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Research Article

Asymptotic equivalence of impulsive dynamic equations on time scales

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Abstract

The asymptotic equivalence of linear and quasilinear impulsive dynamic equations on time scales, as well as two types of linear equations, are proven under mild conditions. To establish the asymptotic equivalence of two impulsive dynamic equations a method has been developed that does not require restrictive conditions, such as the boundedness of the solutions. Not only the time scale extensions of former results have been obtained, but also improved for impulsive differential equations defined on the real line. Some illustrative examples are also provided, including an application to a generalized Duffing equation.

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1. Introduction

Differential equations with impulse effects attract the attention of many researchers in the last two decades due to their efficiency in modeling real phenomena like evolution processes instantly disrupted [6, 21]. The corresponding theory is very fruitful but most of the problems involving impulse effects cannot be solved by using the usual methods of differential equations. On the other hand, there are many phenomena in various fields such as population dynamics, logistics, biology [8, 9, 11], electrical engineering, physics, [10, 12], neural networks [11, 17] which cannot be modeled using only continuous or only discrete dynamical systems. As they contain both continuous and discrete data, such models require simultaneous use of both. In 1988, Stefan Hilger introduced the concept of time scales as a tool to unify the analysis on continuous and discrete sets [13]. Since then, it has been used intensely by many researchers working in different areas to produce solutions for the modeling problems mentioned above. For practical examples and deep knowledge about the qualitative theory of dynamic systems on time scales, we refer the reader to the books [8–12, 14].

Asymptotic equivalence of differential equations is a classical problem investigated widely by many authors. Regarding impulsive differential equations, we refer the reader to the papers [1, 2, 4, 18] involving good results on the asymptotic equivalence. In particular, for studies dealing with asymptotic equivalence of impulsive linear equations, see [1], for linear and quasilinear equations, see [4], and, for quasilinear impulsive differential equation

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and a linear ordinary differential equation, see [2]. Very nice results can be found in the papers [7, 15] about the asymptotic equivalence of non-impulsive dynamic equations on time scales, where [7] deals with the asymptotic equivalence of a linear and a quasilinear dynamic equation, while [15] involves results on both linear dynamic equations and, linear and quasilinear dynamic equations. However, most of the previous studies yields results under very restrictive conditions, such as existence of bounded solutions or the dichotomy condition being met.

To the best of our knowledge, there are no counterparts of the studies mentioned above for the impulsive dynamic equations on time scales in the literature. However, due to complex arguments and calculations encountered when such type of equations handled, it becomes crucial to obtain a correspondence between an equation with complicated terms and a simpler one. Motivated by these reasons, we study asymptotic equivalence of two linear impulsive dynamic equations, as well as a quasilinear and a linear impulsive dynamic equation on time scales. We achieve good results neither by setting the so-called dichotomy condition nor by requiring the existence of bounded solutions. We also provide corollaries for the non-impulsive case and for the particular but crucial time scale $\mathbb{R}$, and we present illustrative examples of our results. In this way, our results become new even for the differential equations defined on the real line.

The paper is organized as follows. The present section contains a short motivation and is introductory. The next section involves basic definitions and concepts utilized in the paper and some related works in the literature. Then, the main problems are introduced. In the third section, previous results on the asymptotic equivalence of linear equations are extended to dynamic equations on time scales. In the fourth section, which contains the main contribution of the paper, mild conditions that guarantee the asymptotic equivalence of quasilinear and linear impulsive dynamic equations are obtained. The fifth section consists of illustrative examples of the theorems, and in the last section, the contribution of the paper to asymptotic equivalence is summarized.

2. Preliminaries

In this section, first we give basic definitions and necessary concept on time scales. Then, we introduce our main problems, following inspiring studies in the literature.

A time scale $\mathbb{T}$ is a nonempty arbitrary closed subset of $\mathbb{R}$. Some of the best known examples are $\mathbb{R}$, $\mathbb{Z}$, the Cantor set $\mathcal{C}$, $q^{\mathbb{N}_0} := \{q^n : n \in \mathbb{N}_0\}$, where $q > 1$ and $h\mathbb{Z} := \{hz : z \in \mathbb{Z}\}$, where $h > 0$. In the following definition, $\mathbb{K}$ denotes a scalar field, either $\mathbb{R}$ or $\mathbb{C}$.

Definition 2.1 ([9]). 1. $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ is the forward jump operator, $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$ is the backward jump operator, $\mu(t) = \sigma(t) - t$ is the graininess at t and $f^\sigma(t) := f(\sigma(t))$.

2. A point $t \in \mathbb{T}$ is right dense if $\sigma(t) = t$, left dense if $\rho(t) = t$, right scattered if $\sigma(t) > t$ and left scattered if $\rho(t) < t$.
3. A function $f : \mathbb{T} \to \mathbb{K}$ is regulated on $\mathbb{T}$ if its right hand limits exist at all right dense points of $\mathbb{T} \setminus \{\sup \mathbb{T}\}$ and left limits exist at all left dense points of $\mathbb{T} \setminus \{\inf \mathbb{T}\}$.
4. A function $f : \mathbb{T} \to \mathbb{K}$ is regressive if $1 + \mu(t)f(t) \neq 0$ for all $t \in \mathbb{T}^\kappa$, where $\mathbb{T}^\kappa$ is the set of points $t \in \mathbb{T}$ such that t is either left dense or non-maximal.
5. A function $f : \mathbb{T} \to \mathbb{K}$ is right dense continuous *(rd-continuous)* if it is regulated and continuous at all right dense points of $\mathbb{T}$. The set of all rd continuous functions on $\mathbb{T}$ is denoted by $C_{rd}(\mathbb{T})$, and the set of all rd-continuous, regressive functions is denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T})$.
6. A function $f : \mathbb{T} \to \mathbb{K}$ is delta-differentiable at $t \in \mathbb{T}^\kappa$ if

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| < \epsilon|\sigma(t) - s|$$

for all $\epsilon > 0$. The delta derivative of f at t is then defined as

$$f^{\Delta}(t) = \lim_{s\to t} \frac{f(s)-f(t)}{s-t} \text{ if } \mu = 0, \quad f^{\Delta}(t) = \frac{f(\sigma(t))-f(t)}{\sigma(t)-t} \text{ if } \mu > 0.$$

7. If $F^{\Delta}(t) = f(t)$ for all $t \in \mathbb{T}^{\kappa}$, F is a delta antiderivative of f on $\mathbb{T}$, and

$$\int_a^b f(t)\Delta t = F(b) - F(a).$$

Below listed some useful formulas:

- $f^{\sigma} = f + \mu f^{\Delta}$,
- Product Rule: $(fg)^{\Delta} = f^{\Delta} g + f^{\sigma} g^{\Delta} = f g^{\Delta} + f^{\Delta} g^{\sigma}$
- Quotient Rule: $(f/g)^{\Delta} = (f^{\Delta} g - f g^{\Delta})/(g g^{\sigma})$,
- $\int_a^b c\Delta t = c(b-a)$, $c \in \mathbb{R}$.

Definition 2.2 ([9, 10])**.** Let $p \in \mathcal{R}$. The generalized exponential function is defined by

$$e_p(t,s) = \exp\Big(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\Big),$$

where $\xi_{\mu}(z)$ is the cylinder transformation defined by $\xi_{\mu}(z) = \mathrm{Log}(1+\mu z)/\mu$ if $\mu \neq 0$ such that $\xi_0(z) = z$ for all $z \in \mathbb{C}$, and Log is the principal logarithm function.
If $\mu(t) \neq 0$, then

$$e_p(t,s) = \exp\Big(\int_s^t \frac{1}{\mu(s)} \log(1+\mu(s)p(s))\Delta s\Big), \quad s,t \in \mathbb{T}.$$

Definition 2.3 ([9])**.** Suppose that $A(t) \in \mathcal{R}$. Then, the unique solution of the initial value problem $x^{\Delta} = A(t)x$, $x(t_0) = I$ is called the matrix exponential function, and is denoted by $e_A(t,t_0)$. Some useful properties of the matrix exponential are listed below:

- $e_0(t,s) = I$ and $e_A(t,t) = I$,
- $e_A(\sigma(t),s) = [1+\mu(t)A(t)]e_A(t,s)$,
- $e_A(t,s)e_A(s,\tau) = e_A(t,\tau)$,
- $(e_A(t,s))^{-1} = e_A(s,t)$.
- If A is a constant matrix, then $e_A(t,t_0) = \begin{cases} e^{A(t-t_0)}, & if\ \mathbb{T} = \mathbb{R}, \\ (A+I)^{(t-t_0)}, & if\ \mathbb{T} = \mathbb{Z}. \end{cases}$

We also recall that the impulse operator Δ is given by $\Delta\varphi|_{t=t_k} = \varphi(t_k+) - \varphi(t_k-)$, and it measures the size of the impulse at t_k, $k \in \mathbb{Z}$.

The asymptotic equivalence of linear impulsive differential equations

$$\begin{cases} y' = [A(t)+B(t)]y, & t \neq t_k, \\ \Delta y = [A_k + B_k]y, & t = t_k \end{cases}$$

and

$$\begin{cases} x' = A(t)x, & t \neq t_k, \\ \Delta x = A_k x, & t = t_k, \end{cases}$$

was proved in [1] using an implementation of Ráb's Lemma [19, 20]. Then, in [4], Bainov et al. proved that the impulsive differential equation

$$\begin{cases} y' = Cy + f(t,y), & t \neq t_k, \\ \Delta y = C_0 y + f_k(y), & t = t_k \end{cases}$$

and the associated linear equation

$$\begin{cases} x' = Cx, & t \neq t_k, \\ \Delta x = C_0 x, & t = t_k, \end{cases} \tag{2.1}$$

are asymptotically equivalent provided that (2.1) has a bounded solution. Later, for the special case $C_0 = 0$, it was proved that the impulsive differential equation

$$\begin{cases} y' = Cy + f(t, y), & t \neq t_k, \\ \Delta y = f_k(y), & t = t_k \end{cases} \tag{2.2}$$

and the corresponding homogeneous ordinary differential equation $x' = Cx$ are asymptotically equivalent by applying a Yakubovich type theorem [23] which does not require existence of bounded solutions.

In [7], the asymptotic equivalence of non-impulsive dynamic equations of the form

$$y^\Delta = A(t)y + f(t) + g(t, y)$$

and the corresponding linear dynamic equation

$$x^\Delta = A(t)x + f(t),$$

was proved under the condition of the existence of bounded solutions. Later, by Kaymakçalan et. al. [15], successful improvements of the Ráb's Lemma and Yakubovich's theorem were obtained to show the equivalence of the dynamic equations

$$y^\Delta = [A(t) + B(t)]y, \quad x^\Delta = A(t)x$$

and, of the equations

$$y^\Delta = Cy + f(t, y), \quad x^\Delta = Cx$$

on arbitrary time scales. To the best of our knowledge, it is the only paper in which the asymptotic equivalence of non-impulsive dynamic equations is shown under weak conditions. In fact, in most of the works in the literature, it is either assumed that the solutions are bounded or the so-called dichotomy condition is valid [3–5, 7, 18, 22].

As far as we know, there is hardly any result obtained for impulsive dynamic equations on time scales. Inspired by the studies mentioned above, in this work we focus on the asymptotic equivalence of the linear systems of impulsive dynamic equation

$$\begin{cases} y^\Delta = [A(t) + B(t)]y, & t \neq t_k, \\ \Delta y = [A_k + B_k]y, & t = t_k \end{cases} \tag{2.3}$$

and

$$\begin{cases} x^\Delta = A(t)x, & t \neq t_k, \\ \Delta x = A_k x, & t = t_k, \end{cases} \tag{2.4}$$

as well as the asymptotic equivalence of the quasilinear system of impulsive dynamic equation

$$\begin{cases} y^\Delta = Cy + f(t, y), & t \neq t_k, \\ \Delta y = C_0 y + f_k(y), & t = t_k \end{cases} \tag{2.5}$$

and the associated linear system of impulsive dynamic equation

$$\begin{cases} x^\Delta = Cx, & t \neq t_k, \\ \Delta x = C_0 x, & t = t_k, \end{cases} \tag{2.6}$$

for $t \in \mathbb{T}$, where $\mathbb{T}$ is an arbitrary time scale with $\sup \mathbb{T} = \infty$; $A(t)$, $B(t)$ are $n \times n$ matrix functions; A_k, B_k, $k \in \mathbb{Z}$ are $n \times n$ matrices whose entries are real sequences; C, C_0 are $n \times n$ constant matrices; f and f_k are n-dimensional vector functions.

Throughout the paper, we let $t_0 \in \mathbb{T}$ to be fixed, and use the notation $[t_0,\infty)_{\mathbb{T}} := \mathbb{T} \cap [t_0,\infty)$. $PLC_{rd}[t_0,\infty)_{\mathbb{T}}$ denotes the set of functions $x : [t_0,\infty)_{\mathbb{T}} \to \mathbb{R}$ such that x is piecewise rd-continuous on $[t_0,\infty)_{\mathbb{T}}$, i.e., $x(t)$ is rd-continuous on each interval $[t_{k-1},t_k)_{\mathbb{T}}$, $x(t_k\pm)$ exist for each $k = 1,2,\dots$, and $x(t_k-) = x(t_k)$. We assume that the points t_k are right dense in $\mathbb{T}$ and they form an increasing sequence $\{t_k\}_{k\geq 1}$, and we use the notations $\underline{n}(t) := \inf\{k : t_k \geq t\}$ and $\overline{n}(t) := \sup\{k : t_k < t\}$.

The definition of asymptotic equivalence for the systems (2.3) and (2.4), or (2.5) and (2.6), adapted from [1], is as follows.

Definition 2.4. The systems (2.3) and (2.4), or (2.5) and (2.6) are asymptotically equivalent if there exists a one to one correspondence between their solutions $y(t)$ and $x(t)$ so that $\lim_{t\to\infty}[y(t) - x(t)] = 0$.

3. Asymptotic equivalence of linear dynamic equations

Let $A(t), B(t) \in PLC_{rd}[t_0,\infty)_{\mathbb{T}}$, where $A(t)$ is regressive and $A_k + I$ is nonsingular. By a solution of the dynamic equations (2.3) or (2.4), we mean a function φ that belongs to the space

$$Y := \{\varphi : [t_0,\infty)_{\mathbb{T}} \to \mathbb{R} |\, \varphi, \varphi^{\Delta} \in PLC_{rd}[t_0,\infty)_{\mathbb{T}}\}$$

and satisfies the equation taken into account.

Let $\phi(t)$ be a fundamental matrix of (2.4). Setting $y = \phi(t,t_0)z$, we obtain

$$\begin{cases} z^{\Delta} = P(t)z, & t \neq t_k \\ \Delta z = P_k z, & t = t_k, \end{cases} \tag{3.1}$$

where $P(t) = \phi(t_0,\sigma(t))B(t)\phi(t,t_0)$ and $P_k = \phi(t_0,t_k)(A_k + I)^{-1}B_k\phi(t_k,t_0)$.

Lemma 3.1. *If*

$$\int_{t_0}^{\infty} \|P(t)\|\Delta t + \sum_{k=\underline{n}(t_0)}^{\infty} \|P_k\| < \infty, \tag{3.2}$$

then

$$\begin{cases} \Psi^{\Delta} = P(t)(\Psi + I), & t \neq t_k,\ t \in [t_0,\infty)_{\mathbb{T}}, \\ \Delta\Psi = P_k(\Psi + I), & t = t_k, \end{cases} \tag{3.3}$$

has a solution $\Psi(t)$ such that $\Psi(t) \to 0$ as $t \to \infty$.

Proof. Let

$$\Psi_0(t) = I, \quad \Psi_j(t) = -\int_t^{\infty} P(s)\Psi_{j-1}(s)\Delta s - \sum_{k=\underline{n}(t)}^{\infty} P_k\Psi_{j-1}(t_k+).$$

From the hypothesis (3.2) there exists $t_* > 0$ such that

$$\int_{t_*}^{\infty} \|P(s)\|\Delta s + \sum_{k=\underline{n}(t_*)}^{\infty} \|P_k\| < \epsilon$$

for a fixed $\epsilon \in (0,1)$. Then, by using the mathematical induction it can be shown that

$$\|\Psi_j(t)\| < \epsilon^j,\ j \in \mathbb{N}.$$

Thus, $\sum_{j=1}^{\infty} \Psi_j(t)$ converges uniformly on the interval $[t_*,\infty)_{\mathbb{T}}$. Setting $\Psi(t) = \sum_{j=1}^{\infty} \Psi_j(t)$ we obtain

$$\Psi(t) = -\int_t^{\infty} P(s)(I + \Psi(s))\Delta s - \sum_{k=\underline{n}(t)}^{\infty} P_k(I + \Psi(t_k+))$$

which shows that $\Psi(t)$ is a solution of (3.3) and $\lim_{t\to\infty} \Psi(t) = 0$. □

Theorem 3.2. *Suppose that* (3.2) *holds, and*

$$\lim_{t\to\infty} \phi(t,t_0)\Psi(t) = 0. \tag{3.4}$$

Then, the impulsive dynamic equations (2.3) *and* (2.4) *are asymptotically equivalent.*

Proof. Fix a sufficiently large t_* so that $u(t) = (I + \Psi(t))c$, where $c \in \mathbb{R}^n$ is a solution of (3.1) and hence $y(t) = \phi(t,t_0)u(t)$ is a solution of (2.3) for $t \geq t_*$. Since $\Psi(t) \to 0$ as $t \to \infty$, there exists $t'_* > t_*$ such that $I + \Psi(t'_*)$ is nonsingular. Define $y^0 = \phi(t'_*, t_0)(I + \Psi(t'_*))c$, $x^0 = \phi(t'_*, t_0)c$, and denote $y(t,c) = y(t, t'_*, y^0)$ and $x(t,c) = x(t, t'_*, x^0)$ to be solutions of (2.3) and (2.4), respectively. Since $I + \Psi(t'_*)$ is nonsingular, and there exists a unique solution of linear impulsive dynamic equation with an initial condition, the equation

$$y^0 = \phi(t'_*, t_0)(I + \Psi(t'_*))\phi(t_0, t'_*)x^0$$

gives an isomorphism between the solutions of (2.3) and (2.4) such that $y(t) = x(t) + \phi(t,t_0)\Psi(t)c$ for $t \in [t_0,\infty)_{\mathbb{T}}$. Hence, from (3.4) it follows that

$$\lim_{t\to\infty} [y(t) - x(t)] = 0.$$

□

Remark 3.3. If we take $\mathbb{T} = \mathbb{R}$ in Theorem 3.2, we recover [1, Theorem 1]. If we drop the impulse effects, i.e., if we put $A_k = B_k = 0$ we get [15, Theorem 2]. Finally, by dropping the impulse effects and setting $\mathbb{T} = \mathbb{Z}$, the above result is reduced to [24, Theorem 2.2]. Therefore, we have extended the theorems mentioned here to dynamic equations on arbitrary time scales with impulse effects.

Remark 3.4. Let $\mathbb{T} = \mathbb{R}$. If $A(t) \equiv A$ and $A_k \equiv \tilde{A}$ are constant and commutative matrices, the state transition matrix of (2.4) becomes

$$\phi(t,t_0) = (I + \tilde{A})^{\overline{n}(t)} e^{A(t-t_0)} = e^{\overline{n}(t)\ln(I+\tilde{A}) + A(t-t_0)}.$$

Thus, $P(t)$ and P_k can be written in more concrete forms.

4. Asymptotic equivalence of linear and quasilinear dynamic equations

Let C and C_0 be constant, $n \times n$ and commutative matrices, where C is regressive and $\det(C_0 + I) \neq 0$, $f(t,y) \in PLC_{rd}([t_0,\infty)_{\mathbb{T}} \times \mathbb{R})$ and $f_k(y) \in PLC(\mathbb{R})$ for each $k \in \mathbb{Z}$. In a similar manner to the previous section, a solution of (2.5) or (2.6) is defined to be a function $\varphi \in Y$ satisfying the equation under consideration. We assume without further mention that $f(t,0) = f_k(0) = 0$.

Lemma 4.1. *Suppose that there exists a function* $\eta(t) \in PLC_{rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{R}^+)$ *and a positive sequence* η_k *such that*

$$|f(t,x_1) - f(t,x_2)| \leq \eta(t)|x_1 - x_2|, \quad |f_k(x_1) - f_k(x_2)| \leq \eta_k|x_1 - x_2| \tag{4.1}$$

and

$$\int_{t_0}^{\infty} \|e_C(t_0,\sigma(t))(C_0+I)^{-\overline{n}(\sigma(t))}\| \; \|(C_0+I)^{\overline{n}(t)} e_C(t,t_0)\| \; \eta(t)\Delta t$$
$$+ \sum_{k=\underline{n}(t_0)}^{\infty} \|e_C(t_0,t_k)(C_0+I)^{-k}\| \; \|(C_0+I)^{k-1} e_C(t_k,t_0)\|\eta_k < \infty. \tag{4.2}$$

Then, every solution of

$$\begin{cases} u^{\Delta} = e_C(t_0,\sigma(t))(C_0+I)^{-\overline{n}(\sigma(t))} f(t,(C_0+I)^{\overline{n}(t)} e_C(t,t_0)u), & t \neq t_k, \\ \Delta u = e_C(t_0,t_k)(C_0+I)^{-k} f_k((C_0+I)^{k-1} e_C(t_k,t_0)u), & t = t_k \end{cases} \tag{4.3}$$

is bounded on $[t_0, \infty)_{\mathbb{T}}$ *and is asymptotically constant. Namely, for each solution* u *of* (4.3) *there exists a constant vector* $c_u \in \mathbb{R}^n$ *such that* $u(t) \to c_u$ *as* $t \to \infty$*.*

Proof. Let $u(t) = u(t, t_0, u_0)$ be the solution of (4.3) satisfying $u(t_0) = u_0$ for some $t_0 \geq 0$. Then, $u(t)$ can be written as

$$u(t) = u_0 + \int_{t_0}^{t} e_C(t_0, \sigma(s))(C_0 + I)^{-\overline{n}(\sigma(s))} f(s, (C_0 + I)^{\overline{n}(s)} e_C(s, t_0) u(s)) \Delta s + \sum_{k=\underline{n}(t_0)}^{\overline{n}(t)} \Big[e_C(t_0, t_k)(C_0 + I)^{-k} f_k((C_0 + I)^{k-1} e_C(t_k, t_0) u(t_k)) \Big]. \tag{4.4}$$

By using (4.1) it follows that

$$|u(t)| \leq |u_0| + \int_{t_0}^{t} \|e_C(t_0, \sigma(s))(C_0 + I)^{-\overline{n}(\sigma(s))}\| \, \|(C_0 + I)^{\overline{n}(s)} e_C(s, t_0)\| \, |u(s)| \eta(s) \Delta s + \sum_{k=\underline{n}(t_0)}^{\overline{n}(t)} \Big[\|e_C(t_0, t_k)(C_0 + I)^{-k}\| \, \|(C_0 + I)^{k-1} e_C(t_k, t_0)\| \, |u(t_k)| \eta_k \Big]. \tag{4.5}$$

Then, by an application of Gronwall's inequality and the hypothesis (4.2), we see that $u(t)$ is bounded. This allows us to define

$$c_u := u_0 + \int_{t_0}^{\infty} e_C(t_0, \sigma(s))(C_0 + I)^{-\overline{n}(\sigma(s))} f(s, (C_0 + I)^{\overline{n}(s)} e_C(s, t_0) u(s)) \Delta s + \sum_{k=\underline{n}(t_0)}^{\infty} \big[e_C(t_0, t_k)(C_0 + I)^{-k} f_k((C_0 + I)^{k-1} e_C(t_k, t_0) u(t_k)) \big].$$

From (4.4) we observe that $u(t) \to c_u$ as $t \to \infty$. Thus, u is asymptotically constant. □

By the following lemma we give the relation between solutions of equations (2.5) and (4.3).

Lemma 4.2. $y(t) = (C_0 + I)^{\overline{n}(t)} e_C(t, t_0) u(t)$ *is a solution of* (2.5) *if and only if* $u(t)$ *is a solution of* (4.3)*.*

Proof. Let $u(t)$ be a solution of (4.3). If $t \neq t_k$, by using the product rule we have

$$\begin{aligned} y^{\Delta}(t) =& (C_0 + I)^{\overline{n}(t)} C e_C(t, t_0) u(t) + (C_0 + I)^{\overline{n}(\sigma(t))} e_C(\sigma(t), t_0) u^{\Delta}(t) \\ =& C y(t) + (C_0 + I)^0 e_C(\sigma(t), \sigma(t)) f(t, y(t)). \end{aligned}$$

Thus, $y(t)$ solves the dynamic equation $y^{\Delta}(t) = Cy(t) + f(t, y(t))$ where $t \neq t_k$. For $t = t_k$, $k = 1, 2, \ldots$ one has $y(t_k) = (C_0 + I)^{k-1} e_C(t_k, t_0) u(t_k)$. Then, from continuity of the exponential matrix $e_C(.,.)$, clearly

$$\begin{aligned} \Delta y|_{t=t_k} =& (C_0 + I)^k e_C(t_k+, t_0)[u(t_k) + e_C(t_0, t_k)(C_0 + I)^{-k} f_k(y(t_k))] - y(t_k) \\ =& [C_0 + I] y(t_k) + f_k(y(t_k)) - y(t_k). \end{aligned}$$

Hence, $y(t)$ is a solution of (2.5).

The sufficiency can be shown similarly. So, we skip the remaining part of the proof. □

Corollary 4.3. *From* (4.2) *one can write that*

$$\int_{t_0}^{t} e_C(t_0,\sigma(s))(C_0+I)^{-\overline{n}(\sigma(s))} f(s,(C_0+I)^{\overline{n}(s)} e_C(s,t_0)u(s))\Delta s$$
$$+\sum_{k=\underline{n}(t_0)}^{\overline{n}(t)} \Big[e_C(t_0,t_k)(C_0+I)^{-k} f_k((C_0+I)^{k-1} e_C(t_k,t_0)u(t_k))\Big] = o(1), \quad t\to\infty.$$

Then, using Lemma 4.2 it can be seen that every solution $y(t)$ *of* (2.5) *satisfies the asymptotic representation*

$$y(t) = (C_0+I)^{\overline{n}(s)} e_C(t,t_0)[c_u + o(1)], \quad t\to\infty.$$

We are now in a position to prove the asymptotic equivalence of the linear and quasilinear impulsive dynamic equations (2.5) and (2.6).

Theorem 4.4. *Suppose that* (4.1) *and* (4.2) *hold, and the matrices* $e_C(t,\tau)$ *and* C_0 *are commutative. If*

$$\lim_{t\to\infty}\Bigg\{\int_t^\infty \|e_C(t,\sigma(s))(C_0+I)^{\overline{n}(t)-\underline{n}(\sigma(s))}\|\ \|(C_0+I)^{\overline{n}(s)} e_C(s,t_0)\|\ \eta(s)\Delta s$$
$$+\sum_{k=\underline{n}(t)}^{\infty} \|e_C(t,t_k)(C_0+I)^{\overline{n}(t)-k}\|\ \|(C_0+I)^{k-1} e_C(t_k,t_0)\|\eta_k\Bigg\} = 0, \tag{4.6}$$

then (2.5) *and* (2.6) *are asymptotically equivalent.*

Proof. Let $y(t)$, $x(t)$ and $u(t)$ be solutions of (2.5), (2.6) and (4.3), respectively. Then, by means of Lemma 4.1 and Lemma 4.2 one has

$$y(t) = (C_0+I)^{\overline{n}(t)} e_C(t,t_0)$$
$$\times\Bigg\{c_u - \int_t^\infty e_C(t_0,\sigma(s))(C_0+I)^{-\overline{n}(\sigma(s))} f(s,(C_0+I)^{\overline{n}(s)} e_C(s,t_0)u(s))\Delta s$$
$$-\sum_{k=\underline{n}(t)}^{\infty} [e_C(t_0,t_k)(C_0+I)^{-k} f_k((C_0+I)^{k-1} e_C(t_k,t_0)u(t_k))]\Bigg\},$$

where c_u is as defined in Lemma 4.1. Any solution of (2.6) is of the form

$$x(t) = e_C(t,t_0)(C_0+I)^{\overline{n}(t)} c_u.$$

Thus, from (4.1) we can write

$$|y(t)-x(t)| \le \int_t^\infty \|e_C(t,\sigma(s))(C_0+I)^{\overline{n}(t)-\overline{n}(\sigma(s))}\|\ \|(C_0+I)^{\overline{n}(s)} e_C(s,t_0)\|\ |u(s)|\eta(s)\Delta s$$
$$+\sum_{k=\underline{n}(t)}^{\infty} \|e_C(t,t_k)(C_0+I)^{\overline{n}(t)-k}\|\|(C_0+I)^{k-1} e_C(t_k,t_0))\|\ |u(t_k)|\eta_k. \tag{4.7}$$

Applying Gronwall's inequality for piecewise continuous functions, from the hypothesis (4.6), we obtain

$$\lim_{t\to\infty}[y(t)-x(t)] = 0.$$

□

Remark 4.5. If the matrices C and C_0 are replaced by nonconstant matrices $C(t)$ and $C_0(t)$, respectively, still Theorem 4.4 is valid.

Corollary 4.6. *For $\mathbb{T} = \mathbb{R}$, the hypotheses* (4.2) *and* (4.6) *turn into*

$$\int_{t_0}^{\infty} \|e^{-\Lambda(t,t_0)}\| \ \|e^{\Lambda(t,t_0)}\|\eta(t)\mathrm{d}t + \sum_{k=\underline{n}(t_0)}^{\infty} \|e^{-\Lambda(t_k,t_0)}(C_0+I)^{-1}\| \ \|e^{\Lambda(t_k,t_0)}\|\eta_k < \infty \tag{4.8}$$

and

$$\lim_{t\to\infty}\left\{ \int_{t}^{\infty} \|e^{\Lambda(t,s)}(C_0+I)^{-1}\| \ \|e^{\Lambda(s,t_0)}\|\eta(s)\mathrm{d}s + \sum_{k=\underline{n}(t)}^{\infty} \|e^{\Lambda(t,t_k)}(C_0+I)^{\overline{n}(t)-1}\|\|e^{\Lambda(t_k,t_0)}\| \ \eta_k \right\} = 0 \tag{4.9}$$

with $\Lambda(t,\tau) = C(t-\tau) + [\overline{n}(t) - \underline{n}(\tau) + 1]\ln(C_0+I)$ and $\Lambda(t_k,\tau) = C(t_k-\tau) + (k-1)\ln(C_0+I)$, where we naturally assume that $\underline{n}(t_0) = 1$ and $\overline{n}(t) = \underline{n}(t)+1$. If (4.8) *and* (4.9) *hold, by Theorem 4.4 we conclude that the system of quasilinear impulsive differential equations*

$$\begin{cases} y' = Cy + f(t,y), & t \neq t_k, \\ \Delta y = C_0 y + f_k(y), & t = t_k, \end{cases} \tag{4.10}$$

and the system of linear impulsive differential equations

$$\begin{cases} x' = Cx, & t \neq t_k, \\ \Delta x = C_0 x, & t = t_k \end{cases} \tag{4.11}$$

are asymptotically equivalent.

We should note that it may not be easy to calculate the matrix exponential $e_C(t,t_0)$ and the matrix power $(C_0+I)^j$. Thereby, we present a theorem with more practical conditions as an alternative method.

Let us assume that λ_j, $j = 1,2,\dots,l$ are eigenvalues of C with multiplicities m_j in the minimal polynomial and denote $m_0 = \max\{m_1, m_2, \dots m_l\}$. Suppose also that $\exists M_0 > 0$ and $\mu_0 \in \mathbb{R}$ such that $(1+\lambda_j\mu(t))^{-1} \leq M_0$ and $(\log|1+\mu(t)\lambda_j|)/\mu(t) \leq \mu_0$, for $t \in [t_0,\infty)_{\mathbb{T}}$. Regarding to C_0 we similarly define α_0, β_0 to be eigenvalues of C_0+I with minimum and maximum real part, and m_{α_0}, m_{β_0} their multiplicities, respectively. Then, we have the following result.

Theorem 4.7. *Suppose that* (4.1) *holds. If*

$$\int_{t_0}^{\infty} F(t)\eta(t)\Delta t + \sum_{k=\underline{n}(t_0)}^{\infty} F_k\eta_k < \infty \tag{4.12}$$

and

$$\lim_{t\to\infty} t^{m_0-1}e^{\mu_0 t} \ \overline{n}(t)^{m_{\beta_0}-1}\beta_0^{\overline{n}(t)}\left[\int_{t}^{\infty} F(s)\eta(s)\Delta s + \sum_{k=\underline{n}(t)}^{\infty} F_k\eta_k\right] = 0, \tag{4.13}$$

where

$$F(t) := \frac{\left(t(\sigma(t))^{n-1}\right)^{m_0-1}e^{\mu_0(n\sigma(t)-\mu(t))}\overline{n}(t)^{m_{\beta_0}-1} \ \overline{n}(\sigma(t))^{m_{\alpha_0}-1}\beta_0^{\overline{n}(t)}}{|e_{tr(C)+\mu(t)\det(C)}(\sigma(t),t_0)|\alpha_0^{\overline{n}(\sigma(t))}},$$

$$F_k := \frac{t_k^{n(m_0-1)}e^{\mu_0 n t_k}k^{m_{\beta_0}+m_{\alpha_0}-2}\beta_0^{k-1}}{|e_{trC}(t_k,t_0)|\alpha_0^k},$$

then, (2.5) *and* (2.6) *are asymptotically equivalent.*

Proof. Since $e_C(t,t_0)$ is $n\times n$, we have

$$\|e_C(t_0,t)\| \le \frac{\|e_C(t,t_0)\|^{n-1}}{|\det(e_C(t,t_0))|}.$$

Then, as it is shown in [15, Lemma 4] via Putzer algorithm on time scales, we can write $|e_{\lambda_j}(t,t_0)| \le e^{\mu_0(t-t_0)}$, which implies that there exists a positive constant K_1 such that

$$\|e_C(t,t_0)\| \le K_1 t^{m_0-1} e^{\mu_0 t} \tag{4.14}$$

and so

$$|e_C(t_0,t)| \le \frac{\left(K_1 t^{m_0-1} e^{\mu_0 t}\right)^{n-1}}{|e_{tr(C)+\mu(t)\det(C)}(t,t_0)|}, \quad t\in[t^*,\infty)_{\mathbb{T}} \tag{4.15}$$

for some $t^*\in[t_0,\infty)_{\mathbb{T}}$. On the other hand, for the powers of C_0+I we can write

$$\|(C_0+I)^k\| \le K_2 k^{m_{\beta_0}-1}\beta_0^k, \qquad \|(C_0+I)^{-k}\| \le K_2 k^{1-m_{\alpha_0}}\alpha_0^{-k}, \tag{4.16}$$

for some $K_2\in\mathbb{R}^+$. Thus, using (4.14), (4.15) and (4.16) in (4.5) we see from (4.12) that the hypothesis (4.2) holds. We further have

$$\begin{aligned}|y(t)-x(t)| \le{}& K t^{m_0-1} e^{\mu_0 t}\ \overline{n}(t)^{m_{\beta_0}-1}\beta_0^{\overline{n}(t)} \\ &\times \Bigg[\int_t^\infty \frac{\left(s(\sigma(s))^{n-1}\right)^{m_0-1} e^{\mu_0(n\sigma(s)-\mu(s))}\overline{n}(s)^{m_{\beta_0}-1}\ \overline{n}(\sigma(s))^{m_{\alpha_0}-1}\beta_0^{\overline{n}(s)}}{|e_{tr(C)+\mu(s)\det(C)}(\sigma(s),t_0)|\alpha_0^{\overline{n}(\sigma(s))}}\eta(s)\Delta s \\ &+ \sum_{k=\underline{n}(t)}^{\infty} \frac{t_k^{n(m_0-1)} e^{\mu_0 n t_k} k^{m_{\beta_0}+m_{\alpha_0}-2}\beta_0^{k-1}}{|e_{trC}(t_k,t_0)|\alpha_0^k}\eta_k\Bigg],\end{aligned}$$

where K is a suitable constant. From (4.13) we conclude that $\lim_{t\to\infty}|y(t)-x(t)|=0$, which means (2.5) and (2.6) are asymptotically equivalent. □

Corollary 4.8. *Let $\mathbb{T}=\mathbb{R}$. Then μ_0 turns out to be the eigenvalue of C with the maximum real part. To have an accordance, let us define α, β to be eigenvalues of C with minimum and maximum real part, and m_α, m_β their multiplicities in the minimal polynomial, respectively. Then, the hypotheses* (4.12) *and* (4.13) *could be replaced with*

$$\begin{aligned}&\int_{t_0}^{\infty} t^{m_\beta+m_\alpha-2} e^{(\beta-\alpha)t}\ \overline{n}(t)^{m_{\beta_0}+m_{\alpha_0}-2}\left(\frac{\beta_0}{\alpha_0}\right)^{\overline{n}(t)}\eta(t)\mathrm{d}t \\ &+ \sum_{k=\underline{n}(t_0)}^{\infty} t_k^{m_\beta+m_\alpha-2} e^{(\beta-\alpha)t_k}\ k^{m_{\beta_0}+m_{\alpha_0}-2}\beta_0^{-1}\left(\frac{\beta_0}{\alpha_0}\right)^k \eta_k < \infty,\end{aligned}$$

and

$$\begin{aligned}\lim_{t\to\infty} e^{\alpha t}\ \overline{n}(t)^{m_{\beta_0}-1}\beta_0^{\overline{n}(t)}\Bigg[&\int_t^\infty (s-t)^{m_\alpha-1}s^{m_\beta-1}e^{(\beta-\alpha)s}\ \overline{n}(s)^{m_{\beta_0}+m_{\alpha_0}-2}\left(\frac{\beta_0}{\alpha_0}\right)^{\overline{n}(s)}\eta(s)\mathrm{d}s \\ &+ \sum_{k=\underline{n}(t)}^{\infty}(t_k-t)^{m_\alpha-1}t_k^{m_\beta-1}e^{(\beta-\alpha)t_k}\ k^{m_{\beta_0}+m_{\alpha_0}-2}\beta_0^{-1}\left(\frac{\beta_0}{\alpha_0}\right)^k\eta_k\Bigg]=0.\end{aligned}$$

Thus, under the above conditions, the impulsive differential equations (4.10) *and* (4.11) *are asymptotically equivalent.*

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Remark 4.9. We recover two former results by Corollary 4.8 and Theorem 4.4. Namely, if we take $C_0=0$ in Corollary 4.8, we obtain [2, Theorem 2], and when we both take $C_0=0$ and drop the impulse effects, Theorem 4.4 reduces to [15, Theorem 5].

5. Examples

In this section, we present some examples to show that our results are applicable. In the first example, we consider a linearly coupled impulsive equation. Such equations are frequently used in modeling many real phenomena such as neural networks, see for example [17] and the references cited therein.

Example 5.1. Let $\mathbb{T} = \mathbb{R}$ and consider the linear system of impulsive equations

$$\begin{cases} y_1^\Delta = e^{-t}y_1 + y_2, \\ y_2^\Delta = -2t^{-2}y_1 + 2t^{-1}y_2, \end{cases} \quad t \neq t_k, \qquad \begin{cases} \Delta y_1 = -\frac{1}{(k+1)^2}y_1, \\ \Delta y_2 = \Big(-\frac{1}{4k^2} + \frac{1}{2^k}\Big)y_2, \end{cases} \quad t = t_k.$$

Letting $t_k = k$, we may rewrite the above system as

$$\begin{cases} y^\Delta = \begin{bmatrix} 0 & 1 \\ -2t^{-2} & 2t^{-1} \end{bmatrix} y + \begin{bmatrix} e^{-t} & 0 \\ 0 & 0 \end{bmatrix} y, & t \neq k, \\ \Delta y = \begin{bmatrix} -\frac{1}{(k+1)^2} & 0 \\ 0 & -\frac{1}{4k^2} \end{bmatrix} y + \begin{bmatrix} 0 & 0 \\ 0 & 2^{-k} \end{bmatrix} y, & t = k \end{cases} \tag{5.1}$$

and consider the associated impulsive system

$$x^\Delta = \begin{bmatrix} 0 & 1 \\ -2t^{-2} & 2t^{-1} \end{bmatrix} x, \quad t \neq k, \qquad \Delta x = \begin{bmatrix} -\frac{1}{(k+1)^2} & 0 \\ 0 & -\frac{1}{4k^2} \end{bmatrix} x, \quad t = k. \tag{5.2}$$

Let $t_0 = 1$. By a simple calculation it can be confirmed that

$$\phi(t,1) = \begin{bmatrix} (2t-t^2)\prod\limits_{k=1}^{\overline{n}(t)} \frac{k^2+2k}{(k+1)^2} & (t^2-t)\prod\limits_{k=1}^{\overline{n}(t)} \frac{k^2+2k}{(k+1)^2} \\ (2-2t)\prod\limits_{k=1}^{\overline{n}(t)} \frac{4k^2-1}{4k^2} & (2t-1)\prod\limits_{k=1}^{\overline{n}(t)} \frac{4k^2-1}{4k^2} \end{bmatrix},$$

where $\phi(t)$ is the fundamental matrix of (5.2). Then

$$P(t) = \phi(1,t)B(t)\phi(t,1) = \begin{bmatrix} \frac{(2-t)(2t-1)}{te^t} & \frac{(t-1)(2t-1)}{te^t} \\ \frac{2(2-t)(t-1)}{te^t} & \frac{2(t-1)^2}{te^t} \end{bmatrix}$$

and

$$P_k = \phi(1,k)(I+A_k)^{-1}B_k\phi(t_k,1) = 2^{-k}\frac{4k}{4k^2-1}\begin{bmatrix} 2(1-k)^2 & (1-k)(2k-1) \\ 2(2-k)(1-k) & (2-k)(2k-1) \end{bmatrix}.$$

Hence, there exist positive constants c_1 and c_2 such that $\|P(t)\| \le c_1te^{-t}$, $\|P_k\| \le c_2k2^{-k}$ which imply the estimate $\|\Psi(t)\| \le Ke^{k2^{-k}-(t+1)e^{-t}} - 1$ for some $K > 0$. Consequently, we have

$$\lim_{t\to\infty} \phi(t,1)\Psi(t) \to 0, \quad t \to \infty.$$

So, by Theorem 3.2, the equations (5.1) and (5.2) are asymptotically equivalent.

Example 5.2. Let $\mathbb{T} = \mathbb{P}_{1,1} = \cup_{j=0}^{\infty}[2j, 2j+1]$. We consider the nonlinear impulsive dynamic system

$$\begin{cases} y^\Delta = \begin{bmatrix} -1 & -2 \\ -2 & -1 \end{bmatrix} y + \begin{bmatrix} y_1 e^{-4t-y_2^2} \\ e^{-4t}\arctan y_2 \end{bmatrix}, & t \neq 2k, \\ \Delta y = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} y + \begin{bmatrix} 2^{-9k}\sin y_1 \\ 2^{-9k}(\cos^2 y_2 - 1) \end{bmatrix}, & t = 2k. \end{cases} \tag{5.3}$$

Clearly, (4.1) holds with $\eta(t) = e^{-4t}$ and $\eta_k = 2^{-9k}$. By direct calculation we have

$$e_C(t,0) = 2^{j-2}\begin{bmatrix} (-1)^j e^{-3(t-j)} + e^{t-j} & (-1)^j e^{-3(t-j)} - 3e^{t-j} \\ (-1)^j e^{-3(t-j)} - e^{t-j} & (-1)^j e^{-3(t-j)} + 3e^{t-j} \end{bmatrix}$$

and

$$(C_0+I)^{\overline{n}(t)} = \frac{1}{2}\begin{bmatrix} 1+(-1)^{\overline{n}(t)} & 1-(-1)^{\overline{n}(t)} \\ 1-(-1)^{\overline{n}(t)} & 1+(-1)^{\overline{n}(t)} \end{bmatrix}.$$

Clearly $e_C(t,0)$ and C_0 are commutative. Now, we may check

$$\begin{aligned}
&\int_0^\infty \|e_C(0,\sigma(t))(C_0+I)^{-\overline{n}(\sigma(t))}\| \; \|(C_0+I)^{\overline{n}(t)}e_C(t,0)\| \; \eta(t)\Delta t \\
&+\sum_{k=1}^{\infty} \|e_C(0,t_k)(C_0+I)^{-k}\| \; \|(C_0+I)^{k-1}e_C(t_k,0)\|\eta_k \\
&\le 32e^6\Big(\int_0^\infty e^{-2t}\Delta t + \sum_{k=1}^{\infty}\Big(\frac{e}{4}\Big)^{4k}\Big) \\
&= 32e^6\Big(\sum_{j=0}^{\infty}\int_{2j}^{2j+1} e^{-2t}\mathrm{d}t + \sum_{j=0}^{\infty} e^{-4j-2} + \sum_{k=1}^{\infty}\Big(\frac{e}{4}\Big)^{4k}\Big).
\end{aligned}$$

The last expression is easily seen to be finite, thus (4.2) holds. Finally, in view of the fact $t\in[2j,2j+1]$ and $k=t/2$ we have the following estimates

$$\begin{aligned}
&\int_t^\infty \|e_C(t,\sigma(s))(C_0+I)^{\overline{n}(t)-\underline{n}(\sigma(s))}\| \; \|(C_0+I)^{\overline{n}(s)}e_C(s,0)\| \; \eta(s)\Delta s \\
&+\sum_{k=\underline{n}(t)}^{\infty} \|e_C(t,t_k)(C_0+I)^{\overline{n}(t)-k}\| \; \|(C_0+I)^{k-1}e_C(t_k,0)\|\eta_k \\
&\le c_1 e^{\frac{t}{2}}\Big(\int_t^\infty e^{-2s}\Delta s + \sum_{k=\underline{n}(t)}^{\infty}\Big(\frac{e}{4}\Big)^{4k}\Big) \\
&\le c_1\Big(e^{\frac{-3t}{2}} + \Big(\frac{e^2}{16}\Big)^{t}\Big).
\end{aligned}$$

Since the above expression converges to zero as t tends to infinity, the hypothesis (4.6) is satisfied as well. Therefore, by Theorem 4.4, the equation (5.3) is asymptotically equivalent to the corresponding homogeneous impulsive dynamic equation

$$x^\Delta = \begin{bmatrix} -1 & -2 \\ -2 & -1 \end{bmatrix} x, \quad t\neq 2k, \qquad \Delta x = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} x, \quad t=2k. \tag{5.4}$$

Example 5.3. Let $\mathbb{T}=\mathbb{P}_{1-\delta,\delta}=\cup_{j=0}^{\infty}[j,j+1-\delta]$ with $0<\delta<1$ and consider the impulsive dynamic system

$$\begin{cases} y^\Delta = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{bmatrix} y + e^{-13t}\begin{bmatrix} y_1(1+t^2+y_3^2)^{-2} \\ \ln(1+y_2^2) \\ y_3 \end{bmatrix}, & t\neq k, \\ \Delta y = \begin{bmatrix} 4 & 4 & 2 \\ 0 & 0 & -1 \\ -1 & -1 & 2 \end{bmatrix} y + \begin{bmatrix} y_1e^{1-10k} \\ e^{-10k}\sin(ky_2) \\ 4^{-8k}y_3(1+y_1^2)^{-1} \end{bmatrix}, & t=k. \end{cases} \tag{5.5}$$

Then, $\sigma(t)=\begin{cases} t, & t\in\cup_{j=1}^{\infty}[j,j+1-\delta) \\ t+\delta & t\in\cup_{j=1}^{\infty}\{j+1-\delta\} \end{cases}$, and so $\mu(t)=\begin{cases} 0, & t\in\cup_{j=1}^{\infty}[j,j+1-\delta) \\ \delta & t\in\cup_{j=1}^{\infty}\{j+1-\delta\} \end{cases}$.
The eigenvalues of the matrix C are 1, 2 and -1. So, $m_0=1$ and $\mu_0=\ln(1+2\delta)/\delta\le 2$

if $\mu(t) = \delta$ while $\mu_0 = 1$ if $\mu(t) = 0$. Moreover, $\alpha_0 = 1$ and $\beta_0 = 4$ are the eigenvalues of $C_0 + I$ with multiplicities $m_{\alpha_0} = 1$ and $m_{\beta_0} = 2$. Thus, we have

$$F(t) \leq \frac{e^{2(3(t+\delta)-\delta)}\overline{n}(t)4^{\overline{n}(t)}}{e_{2-2\delta}(t+\delta,1)} \leq K_1 e^{6t} t4^t, \quad F_k \leq \frac{e^{6k}k4^{k-1}}{e_2(k,1)} \leq K_2 e^{4k} k4^k$$

which hold true for $\mu(t) = 0$ as well. Observing that (4.1) holds with $\eta(t) = e^{-13t}$ and $\eta(k) = e^{-10k}$ and taking $t_0 = 1$ we compute

$$\begin{aligned}\int_1^\infty F(t)\eta(t)\Delta t + \sum_{k=1}^\infty F_k\eta_k \\ \leq \sum_{k=1}^\infty \Big(\int_k^{k+1-\delta} e^{-7t} t4^t \mathrm{d}t + e^{-7(k+1-\delta)}(k+1-\delta)4^{(k+1-\delta)} + e^{-6k}k4^k\Big) \quad (5.6)\end{aligned}$$

which is clearly finite. We further have

$$e^{2t}t4^t\Big(\int_t^\infty F(s)\eta(s)\Delta s + \sum_{k=\underline{n}(t)}^\infty F_k\eta_k\Big) \leq t^2\Big[e^{-t} + \Big(\frac{2}{e}\Big)^{4t}\Big] \to 0, \quad t\to\infty. \quad (5.7)$$

From (5.6) and (5.7), we see that the hypotheses (4.12) and (4.13) hold. Since all the conditions of Theorem 4.7 are fulfilled, we conclude that the equation (5.5) is asymptotically equivalent to the corresponding homogeneous impulsive dynamic system

$$y^\Delta = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{bmatrix} y, \quad t \neq k, \qquad \Delta y = \begin{bmatrix} 4 & 4 & 2 \\ 0 & 0 & -1 \\ -1 & -1 & 2 \end{bmatrix} y, \quad t = k. \quad (5.8)$$

Next, we examine an impulsive dynamic Duffing equation with discontinuous solutions in which the cubic nonlinear term of the classical Duffing equation is replaced by a general nonlinear function. Such equations are used to model many phenomena in several areas, mainly in mechanics and biology, see [16] and the references therein.

Example 5.4. Let $\mathbb{T} = \mathbb{R}$, and consider the impulsive equation

$$\begin{cases} y^{\Delta\Delta} - 2y^\Delta + 2y = \frac{\sin(yt)}{t^4+1}, & t \neq k \\ \Delta y - y = \arctan\left(\frac{y}{t^2}\right), & t = k \\ \Delta y^\Delta - y^\Delta = y^\Delta \ln(1 + 1/(t^2+y^2)), & t = k. \end{cases} \quad (5.9)$$

Letting $y_1 = y$ and $y_2 = y^\Delta$, the equation (5.9) can be reduced to the first order system

$$\begin{cases} y^\Delta = \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix} y + \begin{bmatrix} 0 \\ \sin(y_1 t)/(t^4+1) \end{bmatrix}, & t \neq k, \\ \Delta y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} y + \begin{bmatrix} \arctan\left(\frac{y_1}{t^2}\right) \\ y_2 \ln(1 + 1/(t^2 + y_1^2)) \end{bmatrix}, & t = k. \end{cases} \quad (5.10)$$

Let $t_0 = 1$. Then, we calculate the matrix exponential

$$e_C(t,s) = e^{t-s}\begin{bmatrix} \cos(t-s) & \sin(t-s) \\ -\sin(t-s) & \cos(t-s) \end{bmatrix}$$

and the matrix power $(C_0 + I)^{\overline{n}(t)} = 2^{\overline{n}(t)} I_{2\times 2}$. The hypotheses in (4.1) hold with $\eta(t) = t/(1+t^4)$ and $\eta_k = 1/k^2$. It is not difficult to confirm that (4.2) and (4.6) hold.

So, from Theorem 4.4, we conclude that the equation (5.10) is asymptotically equivalent to the associated system

$$y^\Delta = \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix} y, \quad t \neq k, \qquad \Delta y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} y, \quad t = k.$$

Remark 5.5. Note that neither of the equations (5.2), (5.4), (5.8) nor (5.10) have bounded solutions. So, none of the former results can be applied there even for $\mathbb{T} = \mathbb{R}$.

6. Conclusion

In this paper, the asymptotic equivalence of the two linear dynamic equations on time scales (2.3) and (2.4) was obtained, extending the results for non-impulsive dynamic equations to impulsive case, and former results for impulsive differential equations on $\mathbb{R}$ to arbitrary time scales. The asymptotic equivalence was proved under less restrictive conditions, unlike the already existing literature. Namely, under weak conditions on the nonlinear functions $f(t, y)$, $f_k(y)$ and the coefficient matrices, asymptotic equivalence between the quasilinear impulsive dynamic equation (2.6) and the corresponding homogeneous linear equation (2.6) was proved as well. As an alternative to Theorem 4.4, due to potential difficulties that can be encountered while applying it, Theorem 4.7 with more practical conditions was introduced. It is clearly seen in the stated corollaries that Theorem 4.4 and Theorem 4.7 are new even for the impulsive differential equations defined on $\mathbb{R}$. Thus, the obtained results are not only the time scale extensions of the previous studies but also improvements for impulsive differential equations defined on $\mathbb{R}$. Applications were also provided including a general nonlinear Duffing equation, and a linearly coupled impulsive dynamic equation.

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References

[1] M. U. Akhmet and M. A. Tleubergenova, *On Asymptotic Equivalence of Impulsive Linear Homogenous Differential Systems*, Mat. Zh. **2** (4), pp 15, 2008.

[2] M. U. Akhmet and M. A. Tleubergenova, *Asymptotic Equivalence of a Quasilinear Impulsive Differential Equation and a Linear Ordinary Differential Equation*, Miskolc Math. Notes **8** (2), 117-121, 2007.

[3] D. D. Bainov, A. B. Dishliev and I. M. Stamova, *Asymptotic Equivalence of a Linear System of Impulsive Differential Equations and a System of Impulsive Differential-Difference equations*, Ann. Univ. Ferrara **41**, 45-54, 1995.

[4] D. D. Bainov, S. I. Kostadinov and A. D. Myshkis, *Asymptotic Equivalence of Impulsive Differential Equations in a Banach Space*, Publ. Mat. **34** (2), 249-257, 1990.

[5] D. D. Bainov, S. I. Kostadinov and A. D. Myshkis, *Asymptotic Equivalence of Abstract Impulsive Differential Equations*, Int. J. Theor. Phys. **35**, 383393, 1996.

[6] D. D. Bainov and P. S. Simeonov, *Impulsive Differential Equations: Asymptotic Properties of the Solutions*, World Scientific, 1995.

[7] M. Bohner and D. A. Lutz, *Asymptotic Behavior of Dynamic Equations on Time Scales*, J. Differ. Equ. Appl. **7**, 2150, 2001.

[8] M. Bohner and A. Peterson, *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, 2003.

[9] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Basel, 2001.

[10] M. Bohner and S. G. Georgiev, *Multivariable Dynamic Calculus on Time Scales*, Springer, Switzerland, 2016.

[11] İ. M. Erhan and S. G. Georgiev, *Nonlinear Integral Equations on Time Scales*, Nova Science Publishers, 2019.

[12] S. G. Georgiev, *Integral Equations on Time Scales, Atlantis Studies in Dynamical Systems*, Springer, Newyork, 2016.

[13] S. Hilger, *Analysis on Measure Chains A Unified Approach to Continuous and Discrete Calculus*, Results Math. **18**, 18-56, 1990.

[14] B. Kaymakçalan, V. Lakshmikantham and S. Sivasundaram, *Dynamic Systems on Measure Chains*, Dordrecht, Kluwer, 1996.

[15] B. Kaymakçalan, R. Mert and A. Zafer, *Asymptotic Equivalence of Dynamic Systems on Time Scales*, Discrete Contin. Dyn. Syst. Supplement Volume, 558-567, 2007.

[16] Y. Li and T. Zhang, *On the Existence of Solutions for Impulsive Duffing Dynamic Equations on Time Scales with Dirichlet Boundary Conditions*, Abstr. Appl. Anal. **2010**, 152460, 2010.

[17] J. Lu, D. W. C. Ho, J. Cao and J. Kurths, *Exponential Synchronization of Linearly Coupled Neural Networks With Impulsive Disturbances*, IEEE Trans. Neural Netw. **22** (2), 329-336, 2011.

[18] A. A. Martynyuk, G. T. Stamov and I. Stamova, *Asymptotic Equivalence of Ordinary and Impulsive Operator-Differential Equations*, Commun. Nonlinear Sci. Numer. Simul. **78**, 104891, 2019.

[19] M. Ráb, *Uber Lineare Perturbationen Eines Systems von Linearen Differential- ĺ Gleichungen*, Czech. Math. J. **83**, 222-229, 1958.

[20] M. Ráb, *Note sur les Formules Asymptotiques Pour les Solutions dun Systéme Déquations Différentielles Linéaires*, Czech. Math. J. **91**, 127-129, 1966.

[21] A. M. Samoilenko and N. A. Perestyuk, *Impulsive Differential Equations*, World Scientific, 1995.

[22] P. S. Simeonov and D. D. Bainov, *On the Asymptotic Equivalence of Systems with Impulse Effect*, J. Math. Anal. Appl. **135** (2), 591-610, 1988.

[23] V. A. Yakubovic, *On the Asymptotic Behavior of the Solutions of a System of Differential Equations*, Mat. Sb. (N.S.) **28** (70), 217-240, 1951.

[24] A. Zafer, *On Asymptotic Equivalence of Linear and Quasilinear Difference Equations*, Appl. Anal. **84** (9), 899-908, 2005.