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WHEN DOES A QUOTIENT RING OF A PID HAVE THE CANCELLATION PROPERTY?

Gyu Whan Chang and Jun Seok Oh

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Abstract. An ideal I of a commutative ring is called a cancellation ideal if $IB = IC$ implies $B = C$ for all ideals B and C. Let D be a principal ideal domain (PID), $a, b \in D$ be nonzero elements with $a \nmid b$, $(a, b)D = dD$ for some $d \in D$, $D_a = D/aD$ be the quotient ring of D modulo aD, and $bD_a = (a, b)D/aD$; so bD_a is a nonzero commutative ring. In this paper, we show that the following three properties are equivalent: (i) $\frac{a}{d}$ is a prime element and $a \nmid d^2$, (ii) every nonzero ideal of bD_a is a cancellation ideal, and (iii) bD_a is a field.

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1. Introduction

Let S be a commutative semigroup under multiplication. The zero element of S (if it exists) is an element $0 \in S$ such that $a \cdot 0 = 0 \cdot a = 0$ for all $a \in S$. An element $a \in S$ is said to be *cancellative* if $ab = ac$ implies $b = c$ for all $b, c \in S$. Clearly if S has an identity, then every invertible element of S is cancellative, so the cancellation property is a natural generalization of invertibility. We say that S is *cancellative* if every nonzero element of S is cancellative. Let $S^\bullet = S \setminus \{0\}$. Then $S^\bullet$ is not a semigroup in general, while S is cancellative if and only if $S^\bullet$ is a cancellative semigroup. The cancellation property plays an important role for the study on algebra. For example, assume that $S^\bullet$ is a semigroup. Then (i) S is cancellative if and only if $S^\bullet$ can be embedded in a group (i.e., $S^\bullet$ has a quotient group) [5, Theorem 1.2], (ii) S is torsion-free and cancellative if and only if $S^\bullet$ admits a total order compatible with its semigroup operation [5, Corollary 3.4], and (iii) several kinds of factorization properties of a semigroup (e.g., atomic, factorial, half-factorial, bounded factorization) have been studied under the assumption that it is cancellative (see [3] for a survey).

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Let R be a commutative ring (not necessarily having an identity), $\mathcal{I}(R)$ be the set of ideals of R, and $\mathcal{P}(R)$ be the set of principal ideals of R. Then $\mathcal{I}(R)$ becomes a commutative semigroup with zero element under the usual ideal multiplication, $\mathcal{P}(R)$ is a subsemigroup of $\mathcal{I}(R)$, and if R has an identity, then $\mathcal{I}(R)$ has an identity. We say that an ideal I of R is a *cancellation ideal* if I is cancellative as an element of $\mathcal{I}(R)$. It is easy to see that a principal ideal (a) of R generated by $a \in R$ is a cancellation ideal if and only if a is a regular element of R (i.e., a is not a zero-divisor). Furthermore, if R has an identity distinct from the zero element, then a nonzero ideal I of R is a cancellation ideal if and only if IR_M is a regular principal ideal for all maximal ideals M of R [1, Theorem, p. 2853], and $\mathcal{P}(R)$ is cancellative if and only if R is an integral domain, if and only if $\mathcal{I}(R)^\bullet$ is a semigroup with an identity. It is well known that if R is an integral domain, then (i) $\mathcal{P}(R)$ is factorial if and only if R is a unique factorization domain, (ii) $\mathcal{I}(R)$ is factorial if and only if R is a Dedekind domain [4, Theorem 37.8], (iii) $\mathcal{I}(R)$ is cancellative if and only if R is an almost Dedekind domain (i.e., R_M is a principal ideal domain (PID) for all maximal ideals M of R) [4, Theorem 36.5], and (iv) every nonzero finitely generated ideal of R is a cancellation ideal if and only if R is a Prüfer domain, (i.e., every nonzero finitely generated ideal of R is invertible) [4, Theorem 24.3].

Now let $\mathbb{Z}$ be the ring of integers, m and n be positive integers, $\gcd(m,n)$ denote the greatest common divisor of m and n, $\mathbb{Z}_n$ be the ring of integers modulo n, and $m\mathbb{Z}_n$ be the ideal of $\mathbb{Z}_n$ generated by m; so $m\mathbb{Z}_n$ is a commutative ring. Then $\mathcal{I}(\mathbb{Z}_n) = \mathcal{P}(\mathbb{Z}_n)$, and hence $\mathcal{I}(\mathbb{Z}_n)$ is cancellative if and only if $\mathbb{Z}_n$ is an integral domain, if and only if n is a prime number. Moreover, in [2, Theorem 2.5], the authors showed that if $n \nmid m$, then every nonzero ideal in $m\mathbb{Z}_n$ is a cancellation ideal, i.e., $\mathcal{I}(m\mathbb{Z}_n)$ is a cancellative semigroup, if and only if $\frac{n}{\gcd(n,m)}$ is a prime number and $n \nmid \gcd(n,m)^2$, which motivated the main result of this paper.

Let D be a PID, a and b be nonzero elements of D, and $d \in D$ be such that $(a,b)D = dD$. We define D_a to be the quotient ring D/aD, $d = \gcd(a,b)$, and $bD_a = (a,b)D/aD$. Then bD_a is a commutative ring, d is determined only up to units, and since $\gcd(a,b) = d$, we have that $\gcd(a/d, b/d) = 1$ and $bD_a = dD_a$. In this paper, we show that every nonzero ideal of bD_a is a cancellation ideal if and only if $\frac{a}{\gcd(a,b)}$ is a prime element and $a \nmid \gcd(a,b)^2$. This result is applied in two special cases, i.e., the ring of integers and the polynomial ring over a field. The former case was provided previously in [2, Theorem 2.5]; our work can be viewed as a direct generalization of the results of that paper, passing from $\mathbb{Z}$ to arbitrary PIDs.

2. Results

Let R be a commutative ring with identity. Then two ideals I, J of R are said to be *comaximal* if $I + J = R$, and we say that two elements a, b of R are *comaximal* if the principal ideals aR and bR are comaximal. Clearly, a, b are comaximal if and only if $ar + bs = 1$ for some $r, s \in R$.

Lemma 2.1. *Let D be a PID and $a, b \in D$ be nonzero elements. Then bD_a has an identity if and only if b and $\frac{a}{\gcd(a,b)}$ are comaximal.*

Proof. Let $d = \gcd(a, b)$, $a_1 = \frac{a}{d}$, and $b_1 = \frac{b}{d}$.

($\Rightarrow$) Let $bx + aD$ be the identity of bD_a for some $x \in D$. Then

$$(bx + aD)(b + aD) = b + aD,$$

whence $a \mid b(bx-1)$. Also, $\gcd(a_1, b_1) = 1$ implies $a_1 | bx-1$, and hence $bx+a_1y = 1$ for some $y \in D$. Thus, b and a_1 are comaximal.

($\Leftarrow$) By assumption, $bx + a_1y = 1$ for some $x, y \in D$, and hence $bx = 1 - a_1y$. So, for every $z \in D$, we have

$$\begin{aligned}(bx + aD)(bz + aD) &= (1 - a_1y)(bz) + aD = (bz - a_1byz) + aD \\ &= (bz + aD) - (ab_1yz + aD) \\ &= bz + aD.\end{aligned}$$

Thus, $bx + aD$ is the identity of bD_a. □

We now give the main result of this paper.

Theorem 2.2. *Let D be a PID and $a, b \in D$ be nonzero elements with $a \nmid b$. Then the following statements are equivalent:*

(1) *$\frac{a}{\gcd(a,b)}$ is a prime element and $a \nmid \gcd(a, b)^2$.*
(2) *bD_a is a field.*
(3) *Every nonzero ideal of bD_a is a cancellation ideal.*

Proof. Let $d = \gcd(a, b)$, $a_1 = \frac{a}{d}$, and $b_1 = \frac{b}{d}$. Clearly, $bD_a = dD_a$, $(a_1, b_1)D = D$, a_1 is a nonunit, and $bD_a \neq (0)$ because $a \nmid b$.

(1) $\Rightarrow$ (2) Note that bD_a is a commutative ring; so bD_a is a field if and only if bD_a has an identity and bD_a does not have a proper nonzero ideal.

We first show that bD_a has an identity. Note that $a = da_1$ and $b = db_1$; so $a \nmid d^2$ implies $a_1 \nmid d$. Hence, $a_1 \nmid b$ because $\gcd(a_1, b_1) = 1$ and $b = db_1$. Since $a_1 = \frac{a}{d}$ is a prime element by assumption, it follows that a_1 and b are comaximal. Hence, by Lemma 2.1, bD_a has an identity. Next, let A be an ideal of bD_a. Then

$D_aA = D_a(bD_aA) = bD_aA = A$ because bD_a has an identity. Thus, A is an ideal of D_a, so there exists $e \in D$ such that

$$aD \subseteq eD \subseteq dD \quad \text{and} \quad A = eD/aD \subseteq bD_a.$$

Hence, $e = dx$ and $a = ey$ for some $x, y \in D$, and thus $a = dxy$ or $a_1 = xy$. By assumption, $a_1 = xy$ is a prime element of D, whence either x or y is a unit of D. If x is a unit, then $eD = dD$, and hence $A = eD/aD = dD_a = bD_a$. If y is a unit, then $eD = aD$, whence $A = eD/aD = aD/aD$ is the zero ideal of bD_a. Therefore, bD_a does not have a proper nonzero ideal.

(2) $\Rightarrow$ (3) Clear.

(3) $\Rightarrow$ (1) If $a \mid d^2$, then $\big(dD_a\big)^2 = d^2D_a = (0)$, and since dD_a is a cancellation ideal in bD_a, $dD_a = (0)$, a contradiction. Thus, $a \nmid d^2$.

Next, assume to the contrary that $a_1 = pq$ for some nonunit elements p, q of D. Let $I = pdD_a$ and $J = qdD_a$. Then I and J are ideals of dD_a. If $I = (0)$, then $pd + aD = aD$, and hence $a \mid pd$. Note that $a = a_1d$; so $a_1 \mid p$, and thus q is a unit, a contradiction. Similarly, we have $J \neq (0)$. However,

$$IJ = pdqdD_a = a_1d^2D_a = adD_a = (0).$$

Thus, I and J are not cancellation ideals, a contradiction. $\square$

As a corollary of Theorem 2.2, we have the following result.

Corollary 2.3. *Let D be a PID and $a \in D$ be a nonzero element. Then every nonzero ideal in D_a is a cancellation ideal if and only if a is a prime element.*

We have two applications of Theorem 2.2: one is to the ring of integers and the other is to the polynomial ring over a field.

Corollary 2.4. *Let $n, m \in \mathbb{Z}$ be positive integers with $n \nmid m$.*

(1) ([2, Theorem 2.5]) *Every nonzero ideal in $m\mathbb{Z}_n$ is a cancellation ideal if and only if $\frac{n}{\gcd(n,m)}$ is a prime number and $n \nmid \gcd(n,m)^2$.*

(2) ([2, Corollary 2.6]) *Every nonzero ideal in $\mathbb{Z}_n$ is a cancellation ideal if and only if n is a prime number.*

Corollary 2.5. *Let F be a field, X be an indeterminate over F, $F[X]$ be the polynomial ring over F, $f, g \in F[X]$ be nonzero polynomials with $f \nmid g$ in $F[X]$, and $(f,g)F[X] = hF[X]$ for some $h \in F[X]$.*

(1) *Every nonzero ideal in $(f,g)F[X]/fF[X]$ is a cancellation ideal if and only if $\frac{f}{h}$ is irreducible over F and $f \nmid h^2$ in $F[X]$.*

(2) *Every nonzero ideal in $F[X]/fF[X]$ is a cancellation ideal if and only if f is irreducible over F.*

Proof. This follows directly from Theorem 2.2 and Corollary 2.3, because an irreducible polynomial of $F[X]$ is a prime element. $\square$

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Gyu Whan Chang
Department of Mathematics Education
Incheon National University
22012 Incheon, Republic of Korea
e-mail: whan@inu.ac.kr

Jun Seok Oh (Corresponding Author)
Department of Mathematics Education
Jeju National University
63243 Jeju, Republic of Korea
e-mail: junseok.oh@jejunu.ac.kr