])**.** *Let (X, δ) be an Efremovič proximity space and $A, B, C, D \subseteq X$. If $A\ \delta\ B$, $A \subseteq C$ and $B \subseteq D$, then $C\ \delta\ D$.*

Lemma 2.5 is also true for the descriptive Efremovič proximity relation δ_Φ.

Definition 2.6. Let (X, τ) be a topological space and $A \subseteq X$. Closure of A in (X, τ), which we denote with clA, is the intersection of all closed sets containing A or the smallest closed set containing A.

Definition 2.7. Let (X, τ_X) and (Y, τ_Y) be topological spaces. A function $f : X \to Y$ is said to be continuous if the inverse image of every open subset of Y is open in X. In other words, if $V \in \tau_Y$, then its inverse image $f^{-1}(V) \in \tau_X$.

Theorem 2.8. *Let (X, τ_X) and (Y, τ_Y) be topological spaces and $f : X \to Y$ be a function. Then f is continuous iff inverse image of every closed subset of Y is closed in X.*

Theorem 2.9 ([12])**.** *If a subset A of a proximity space (X, δ) is defined to be closed iff $x\delta A$ implies $x \in A$, then the collection of complements of all closed subsets so defined yields a topology $\tau = \tau(\delta)$ on X.*

A proximity relation δ on X induces a topology $\tau = \tau(\delta)$ on X if one defines the closure clA of A to be the intersection of all closed subsets containing A.

Definition 2.10 ([12])**.** If there is a topology τ and a proximity relation δ on a set X such that $\tau = \tau(\delta)$, then τ and δ are said to be compatible.

Lemma 2.11 ([12])**.** *If (X, δ) be a proximity space and $A \subseteq X$, then $A \in \tau(\delta)$ iff $x\underline{\delta}(X\backslash A)$ for every $x \in A$.*

Definition 2.12 ([12])**.** Let (X, δ) be a proximity space and $A \subseteq X$. A subset B of X is a δ-neighbourhood of A iff $A\underline{\delta}(X\backslash B)$ and denoted by $A \ll B$.

Lemma 2.13 ([12])**.** *Let (X, δ) be a proximity space, clA and $IntA$ denote, respectively, the closure and interior of A in $\tau(\delta)$. Then*
(i) $A \ll B$ implies $clA \ll B$,
(ii) $A \ll B$ implies $A \ll IntB$.

Therefore $A \subseteq IntB$, showing that a δ-neighbourhood is a topological neighbourhood.

Definition 2.14 ([12])**.** Let (X, δ) be a proximity space and $A, B \subseteq X$. A function $f : X \longrightarrow X$ is called a proximally continuous mapping or δ-continuous mapping iff

$$A\delta B \text{ implies } f(A)\,\delta f(B).$$

Equivalently, f is a δ-continuous mapping iff

$$C \ll D \text{ implies } f^{-1}(C) \ll f^{-1}(D)$$

for $C, D \subseteq X$.

Theorem 2.15 ([12])**.** *A δ-continuous mapping $f : X \longrightarrow X$ is continuous with respect to $\tau(\delta)$.*

3. δ-groups

Definition 3.1. Let (X, δ) be a proximity space and $A \subseteq X$. A δ-approximation of A is defined as

$$A_\delta = \{x \in X \mid x\delta A\}.$$

If we consider (X, δ_Φ) descriptive proximity space instead of (X, δ), then δ_Φ-approximation of A is defined as

$$A_{\delta_\Phi} = \{x \in X \mid x\delta_\Phi A\}.$$

Definition 3.2. Let (X, δ) be a proximity space and "$\cdot$" be a binary operation on X. $G \subseteq X$ is called a δ-groupoid if $x \cdot y \in G_\delta$ for all $x, y \in G$.

Definition 3.3. Let (X, δ) be a proximity space and "$\cdot$" be a binary operation on X. $G \subseteq X$ is called a δ-group if the followings are true:

(δG_1) For all $x, y \in G$, $x \cdot y \in G_\delta$,
(δG_2) For all $x, y, z \in G$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ property holds in G_δ,
(δG_3) There exists $e \in G_\delta$ such that $x \cdot e = e \cdot x = x$ for all $x \in G$ (e is called the δ-identity element of G),
(δG_4) There exists $y \in G$ such that $x \cdot y = y \cdot x = e$ for all $x \in G$ (y is called the inverse of x in G and denoted as x^{-1}).

A subset S of X is called a δ-semigroup if
(δS_1) $x \cdot y \in S_\delta$ for all $x, y \in S$,
(δS_2) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ property holds in S_δ for all $x, y, z \in S$
properties are satisfied.

If δ-semigroup have a δ-identity element $e \in S_\delta$ such that $x \cdot e = e \cdot x = x$ for all $x \in S$, then S is called a δ-monoid.

If $x \cdot y = y \cdot x$ property holds in G_δ for all $x, y \in G$, then G is called commutative δ-group.

Similarly, concepts of δ_Φ-groupoid and δ_Φ-group can be defined.

Definition 3.4. Let (X, τ) be a topological space endowed with δ_Φ and $A \subseteq X$. Descriptive closure of A in (X, τ), which we denote with $cl_\Phi A$, is the descriptive intersection of all closed sets containing A.

4. Semitopological δ-groups

Let (X, τ) be a topological space and τ be a topology compatible with the proximity relation δ. This topology is denoted by $\tau(\delta)$ and it is called a δ-topology. Moreover, $(X, \tau(\delta))$ is called a δ-topological space. Also, if δ_Φ is consider instead of δ, $(X, \tau(\delta_\Phi))$ is a δ_Φ-topological space.

Let A be a subset of X. The subspace topology on A is defined by

$$\tau_A(\delta) = \{A \cap U \mid U \in \tau(\delta)\} \text{ or } \tau_A(\delta_\Phi) = \{A \cap U \mid U \in \tau(\delta_\Phi)\}.$$

Let A_1, A_2 be subspaces of X and $\mathcal{A} = A_1 \times A_2$. A product topology on X is a topology generated by subsets of the form $p_1^{-1}(U_1)$ and $p_2^{-1}(U_2)$, where U_1, U_2 are open sets of A_1, A_2 and p_1, p_2 are canonical projections by $p_1 : \mathcal{A} \to A_1$, $p_2 : \mathcal{A} \to A_2$, respectively. It is clear that the canonical projections p_1 and p_2 are continuous.

Theorem 4.1. *Let X be a δ-topological space and $A \subseteq X$. Then $clA = A_\delta$.*

Proof. If clA denotes the intersection of all closed sets containing A and $A_\delta = \{x \in X \mid x\delta A\}$, then we must show that $clA = A_\delta$. If $x \in A_\delta$, then $x\delta A$. From Lemma 2.5, this implies $x\delta clA$ and since clA is closed, $x \in clA$. Thus $A_\delta \subseteq clA$. To prove the reverse inclusion it suffices to prove that A_δ is closed, i.e., $x\delta A_\delta$ implies $x \in A_\delta$. Assuming $x \notin A_\delta$, then $x\underline{\delta}A$ so that, by the Axiom (A_6), there is a set E such that $x\underline{\delta}E$ and $E^c\underline{\delta}A$. Thus no point of E^c is proximal to A, i.e., $A_\delta \subseteq E$ which together with $x\underline{\delta}E$ implies that $x\underline{\delta}A_\delta$. $\square$

Theorem 4.2. *Let X be a δ_Φ-topological space and $A \subseteq X$. Then $cl_\Phi A = A_{\delta_\Phi}$.*

Proof. If $cl_\Phi A$ denotes the descriptive intersection of all closed sets containing A and $A_{\delta_\Phi} = \{x \in X \mid x\delta_\Phi A\}$, then we must show that $cl_\Phi A = A_{\delta_\Phi}$. If $x \in A_{\delta_\Phi}$, then $x\delta_\Phi A$. From Lemma 2.5, this implies $x\delta_\Phi (cl_\Phi A)$ and since $cl_\Phi A$ is closed, $x \in cl_\Phi A$. Thus $A_{\delta_\Phi} \subseteq cl_\Phi A$. To prove the reverse inclusion it suffices to prove that A_{δ_Φ} is closed, i.e., $x\delta_\Phi A_\delta$ implies $x \in A_{\delta_\Phi}$. Assuming $x \notin A_{\delta_\Phi}$, then $x\underline{\delta}_\Phi A$ so that, by the Axiom (A_6), there is a set E such that $x\underline{\delta}_\Phi E$ and $E^c\underline{\delta}_\Phi A$. Thus no point of E^c is descriptive proximal to A, i.e., $A_{\delta_\Phi} \subseteq E$ which together with $x\underline{\delta}_\Phi E$ implies that $x\underline{\delta}_\Phi A_\delta$. $\square$

Theorem 4.3. *Let X be a δ-topological space, A_1, A_2 be subspaces of X and $\mathcal{A} = A_1 \times A_2$. Then product space $(\mathcal{A}, \tau)$ on X is a δ-topological space.*

Proof. To show that $(\mathcal{A}, \tau)$ is a δ-topological space, it is sufficient to prove that is compatible with the given topology, that is, $(x, y)\,\delta\mathcal{B}$ if and only if $(x, y) \in cl\mathcal{B}$ for $(x, y) \in \mathcal{A}$ and $B_1 \times B_2 = \mathcal{B} \subseteq \mathcal{A}$ such that $B_1 \subseteq A_1$ and $B_2 \subseteq A_2$. From Theorem 4.1,

$$\begin{aligned} (x, y) \in cl\mathcal{B} &\iff (x, y) \in \mathcal{B}_\delta \\ &\iff (x, y)\,\delta\mathcal{B}. \end{aligned}$$

$\square$

Let X be a δ_Φ-topological space, A_1, A_2 be subspaces of X and $\mathcal{A} = A_1 \times A_2$. From Theorem 4.2, it is easily shown that product space $(\mathcal{A}, \tau)$ on X is a δ_Φ-topological space.

Definition 4.4. Let X be a δ-topological space, $G, G_\delta \subseteq X$ be subspaces of X and G be a δ-group with the multiplication on X. G is called semitopological δ-group if the multiplication $G \times G \longrightarrow G_\delta$, $(x, y) \mapsto xy$ is continuous (in each variable separately).

Definition 4.5. Let X be a δ-topological space, $G, G_\delta \subseteq X$ be subspaces of X and G be a δ-group with the multiplication on X. G is called topological δ-group if the multiplication $G \times G \longrightarrow G_\delta$, $(x, y) \mapsto xy$ and inversion $G \longrightarrow G$, $x \mapsto x^{-1}$ are continuous.

Moreover, in Definitions 4.4 and 4.5 if δ_Φ is considered instead of δ, G is a semitopological δ_Φ-group and G is a topological δ_Φ-group, respectively.

Digital images consist of a certain number of pixels. Each pixel is named according to its row and column, due to its location. Pixels can be processed by taking their color into account along with their location. In Example 4.6, a topological δ_Φ-group sample will be investigated on a 36-pixel digital image.

Example 4.6. Let X be a digital image endowed with descriptive proximity relation δ_Φ and consists of 36 pixels as in Figure 1. In digital image X, each of pixels is a singleton and so it is closed set as in the discret topology. Similarly, subsets of digital images and δ_Φ-approximations of these subsets are also closed sets. If we consider $\tau(\delta_\Phi)$ as a set of these closed sets, then $\tau(\delta_\Phi)$ is a topology compatible with the relation δ_Φ. Hence digital image X is a δ_Φ-topological space.

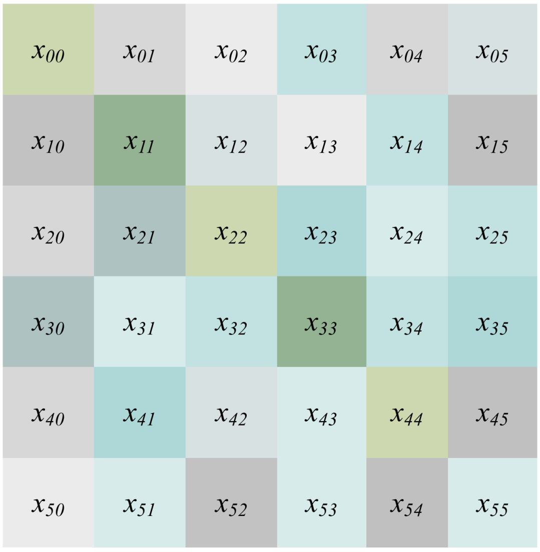

Figure 1. Digital image X

A pixel x_{ij} is an element at position (i, j) in X. Let φ be a probe function that represent RGB colour of each pixel are given in Table 1.

Table 1. RGB colour of each pixel

	Red	*Green*	*Blue*		*Red*	*Green*	*Blue*		*Red*	*Green*	*Blue*
x_{00}	205	216	176	x_{20}	215	215	215	x_{40}	215	215	215
x_{01}	215	215	215	x_{21}	174	194	194	x_{41}	174	215	215
x_{02}	235	235	235	x_{22}	205	216	176	x_{42}	215	225	225
x_{03}	194	225	225	x_{23}	174	215	215	x_{43}	215	235	235
x_{04}	100	168	255	x_{24}	215	235	235	x_{44}	205	216	176
x_{05}	215	225	225	x_{25}	194	225	225	x_{45}	192	192	192
x_{10}	194	194	194	x_{30}	174	194	194	x_{50}	235	235	235
x_{11}	147	179	147	x_{31}	215	235	235	x_{51}	215	235	235
x_{12}	215	225	225	x_{32}	194	225	225	x_{52}	192	192	192
x_{13}	235	235	235	x_{33}	147	179	147	x_{53}	215	235	235
x_{14}	194	225	225	x_{34}	194	225	225	x_{54}	192	192	192
x_{15}	192	192	192	x_{35}	174	215	215	x_{55}	215	235	235

Let

$$\begin{aligned} * : X \times X &\longrightarrow X \\ (x_{ij}, x_{kl}) &\longmapsto x_{ij} * x_{kl} = x_{pr},\ i + k = p \bmod (5) \text{ and } j + l = r \bmod (5) \end{aligned}$$

be a binary operation on X and $G = \{x_{22}, x_{33}\}$ be a subimage of X.

We can compute the δ_Φ-approximation of G by using the Definition 3.1. $G_{\delta_\Phi} = \{x \in X \mid x\delta_\Phi G\}$, where $Q(G) = \{\Phi(x_{ij}) \mid x_{ij} \in G\}$. Then $Q(\{x_{ij}\}) \cap Q(G) \neq \emptyset$ such that $x_{ij} \in X$. From Table 1, we obtain

$$\begin{aligned} \mathfrak{Q}(G) &= \{\Phi(x_{22}), \Phi(x_{33})\} \\ &= \{(205, 216, 176), (147, 179, 147)\}. \end{aligned}$$

Hence we get $G_{\delta_\Phi} = \{x_{00}, x_{11}, x_{22}, x_{33}, x_{44}\}$. Obviously, $G, G_{\delta_\Phi} \subseteq X$ are subspaces of $(X, \tau(\delta_\Phi))$ with similar considerations as in X.

The operation we will consider for G to be a δ_Φ-group is as follows:

$$\begin{aligned} * : G \times G &\longrightarrow G_{\delta_\Phi} \\ (x_{ij}, x_{kl}) &\longmapsto x_{ij} * x_{kl} = x_{pr},\ i + k = p \bmod (5) \text{ and } j + l = r \bmod (5). \end{aligned}$$

Since

($\delta_\Phi G_1$) For all $x_{ij}, x_{kl} \in G$, $x_{ij} * x_{kl} \in G_{\delta_\Phi}$,

($\delta_\Phi G_2$) For all $x_{ij}, x_{kl}, x_{mn} \in G$, $(x_{ij} * x_{kl}) * x_{mn} = x_{ij} * (x_{kl} * x_{mn})$ property holds in G_{δ_Φ},

($\delta_\Phi G_3$) For all $x_{ij} \in G$, $x_{ij} * x_{00} = x_{00} * x_{ij} = x_{ij}$ and hence $x_{00} \in G_{\delta_\Phi}$ is a proximal identity element of G,

($\delta_\Phi G_4$) Since $x_{22} * x_{33} = x_{33} * x_{22} = x_{00}$, $x_{22}^{-1} = x_{33}$ and $x_{33}^{-1} = x_{22}$, that is, x_{33}, x_{22} are inverses of x_{22}, x_{33}, respectively

are satisfied, the subimage G of the image X is indeed a δ_Φ-group in descriptive proximity space (X, δ_Φ) with the operation " $*$ ". Also, since $x_{ij} * x_{kl} = x_{kl} * x_{ij}$ for all $x_{ij}, x_{kl} \in G$ property holds in G_{δ_Φ}, G is a commutative δ_Φ-group.

Furthermore, since the operation $G \times G \longrightarrow G_{\delta_\Phi}$, $(x, y) \mapsto x * y$ and the inversion $G \longrightarrow G$, $x \mapsto x^{-1}$ are continuous, then G is a topological δ_Φ-group.

Let $x, y \in G$ and W, W' be neighbourhoods of $xy \in G_\delta$ and $x^{-1} \in G$, respectively. The multiplication is continuous iff there exist neighbourhoods U of x and V of y in G such that $UV \subseteq W \subseteq G_\delta$, where $UV := \{uv \,|u \in U, v \in V\}$. The inversion is continuous iff there exists a neighbourhood U of $x \in G$ such that $U^{-1} \subseteq W'$, where $U^{-1} := \{u^{-1} \,|u \in U\}$.

Theorem 4.7. *Every topological δ-group is a semitopological δ-group.*

Proof. In topological δ-group $G \subseteq X$, since the multiplication $G \times G \longrightarrow G_\delta$, $(x, y) \mapsto xy$ and inversion $G \longrightarrow G$, $x \mapsto x^{-1}$ are continuous, obviously $G \subseteq X$ is also a semitopological δ-group. □

Theorem 4.8. *Let g be a fixed element of a semitopological δ-group G. Then the onto mappings $r_g : G_\delta \longrightarrow G_\delta$, $x \mapsto xg$ and $l_g : G_\delta \longrightarrow G_\delta$, $x \mapsto gx$ are homeomorphisms for all $x \in G$.*

Proof. It is clear that r_g is an one to one and onto mapping. Let W be a neighbourhood of xg. Since G is a semitopological δ-group, there exists a neighbourhood U of x such that $Ug \subseteq W$. Therefore r_g is continuous. Moreover, it is easy to see that the inverse r_g^{-1} of r_g is the mapping $x \mapsto xg^{-1}$ which is continuous by the same argument as above. Consequently, r_g is a homeomorphism. Similarly, l_g is also a homeomorphism. □

In Theorem 4.8, r_g and l_g are called the right and left δ-translations of G, respectively. It is obvious that $r_g^{-1} = r_{g^{-1}}$ and $l_g^{-1} = l_{g^{-1}}$.

Corollary 4.9. *Let X be a δ-topological space, $F \subseteq X$ be a closed, $P \subseteq X$ be an open, A be an arbitrary subset of a semitopological δ-group G and $g \in G$. Then*
(i) Fg and gF are closed.
(ii) Pg, gP, AP and PA are open.

Proof. Since the mappings are homeomorphisms from Theorem 4.8, (i) is obvious. Similarly, Pg and gP are open in (ii). Since $AP = \bigcup\limits_{g \in G} gP$, $PA = \bigcup\limits_{g \in G} Pg$ and the union of open sets is open, AP and PA are open. □

Corollary 4.10. *Let G be a semitopological δ-group. For any $g, h \in G$, there exists a homeomorphism $\rho : G \longrightarrow G$ such that $\rho(g) = h$.*

Proof. Let $g^{-1}h = a \in G$ and consider the mapping $\rho : x \mapsto xa$ from Theorem 4.8. Hence ρ is a homeomorphism by Theorem 4.8 and $\rho(g) = h$. □

If Corollary 4.10 is true, then a δ-topological space is called a homogeneous.

Corollary 4.11. *Let G be a semitopological δ-group, A be an arbitrary subset of G and $g \in G$. Then $cl(gA) = g(clA)$, $cl(Ag) = (clA)g$ and $cl(A^{-1}) = (clA)^{-1}$.*

Proof. It is clear from Theorem 4.8. □

Definition 4.12. Let X be a δ-topological space and δ-neighbourhood of $x \in X$ is denoted by U_x^δ. Then $\left\{U_x^\delta\right\} = \{B | \{x\} \ll B\}$ is called a fundamental system of open δ-neighbourhoods of $x \in X$.

Theorem 4.13. *Let G be a semitopological δ-group and e be an identity of G. If $\left\{U_e^\delta\right\}$ is a fundamental system of open δ-neighbourhoods of e, then $\left\{gU_e^\delta\right\}$ and $\left\{U_e^\delta g\right\}$, where g runs over G, form bases of the δ-topology of G.*

Proof. Let $g \in G$ and W be an open neighbourhood of g. From Theorem 4.8, since $l_g^{-1} : x \mapsto g^{-1}x$ is a homeomorphism in x, $l_g^{-1}(W) = g^{-1}W$ is an open set containing e and hence there exists a U_e^δ in $\left\{U_e^\delta\right\}$ such that $U_e^\delta \subset g^{-1}W$. This implies $gU_e^\delta \subset W$ which proves that $\left\{gU_e^\delta\right\}$ is a base of the δ-topology on G. Similarly, it is clear that $\left\{U_e^\delta g\right\}$ is also a base of the δ-topology on G. □

It is shown in Theorem 4.14 that the converse of the Theorem 4.13 is also true.

Theorem 4.14. *Let a filter base $\left\{U_e^\delta\right\}$ be given so that each U_e^δ contains e and for each U_e^δ and each $x \in U_e^\delta$, there exist V and W in $\left\{U_e^\delta\right\}$ such that $xV \subset U_e^\delta$ and $Wx \subset U_e^\delta$. Then there exists a δ-topology on G so that G, endowed with this δ-topology, is a semitopological δ-group.*

Proof. Let $\{\mathcal{U}\}$ be a non-empty family of $\mathcal{U} = \bigcap\limits_{i=1}^{n} \left(U_e^\delta\right)_i$. Furthermore, $g\mathcal{U} = g\bigcap\limits_{i=1}^{n} \left(U_e^\delta\right)_i = \bigcap\limits_{i=1}^{n} g\left(U_e^\delta\right)_i$ for any $g \in G$. If $g \in \bigcap\limits_{i=1}^{n} \left(U_e^\delta\right)_i$, then there exists a V_i for each i, $1 \leq i \leq n$, such that $gV_i \subset \left(U_e^\delta\right)_i$ and hence $g\bigcap\limits_{i=1}^{n} V_i = \bigcap\limits_{i=1}^{n} gV_i \subset \bigcap\limits_{i=1}^{n} \left(U_e^\delta\right)_i$. This shows that the family $\{\mathcal{U}\}$ also satisfies the conditions assumed for the filter base $\left\{U_e^\delta\right\}$. By the definition of a subbase, the family of finite intersections of the family $\left\{gU_e^\delta\right\}$, where g runs over G, forms a base of the δ-topology on G.

Now if $h \in \bigcap\limits_{i=1}^{n} g\left(U_e^\delta\right)_i$, then $g^{-1}h \in \left(U_e^\delta\right)_i$ for each i, $1 \leq i \leq n$, and by assumption, there exists a $V_i \in \left\{U_e^\delta\right\}$ such that $g^{-1}hV_i \subset \left(U_e^\delta\right)_i$ or $hV_i \subset g\left(U_e^\delta\right)_i$. Therefore $h\bigcap\limits_{i=1}^{n} V_i =$

$\bigcap_{i=1}^{n} hV_i \subset \bigcap_{i=1}^{n} g\left(U_e^{\delta}\right)_i$. Then $\{h\mathcal{U}\}$, where $\mathcal{U}$ runs over $\{\mathcal{U}\}$, forms a fundamental system of open δ-neighbourhoods of $h \in G$. Similary, $\{\mathcal{U}h\}$ is also a fundamental system of open δ-neighbourhoods of $h \in G$.

Lastly, we have to show that G is a semitopological δ-group. Let we consider the mapping $G \times G \longrightarrow G_\delta$, $(g,h) \mapsto gh$. Assume that g is fixed and let $\mathcal{U}$ be any member in $\{\mathcal{U}\}$. Then $gh\mathcal{U}$ is a member of a fundamental system of open δ-neighbourhoods of gh. Since $h \in h\mathcal{U}$ and $h\mathcal{U}$ is a δ-neighbourhood of h as shown in the previous paragraph, $g(h\mathcal{U}) \subset gh\mathcal{U}$. Therefore the mapping $(g,h) \mapsto gh$ is continuous in h, while g is kept fixed. Similarly, one proves the mapping $(g,h) \mapsto gh$ is continuous in g by considering $\mathcal{U}gh$ as a δ-neighbourhood of gh. As a result, G is a semitopological δ-group. □

References

[1] A. Di Concilio, C. Guadagni, J. F. Peters and S. Ramanna, *Descriptive Proximities: Properties and interplay between classical proximities and overlap*, Mathematics in Computer Science **12** (1), 91-106, 2018.

[2] V. A. Efremovič, *Infinitesimal spaces*, Doklady Akad. Nauk SSSR (N. S.) (Russian) **76**, 341-343, 1951.

[3] V. A. Efremovič, *The geometry of proximity I*, Mat. Sb. (N. S.) (Russian) **31** (73), 189-200, 1952.

[4] T. Husein, *Introduction to Topological Groups*, W.B. Sauders Company, 1966.

[5] E. İnan, *Approximately semigroups and ideals: An algebraic view of digital images*, Afyon Kocatepe University Journal of Science and Engineering **17**, 479-487, 2017.

[6] E. İnan, *Approximately subgroups in proximal relator spaces*, Adıyaman University Journal of Science **8** (1), 24-41, 2018.

[7] E. İnan, *Approximately groups in proximal relator spaces*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. **68** (1), 572-582, 2019.

[8] E. İnan, *Approximately rings in proximal relator spaces*, Turkish J. Math. **43**, 2941-2953, 2019.

[9] E. İnan and M. Uçkun, *Approximately* Γ*-semigroups in proximal relator spaces*, Appl. Algebra Eng. Commun. Comput. **30** (4), 299-311, 2019.

[10] M. Kovăr, A new causal topology and why the universe is co-compact, arXiv: 1112.0817 [math-ph], 1-15, 2011.

[11] S. A. Naimpally and J. F. Peters, *Topology with Applications: Topological Spaces via Near and Far*, World Scientific, 2013.

[12] S. A. Naimpally and B. D. Warrack, *Proximity Spaces*, Cambridge University Press, 1970.

[13] J. F. Peters and S. A. Naimpally, *Applications of near sets*, Notices Amer. Math. Soc. **59** (4), 536-542, 2012.

[14] M. Uçkun, *Approximately* Γ*-rings in proximal relator spaces*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. **68** (2), 1780-1796, 2019.