Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 52 (3) (2023), 585 – 595
DOI : 10.15672/hujms.1104784

Research Article

The Stević-Sharma operator on the Lipschitz space into the logarithmic Bloch space

Hamid Vaezi*[1], Soran Mahmoudfakheh[2]

[1] Department of Pure Mathematics, Faculty of Mathematics, Statistics and Computer Sciences, University of Tabriz, Tabriz, Iran

[2] University of Tabriz, Tabriz, Iran

Abstract

In this paper, we study the boundedness and compactness of the Stevic-Sharma operator on the Lipschitz space into the logarithmic Bloch space. Also, we give an estimate for the essential norm of the above operator.

Mathematics Subject Classification (2020). 30H30, 30H99, 47B33.

Keywords. Stević-Sharma operator, Lipschtiz space, logarithmic Bloch space, weighted composition operator.

1. Introduction

Let X and Y be Banach spaces of analytic functions on a domain Ω in $\mathbb{C}$, u an analytic function onΩ and φ be an analytic function mapping Ω into itself. The weighted composition operator with symbols u and φ from X to Y is the operator uC_φ with range in Y defined by

$$uC_\varphi f = M_u C_\varphi f = u(f \circ \varphi), \quad f \in X,$$

where M_u is the multiplication operator with symbol u and C_φ is the composition operator with symbol φ. We refer the interested reader to [7] and [17] for the theory of composition operators and to [6, 13, 18, 20, 21] for (weighted) composition on some spaces of analytic functions. For essential norm of (generalized) weighted composition operators from some spaces of analytic functions into nth weighted type spaces, we refer for example to [1,2,14].

Let $\mathbb{D}$ be the open unit disc in the complex plane $\mathbb{C}$, $H(\mathbb{D})$ the space of analytic functions on $\mathbb{D}$ and $H^\infty = H^\infty(\mathbb{D})$ denote the space of bounded analytic functions f on $\mathbb{D}$ with norm $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|$.

The Lipschitz space Lip_α $(with \quad 0 < \alpha < 1)$ is the space of functions $f \in H(\mathbb{D})$ satisfying the Lipschitz condition of order α, i.e, there exists a constant $C > 0$ such that $|f(z) - f(w)| < C|z - w|^\alpha \quad z, w \in \mathbb{D}$. Such functions f extend continuously to the closure of the disc.

*Corresponding Author.
Email addresses: hvaezi@tabrizu.ac.ir (H. Vaezi), s.mahmod@tabrizu.ac.ir (S. Mahmoudfakheh)
Received: 20.04.2022; Accepted: 09.10.2022

The quantity

$$\|f\|_{Lip_\alpha} = |f(0)| + sup\Big\{\frac{|f(z)-f(w)|}{|z-w|^\alpha}, \quad z,w \in \mathbb{D}, z \neq w\Big\}$$

defines a norm on Lip_α. Let $f \in Lip_\alpha$ and set

$$C = \sup\{\frac{|f(z)-f(w)|}{|z-w|^\alpha}, \quad z,w \in \mathbb{D}, z \neq w\}.$$

Then, for $z \in \mathbb{D}$, we have

$$|f(z)| \leq |f(0)| + C|z|^\alpha < C|z-w|^\alpha \leq \|f\|_{Lip_\alpha}.$$

Thus, taking the supremum over $\mathbb{D}$, we obtain $\|f\|_\infty \leq \|f\|_{Lip_\alpha}$.

A function $f \in H(\mathbb{D})$ is said to belong to the Bloch-type space, denoted by $\mathcal{B}^\alpha$, if $\beta_f = \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f'(z)| < \infty$. Under the seminorm $f \to \beta_f$, $\mathcal{B}^\alpha$ is conformally invariant, and the norm defined by $\|f\|_{\mathcal{B}^\alpha} = |f(0)| + \beta_f$ yields a Banach space structure on $\mathcal{B}^\alpha$.

By a theorem of Hardy and Littlewood [12], the elements of Lip_α are characterized by the following Bloch-type condition: A function $f \in H(\mathbb{D})$ belongs to Lip_α if and only if

$$\alpha(f) = \sup_{z\in\mathbb{D}}(1-|z|^2)^{1-\alpha}|f'(z)| < \infty. \tag{1.1}$$

Moreover,

$$\|f\|_{Lip_\alpha} \asymp |f(0)| + \alpha(f). \tag{1.2}$$

Composition operators uC_φ between Lip_α and Zygmund space were studied by Colonna and Li in [6]. The spaces Lip_α and the Zygmund space play an important role in connection to the theory of the H^p spaces when $0 < p < 1$. For more information on these and other facts regarding these spaces, we refer the interested reader to [8] and [9].

The logarithmic Bloch space is defined as follows [22, 23]:

$$\mathcal{B}_{log} = \Big\{f \in H(D) : \|f\| = \sup_{z\in\mathbb{D}}(1-|z|^2)\log\frac{2}{1-|z|^2}|f'(z)| < \infty\Big\}.$$

The space $\mathcal{B}_{log}$ is a Banach space under the norm $\|f\|_{\mathcal{B}_{log}} = |f(0)| + \|f\|$. The space $\mathcal{B}_{log}$ arises in connection to the study of certain operators with symbols. Arazy in [3] proved that the multiplication operator M_ψ is bounded on the Bloch space if and only if $\psi \in H^\infty \cap \mathcal{B}_{log}$. The space $\mathcal{B}_{log}$ appeared in the study of boundedness of the Hankel operators on the Bergman space. Attele [4] proved that for $f \in L^2_a(\mathbb{D})$, the Hankel operator $H_f : L^1_a(\mathbb{D}) \to L^1(\mathbb{D})$ is bounded if and only if $\|f\|_{\mathcal{B}_{log}} < \infty$. For recent papers on some operators on $\mathcal{B}_{log}$, see, for example, [5, 10, 11, 16, 23]

The composition, multiplication, and differentiation operator on $H(\mathbb{D})$ are defined as follows:

$$\begin{aligned}(C_\varphi f)(z) &= (f \circ \varphi)(z), \quad z \in H(\mathbb{D}),\\ (M_\psi f)(z) &= \psi(z)(f)(z), \quad z \in H(\mathbb{D}),\\ (Df)(z) &= f'(z), \quad z \in H(\mathbb{D}).\end{aligned} \tag{1.3}$$

The differentiation operator is typically unbounded on many analytic function spaces. For $\psi_1, \psi_2 \in H(\mathbb{D})$, let

$$T_{\psi_1,\psi_2,\varphi}f(z) = \psi_1(z)f(\varphi(z)) + \psi_2(z)f'(\varphi(z)), \quad f \in H(\mathbb{D}).$$

The operator $T_{\psi_1,\psi_2,\varphi}$ was studied by Stevic and co-workers for the first time in [18, 24]. This operator is related to the various products of multiplication, composition, and differentiation operators. It is clear that all products of composition, multiplication, and differentiation operators in the following several ways can be obtained from the operator $T_{\psi_1,\psi_2,\varphi}$ by fixing ψ_1, ψ_2.

$$M_\psi C_\varphi D = T_{0,\psi,\varphi}, \quad M_\psi D C_\varphi = T_{0,\psi\varphi',\varphi}, \quad C_\varphi M_\psi D = T_{0,\psi\circ\varphi,\varphi};$$
$$D M_\psi C_\varphi = T_{\psi',\psi\varphi,\varphi}, \quad C_\varphi D M_\psi = T_{\psi'\circ\varphi,\psi\varphi,\varphi}, \quad D C_\varphi M_\psi = T_{\psi'\circ\varphi\varphi',(\psi\circ\varphi)\varphi',\varphi}.$$

The purpose of this paper is to study the boundedness and compactness of the operator $T_{\psi_1,\psi_2,\varphi}$ from Lip_α space to the logarithmic Bloch space $\mathcal{B}_{log}$.

The following criterion for the compactness follows by standard arguments (see, e.g., the proofs of the corresponding lemmas in [7] or [19]. The details will not be pursued here.

Lemma 1.1. *Suppose$\psi_1, \psi_2 \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. Then $T_{\psi_1,\psi_2,\varphi} : Lip_\alpha \to \mathcal{B}_{\log}$ is compact if and only if it is bounded and for any bounded sequence $\{f_n\}$ in Lip_α which converges to zero uniformly on $\overline{\mathbb{D}}$ as $n \to \infty$, $\|T_{\psi_1,\psi_2,\varphi} f_n\|_{\mathcal{B}_{\log}} \to 0$ as $n \to \infty$.*

Constants are denoted by C and K in this paper, they are positive and not necessarily the same in each occurrence. The notation $a \preceq b$ means that there is a positive constant C such that $a \le Cb$. We say that $a \asymp b$ if both $a \preceq b$ and $b \preceq a$ hold.

2. Boundednes of the operator $T_{\psi_1,\psi_2,\varphi} : Lip_\alpha \to \mathcal{B}_{\log}$

In this section we give necessary and sufficient conditions for the boundedness of the operator $T_{\psi_1,\psi_2,\varphi} : Lip_\alpha \to \mathcal{B}_{\log}$.

Theorem 2.1. *Let $\psi_1, \psi_2 \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. The operator $T_{\psi_1,\psi_2,\varphi} : Lip_\alpha \to \mathcal{B}_{\log}$ is bounded if and only if the following quantities are finite:*

$$M_1 = \sup_{z\in\mathbb{D}} (1-|z|^2)\log\frac{2}{1-|z|}|\psi_1'(z)|,$$
$$M_2 = \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)\log\frac{2}{1-|z|}|\psi_1(z)\varphi'(z)+\psi_2'(z)|}{(1-|\varphi(z)|^2)^{1-\alpha}}$$

and

$$M_3 = \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)\log\frac{2}{1-|z|}|\psi_2(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{2-\alpha}}.$$

Proof. Sufficiency. For any $f \in Lip_\alpha$,

$$(1-|z|^2)\log\frac{2}{1-|z|}|(T_{\psi_1,\psi_2,\varphi} f)'(z)| \le (1-|z|^2)\log\frac{2}{1-|z|}|\psi_1'(z)(f(\varphi(z))|$$
$$+ (1-|z|^2)\log\frac{2}{1-|z|}|(\psi_1(z)\varphi'(z)+\psi_2'(z))f'(\varphi(z)|$$
$$+ (1-|z|^2)\log\frac{2}{1-|z|}|(\psi_2(z)\varphi'(z)|)f''(\varphi(z))|$$
$$\le C\|f\|_{Lip_\alpha}\Big((1-|z|^2)\log\frac{2}{1-|z|}|\psi_1'(z)|$$
$$+ \frac{(1-|z|^2)\log\frac{2}{1-|z|}|\psi_1(z)\varphi'(z)+\psi_2'(z)|}{(1-|\varphi(z)|^2)^{1-\alpha}} + \frac{(1-|z|^2)\log\frac{2}{1-|z|}|\psi_2(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{2-\alpha}}\Big) < \infty,$$

where in the last inequality we have used (1.1) and the following well-known characterization of Bloch-type functions (see [25]):

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^{1-\alpha}|f'(z)| \asymp |f'(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)^{2-\alpha}|f''(z)|. \tag{2.1}$$

Necessity. Assume that $T_{\psi_1,\psi_2,\varphi} : Lip_\alpha \to \mathcal{B}$ is bounded. It means that there exists a constant C such that $\|T_{\psi_1,\psi_2,\varphi} f\|_{\mathcal{B}_{\log}} \le C\|f\|_{Lip_\alpha}$ for all $f \in Lip_\alpha$. For $f(z) = 1 \in Lip_\alpha$,

$$M_1 = \sup_{z\in\mathbb{D}}(1-|z|^2)\log\frac{2}{1-|z|}|\psi_1'(z)| < \infty. \tag{2.2}$$

For $f(z) = z \in Lip_\alpha$,

$$M = \sup_{z\in\mathbb{D}}(1-|z|^2)\log\frac{2}{1-|z|}|\psi_1'(z)\varphi(z)+\psi_1(z)\varphi'(z)+\psi_2'(z)| < \infty. \tag{2.3}$$

By (2.2), (2.3), the triangle inequality and the fact that $|\varphi| < 1$, we obtain

$$N_1 = \sup_{z\in\mathbb{D}}(1-|z|^2)\log\frac{2}{1-|z|}|\psi_1(z)\varphi'(z)+\psi_2'(z)| < \infty. \tag{2.4}$$

For $f(z) = z^2 \in Lip_\alpha$,

$$\begin{aligned} N_2 = \sup_{z\in\mathbb{D}}&(1-|z|^2)\log\frac{2}{1-|z|}|\psi_1'(z)\varphi^2(z)+2(\psi_1(z)\varphi'(z)\\ &+\psi_2'(z))\varphi(z)+2\psi_2\varphi'(z)| < \infty. \end{aligned} \tag{2.5}$$

By (2.2), (2.4), (2.5), the triangle inequality and the boundedness of the function $\varphi(z)$, we obtain

$$N_3 = \sup_{z\in\mathbb{D}}(1-|z|^2)\log\frac{2}{1-|z|}|\psi_2(z)\varphi'(z)| < \infty. \tag{2.6}$$

For a fixed $a \in \mathbb{D}$ and for $z \in \mathbb{D}$, set

$$f_a(z) = \frac{-(1-|a|^2)}{(1-\overline{a}z)^{1-\alpha}} + \frac{b(1-|a|^2)^2}{(1-\overline{a}z)^{2-\alpha}} + \frac{c(1-|a|^2)^3}{(1-\overline{a}z)^{3-\alpha}}. \tag{2.7}$$

A direct calculation shows that $f_a \in Lip_\alpha$,

$$\begin{aligned} f_a'(z) = \overline{a}\Big(&\frac{-(1-\alpha)(1-|a|^2)}{(1-\overline{a}z)^{1-\alpha}}\\ &+\frac{b(2-\alpha)(1-|a|^2)^2}{(1-\overline{a}z)^{1-\alpha}}+\frac{c(3-\alpha)(1-|a|^2)^3}{(1-\overline{a}z)^{1-\alpha}}\Big) \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} f_a''(z) = \overline{a}^2\Big(&\frac{-(1-\alpha)(2-\alpha)(1-|a|^2)}{(1-\overline{a}z)^{2-\alpha}}\\ &+\frac{b(2-\alpha)(3-\alpha)(1-|a|^2)^2}{(1-\overline{a}z)^{2-\alpha}}+\frac{c(3-\alpha)(4-\alpha)(1-|a|^2)^3}{(1-\overline{a}z)^{2-\alpha}}\Big). \end{aligned} \tag{2.9}$$

Taking b, c in (2.7) such that $f_w(w) = f_w''(w) = 0$, then $f_a'(a) = \frac{C_1\overline{a}}{(1-|a|^2)^{1-\alpha}}$, where $C_1 \neq 0$. Thus, for $a \in \mathbb{D}$,

$$\begin{aligned} C \geq \|T_{\psi_1,\psi_2,\varphi}f\| &\geq \sup_{z\in\mathbb{D}}(1-|z|^2)\log\frac{2}{1-|z|}|(T_{\psi_1,\psi_2,\varphi}f_{\varphi(a)})'(z)|\\ &= \sup_{z\in\mathbb{D}}(1-|z|^2)\log\frac{2}{1-|z|}\Big(\Big|\psi_1'(z)(f_{\varphi(a)}(\varphi(z))\\ &+(\psi_1(z)\varphi'(z)+\psi_2'(z))f_{\varphi(a)}'(\varphi(z)\\ &+(\psi_2(z)\varphi'(z)|)f_{\varphi(a)}''(\varphi(z))\Big|\Big)\\ &\geq (1-|a|^2)\log\frac{2}{1-|a|}\Big(\Big|\psi_1'(a)(f_{\varphi(a)}(\varphi(a))\\ &+(\psi_1(a)\varphi'(a)+\psi_2'(a))f_{\varphi(a)}'(\varphi(a)\\ &+(\psi_2(a)\varphi'(a)|)f_{\varphi(a)}''(\varphi(a))\Big|\Big)\\ &= \frac{|C||\varphi(a)|(1-|z|^2)\log\frac{2}{1-|a|}|\psi_1(a)\varphi'(a)+\psi_2'(a)|}{(1-|\varphi(a)|^2)^{1-\alpha}}. \end{aligned} \tag{2.10}$$

From (2.10),

$$\begin{aligned}\sup_{\frac{1}{2}<|\varphi(a)|<1} &\frac{(1-|a|^2)\log\frac{2}{1-|a|}|\psi_1(a)\varphi'(a)+\psi_2'(a)|}{(1-|\varphi(a)|^2)^{1-\alpha}}\\ &\le 2\sup_{\frac{1}{2}<|\varphi(a)|<1}\frac{(1-|a|^2)\log\frac{2}{1-|a|}|\psi_1(a)\varphi'(a)+\psi_2'(a)||\varphi(a)|}{(1-|\varphi(a)|^2)^{1-\alpha}}\\ &\le 2\sup_{a\in\mathbb{D}}\frac{(1-|a|^2)\log\frac{2}{1-|a|}|\psi_1(a)\varphi'(a)+\psi_2'(a)||\varphi(a)|}{(1-|\varphi(a)|^2)^{1-\alpha}}\le 2C.\end{aligned}\tag{2.11}$$

According to (2.4),

$$\begin{aligned}\sup_{|\varphi(a)|\le\frac{1}{2}}&(1-|a|^2)\frac{\log\frac{2}{1-|a|}|\psi_1(z)\varphi'(a)+\psi_2'(a)|}{(1-|\varphi(a)|^2)^{1-\alpha}}\\ &\le\frac{4}{3}\sup_{a\in\mathbb{D}}(1-|a|^2)\log\frac{2}{1-|a|}|\psi_1(a)\varphi'(z)+\psi_2'(a)|\le\frac{4}{3}N_1.\end{aligned}\tag{2.12}$$

For a fixed $a\in\mathbb{D}$ and for $z\in\mathbb{D}$, set

$$g_a(z)=\frac{c(1-|a|^2)}{(1-\overline{a}z)^{1-\alpha}}+\frac{d(1-|a|^2)^2}{(1-\overline{a}z)^{2-\alpha}}-\frac{(1-|a|^2)^3}{(1-\overline{a}z)^{3-\alpha}}.\tag{2.13}$$

Then

$$g_a'(z)=\overline{a}\Big(\frac{c(1-\alpha)(1-|a|^2)}{(1-\overline{a}z)^{1-\alpha}}+\frac{d(2-\alpha)(1-|a|^2)^2}{(1-\overline{a}z)^{1-\alpha}}-\frac{(3-\alpha)(1-|a|^2)^3}{(1-\overline{a}z)^{1-\alpha}}\Big)\tag{2.14}$$

and

$$\begin{aligned}g_a''(z)=\overline{a}^2\Big(\frac{c(1-\alpha)(2-\alpha)(1-|a|^2)}{(1-\overline{a}z)^{2-\alpha}}&+\frac{d(2-\alpha)(3-\alpha)(1-|a|^2)^2}{(1-\overline{a}z)^{2-\alpha}}\\ &-\frac{(3-\alpha)(4-\alpha)(1-|a|^2)^3}{(1-\overline{a}z)^{2-\alpha}}\Big).\end{aligned}\tag{2.15}$$

We can take two constants c,d in (2.13) such that $g_w(w)=g_w'(w)=0$. Then

$$g_a''(a)=\frac{C_2\overline{a}^2}{(1-|a|^2)^{2-\alpha}},\tag{2.16}$$

where $C_2\neq 0$. Thus, for $a\in\mathbb{D}$,

$$\begin{aligned}C\ge\|T_{\psi_1,\psi_2,\varphi}f_{\varphi(a)}\|&\ge\sup_{z\in\mathbb{D}}(1-|z|^2)\log\frac{2}{1-|z|}|(T_{\psi_1,\psi_2,\varphi}f_{\varphi(a)})'(z)|\\ &=\sup_{z\in\mathbb{D}}(1-|z|^2)\log\frac{2}{1-|z|}\Big(\Big|\psi_1'(z)(f_{\varphi(a)}(\varphi(z))|\\ &+(1-|z|^2)\log\frac{2}{1-|z|}|(\psi_1(z)\varphi'(z)+\psi_2'(z))f'_{\varphi(a)}(\varphi(z)|\\ &+(1-|z|^2)\log\frac{2}{1-|z|}|(\psi_2(z)\varphi'(z)|)f''_{\varphi(a)}(\varphi(z))\Big|\Big)\\ &\ge(1-|a|^2)\log\frac{2}{1-|a|}\Big(\Big|\psi_1'(a)(f_{\varphi(a)}(\varphi(a))|\\ &+(1-|a|^2)\log\frac{2}{1-|a|}|(\psi_1(a)\varphi'(a)+\psi_2'(a))f'_{\varphi(a)}(\varphi(a)|\\ &+(1-|a|^2)\log\frac{2}{1-|a|}|(\psi_2(a)\varphi'(a)|)f''_{\varphi(a)}(\varphi(a))\Big|\Big)\\ &=\frac{|C_2||\varphi^2(a)|(1-|a|^2)\log\frac{2}{1-|a|}|\psi_2(a)\varphi'(a)|}{(1-|\varphi(a)|^2)^{2-\alpha}}.\end{aligned}\tag{2.17}$$

From (2.17),

$$\begin{aligned}
&\sup_{\frac{1}{2}<|\varphi(a)|<1} \frac{(1-|a|^2)\log\frac{2}{1-|a|}|\psi_2(a)\varphi'(a)|}{(1-|\varphi(a)|^2)^{2-\alpha}} \\
&\leq 4 \sup_{\frac{1}{2}<|\varphi(a)|<1} \frac{|\varphi^2(a)|(1-|z|^2)\log\frac{2}{1-|z|}|\psi_2(z)\varphi'(a)|}{(1-|\varphi(a)|^2)^{2-\alpha}} \qquad (2.18)\\
&\leq 4\sup_{a\in\mathbb{D}}(1-|a|^2)\frac{|\varphi^2(a)|\log\frac{2}{1-|a|}|\psi_2(a)\varphi'(a)|}{(1-|\varphi(a)|^2)^{2-\alpha}} \leq C.
\end{aligned}$$

According to (2.6),

$$\begin{aligned}
&\sup_{|\varphi(a)|\leq\frac{1}{2}} \frac{(1-|z|^2)\log\frac{2}{1-|z|}|\psi_2(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{2-\alpha}} \\
&\leq \frac{4}{3}\sup_{1/2<|\varphi(a)|<1}(1-|z|^2)\log\frac{2}{1-|z|}|\psi_2(z)\varphi'(z)| \qquad (2.19)\\
&\leq \frac{4}{3}\sup_{a\in\mathbb{D}}(1-|z|^2)\log\frac{2}{1-|z|}|\psi_2(z)\varphi'(z)| \leq \frac{4}{3}N_3.
\end{aligned}$$

□

3. Compactness of the operator $T_{\psi_1,\psi_2,\varphi} : Lip_\alpha \to \mathcal{B}_{\log}$

In this section we study the compactness of the operator $T_{\psi_1,\psi_2,\varphi}$ from Lip_α into $\mathcal{B}_{\log}$.

Theorem 3.1. *Let $\psi_1, \psi_2 \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. The operator $T_{\psi_1,\psi_2,\varphi} : Lip_\alpha \to \mathcal{B}_{\log}$ is compact if and only if it is bounded and*

$$\lim_{|\varphi(z)|\to 1}(1-|z|^2)\log\frac{2}{1-|z|}|\psi_1'(z)| = 0, \qquad (3.1)$$

$$\lim_{|\varphi(z)|\to 1}\frac{(1-|z|^2)\log\frac{2}{1-|z|}|\psi_1(z)\varphi'(z)+\psi_2'(z)|}{(1-|\varphi(z)|^2)^{1-\alpha}} = 0 \qquad (3.2)$$

and

$$\lim_{|\varphi(z)|\to 1}\frac{(1-|z|^2)\log\frac{2}{1-|z|}|\psi_2(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{2-\alpha}} = 0. \qquad (3.3)$$

Proof. Suppose that $T_{\psi_1,\psi_2,\varphi} : Lip_\alpha \to \mathcal{B}$ is bounded, (3.1), (3.2) and (3.3) hold. Let $\{f_k\}$ be a bounded sequence in Lip_α which convergence to zero uniformly on $\overline{\mathbb{D}}$ as $k \to \infty$. It suffices, in view of Lemma (1.1), to show that

$$\|T_{\psi_1,\psi_2,\varphi}f_k\|_{\mathcal{B}_{\log}} \to 0, \quad k\to\infty. \qquad (3.4)$$

For any $\epsilon > 0$, there exists $\delta \in (0,1)$ such that if $\delta < |\varphi(z)| < 1$ then

$$(1-|z|^2)\log\frac{2}{1-|z|}|\psi_1'(z)| < \epsilon, \qquad (3.5)$$

$$\frac{(1-|z|^2)\log\frac{2}{1-|z|}|\psi_1(z)\varphi'(z)+\psi_2'(z)|}{(1-|\varphi(z)|^2)^{1-\alpha}} < \epsilon \qquad (3.6)$$

and

$$\frac{(1-|z|^2)\log\frac{2}{1-|z|}|\psi_2(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{2-\alpha}} < \epsilon. \qquad (3.7)$$

From the boundedness of the operator $T_{\psi_1,\psi_2,\varphi} : Lip_\alpha \to \mathcal{B}_{\log}$ and the proof of Theorem (2.1),the relations (2.2), (2.4) and (2.6) hold. Since $f_k \to 0$ uniformly on $\overline{\mathbb{D}}$, Cauchy estimate shows that $\{f'_k\}$ and $\{f''_k\}$ converge to 0 uniformly on compact subsets of $\mathbb{D}$. There exists a $N_0 \in \mathbb{N}$ such that $k \geq N_0$ implies that

$$\begin{aligned}
&\sup_{|\varphi(z)|\leq\delta}(1-|z|^2)\log\frac{2}{1-|z|}|(T_{\psi_1,\psi_2,\varphi}f_k)'(z)| \\
&\leq \sup_{|\varphi(z)|\leq\delta}(1-|z|^2)\log\frac{2}{1-|z|}|\psi_1'(z)(f(\varphi(z))| \\
&+ \sup_{|\varphi(z)|\leq\delta}(1-|z|^2)\log\frac{2}{1-|z|}|(\psi_1(z)\varphi'(z)+\psi_2'(z))f_k'(\varphi(z)| \\
&+ \sup_{|\varphi(z)|\leq\delta}(1-|z|^2)\log\frac{2}{1-|z|}|(\psi_2(z)\varphi'(z)|)f_k''(\varphi(z))| \\
&\leq M\sup_{|\varphi(z)|\leq\delta}\sup f_k\varphi(z) + N_1\sup_{|\varphi(z)|<\delta}\sup f_k'\varphi(z) + N_3\sup_{|\varphi(z)|\leq\delta}\sup f_k''\varphi(z) \leq C\epsilon.
\end{aligned} \tag{3.8}$$

From (3.1), (3.2), (3.3), (3.8),(1.1) and (2.1),

$$\begin{aligned}
&\sup_{z\in\mathbb{D}}(1-|z|^2)\log\frac{2}{1-|z|}|(T_{\psi_1,\psi_2,\varphi}f)'(z)| \leq C\epsilon \\
&+ \sup_{\delta<|\varphi(z)|<1}(1-|z|^2)\log\frac{2}{1-|z|}|\psi_1'(z)\|f\|_{Lip_\alpha} \\
&+ \sup_{\delta<|\varphi(z)|<1}\frac{(1-|z|^2)\log\frac{2}{1-|z|}|\psi_1(z)\varphi'(z)+\psi_2'(z)|}{(1-|\varphi(z)|^2)^{1-\alpha}}\|f'\|_{Lip_\alpha} \\
&+ \sup_{\delta<|\varphi(z)|<1}\frac{(1-|z|^2)\log\frac{2}{1-|z|}|\psi_2(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{2-\alpha}}\|f''\|_{Lip_\alpha} \leq K\epsilon,
\end{aligned} \tag{3.9}$$

where K is constant. It follows that the operator $T_{\psi_1,\psi_2,\varphi} : Lip_\alpha \to \mathcal{B}_{\log}$ is compact.

Conversely it is clear that the compactness of $T_{\psi_1,\psi_2,\varphi} : Lip_\alpha \to \mathcal{B}_{\log}$ implies its boundedness. Let $\{z_k\}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_k)| \to 1$ as $k \to \infty$. We can use the test functions $f_k(z) = f_{\varphi(z_k)}(z)$, where f_a is defined in (2.7). We have

$$f_k'(\varphi(z_k)) = \frac{C_1\overline{\varphi(z_k)}}{(1-|\varphi(z_k)|^2)^{1-\alpha}} \quad \text{and} \quad f_k(\varphi(z_k)) = f_a''(\varphi(z_k)) = 0.$$

Since $\{f_k\}$ converges to 0 uniformly on $\overline{\mathbb{D}}$, hence $\{f_k\}$ converges to 0 uniformly on compact subsets of $\mathbb{D}$. Then $\{f_k\}$ is a bounded sequence in Lip_α which converges to 0 uniformly on $\overline{\mathbb{D}}$. By Lemma (1.1) we obtain $\lim_{k\to\infty}\|T_{\psi_1,\psi_2,\varphi}f_k\|_{\mathcal{B}_{\log}} = 0$. Thus,

$$\begin{aligned}
&\frac{|C_1||\overline{\varphi(z_k)}|(1-|z_k|^2)\log\frac{2}{1-|z_k|}|\psi_1(z_k)\varphi'(z_k)+\psi_2'(z_k)|}{(1-|\varphi(z_k)|^2)^{1-\alpha}} \\
&\leq \|T_{\psi_1,\psi_2,\varphi}f_k\|_{\mathcal{B}_{\log}} \to 0, \qquad k\to\infty.
\end{aligned} \tag{3.10}$$

By (3.10) and since $|\varphi(z_k)| \to 1$ as $k \to \infty$, so

$$\lim_{k\to 1}\frac{(1-|z|^2)\log\frac{2}{1-|z|}|\psi_1(z)\varphi'(z)+\psi_2'(z)|}{(1-|\varphi(z)|^2)^{1-\alpha}} = 0.$$

Taking $f_k(z) = 1$, we obtain $\sup_k(1-|z_k|^2)\log\frac{2}{1-|z_k|}|\psi_1'(z_k)| \leq \|T_{\psi_1,\psi_2,\varphi}f_k\|_{\mathcal{B}_{\log}} \to 0$. Analogously, (3.3) can be proved by choosing the test function $g_k(z) = g_{\varphi(z_k)}(z)$, g_k is defined in (2.13). This completes the proof of the theorem. □

4. Essential norm of the operator $T_{\psi_1,\psi_2,\varphi} : Lip_\alpha \to \mathcal{B}_{\log}$

In this section, we give an estimate for the essential norm of the operator $T_{\psi_1,\psi_2,\varphi} : Lip_\alpha \to \mathcal{B}_{\log}$.

Theorem 4.1. *Let $\psi_1, \psi_2 \in H(\mathbb{D})$, φ be an analytic self-map of $\mathbb{D}$ and $T_{\psi_1,\psi_2,\varphi} : Lip_\alpha \to \mathcal{B}_{\log}$ is bounded. Then*

$$\|T_{\psi_1,\psi_2,\varphi} f\|_{e,Lip_\alpha\to\mathcal{B}_{\log}} \approx \max\{A_1, A_2\},$$

where

$$A_j := \limsup_{|a|\to 1} \|T_{\psi_1,\psi_2,\varphi}\Big(\frac{(1-|a|^2)^j}{(1-\bar{a}z)^{j-\alpha}}\Big)\|_{\mathcal{B}_{\log}}, \quad j = 1, 2.$$

Proof. First we prove that $\max\{A_1, A_2\} \leq \|T_{\psi_1,\psi_2,\varphi}\|_{e,Lip_\alpha\to\mathcal{B}_{\log}}$. Let $a \in \mathbb{D}$. Define

$$f_{a,j}(z) = \frac{(1-|a|^2)^j}{(1-\bar{a}z)^{j-\alpha}}.$$

It is easy to check that $f_{a,j} \in Lip_\alpha$ for all $a \in \mathbb{D}$ and $f_{a,j}$ converges uniformly to 0 on compact subset of Lip_α as $|a| \to 1$ Thus, for any compact operator $K : \mathrm{Lip}_\alpha \to \mathcal{B}_{\log}$, we have

$$\lim_{|a|\to 1} \|K f_{a,j}\|_{\mathcal{B}_{\log}} = 0, \quad j = 1, 2.$$

Hence,

$$\begin{aligned} \|T_{\psi_1,\psi_2,\varphi} - K\|_{Lip_\alpha\to\mathcal{B}_{\log}} &\gtrsim \limsup_{|a|\to 1} \|(T_{\psi_1,\psi_2,\varphi} - K) f_{a,j}\|_{\mathcal{B}_{\log}} \\ &\gtrsim \limsup_{|a|\to 1} \|T_{\psi_1,\psi_2,\varphi} f_{a,j}\|_{\mathcal{B}_{\log}} - \limsup_{|a|\to 1} \|K f_{a,j}\|_{\mathcal{B}_{\log}} = A_j. \end{aligned}$$

Therefore, based on the definition of the essential norm, we obtain

$$\|T_{\psi_1,\psi_2,\varphi}\|_{e,Lip_\alpha\to\mathcal{B}_{\log}} = \inf_K \|T_{\psi_1,\psi_2,\varphi} - K\|_{Lip_\alpha\to\mathcal{B}_{\log}} \gtrsim A_j, \quad j = 1, 2.$$

Now, we prove that

$$\|T_{\psi_1,\psi_2,\varphi} f\|_{e,Lip_\alpha\to\mathcal{B}_{\log}} \lesssim \max\{A_1, A_2\}.$$

For $r \in [0, 1)$, set $K_r : H(\mathbb{D}) \to H(\mathbb{D})$ by $(K_r f)(z) = f_r(z) = f(rz)$. It is obvious that $f_r - f \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $r \to 1$. Moreover, the operator K_r is compact on $\mathcal{B}$ and $\|K_r\|_{\mathcal{B}\to\mathcal{B}} \leq 1$(see[15]). By a similar argument can be proved that the operator K_r is compact on Lip_α and $\|K_r\|_{Lip_\alpha\to Lip_\alpha} \leq 1$. Let $\{r_j\} \subset (0, 1)$ be a sequence such that $r_j \to 1$ as $j \to \infty$. Then for all positive integer j, the operator $T_{\psi_1,\psi_2,\varphi} K_{r_j} : \mathrm{Lip}_\alpha \to \mathcal{B}_{\log}$ is compact. By the definition of the essential norm, we get

$$\|T_{\psi_1,\psi_2,\varphi}\|_{e,Lip_\alpha\to\mathcal{B}_{\log}} \leq \limsup_{j\to\infty} \|T_{\psi_1,\psi_2,\varphi} - T_{\psi_1,\psi_2,\varphi} K_{r_j}\|_{Lip_\alpha\to\mathcal{B}_{\log}}.$$

For any $f \in Lip_\alpha$ such that $\|f\|_{Lip_\alpha} \le 1$,

$$
\begin{aligned}
\|(T_{\psi_1,\psi_2,\varphi} - T_{\psi_1,\psi_2,\varphi}K_{r_j})f\|_{\mathcal{B}_{\log}} &\le |(T_{\psi_1,\psi_2,\varphi}f(0)| + |(f-f_{r_j})'(\varphi(0))g(0)| \\
&+ \sup_{z\in\mathbb{D}}(1-|z|^2)\log\frac{2}{1-|z|}|\psi_1'(z)|(f-f_{r_j})(\varphi(z))| \\
&+ \sup_{z\in\mathbb{D}}(1-|z|^2)\log\frac{2}{1-|z|}|(\psi_1(z)\varphi'(z)+\psi_2'(z))(f-f_{r_j})'(\varphi(z))| \\
&+ \sup_{z\in\mathbb{D}}(1-|z|^2)\log\frac{2}{1-|z|}|(\psi_2(z)\varphi'(z))(f-f_{r_j})''(\varphi(z))| \\
&\le \underbrace{\limsup_{j\to\infty}\sup_{|\varphi(z)|\le r_N}(1-|z|^2)\log\frac{2}{1-|z|}|\psi_1'(z)|(f-f_{r_j})(\varphi(z))|}_{M_1} \\
&+ \underbrace{\limsup_{j\to\infty}\sup_{|\varphi(z)|> r_N}(1-|z|^2)\log\frac{2}{1-|z|}|\psi_1'(z)|(f-f_{r_j})(\varphi(z))|}_{M_2} \\
&+ \underbrace{\limsup_{j\to\infty}\sup_{|\varphi(z)|\le r_N}(1-|z|^2)|(\psi_1(z)\varphi'(z)+\psi_2'(z))(f-f_{r_j})'(\varphi(z))|}_{M_3} \\
&+ \underbrace{\limsup_{j\to\infty}\sup_{|\varphi(z)|> r_N}(1-|z|^2)|(\psi_1(z)\varphi'(z)+\psi_2'(z))(f-f_{r_j})'(\varphi(z))|,}_{M_4} \\
&+ \underbrace{\limsup_{j\to\infty}\sup_{|\varphi(z)|> r_N}(1-|z|^2)|(\psi_2(z)\varphi'(z))(f-f_{r_j})''(\varphi(z))|}_{M_5} \\
&+ \underbrace{\limsup_{j\to\infty}\sup_{|\varphi(z)|> r_N}(1-|z|^2)|(\psi_2(z)\varphi'(z))(f-f_{r_j})''(\varphi(z))|,}_{M_6}
\end{aligned}
$$

where $N \in \mathbb{N}$ is large enough such that $r_j \ge \frac{1}{2}$ for all $j \in \mathbb{N}$. Since $T_{\psi_1,\psi_2,\varphi} : Lip_\alpha \to \mathcal{B}_{\log}$ is bounded, by (2.3) and (2.6), we have

$$\widetilde{F_1} = \sup_{z\in\mathbb{D}}(1-|z|^2)\log\frac{2}{1-|z|}|\psi_1'(z)| < \infty,$$

$$\widetilde{F_2} = \sup_{z\in\mathbb{D}}(1-|z|^2)\log\frac{2}{1-|z|}|\psi_1(z)\varphi'(z)+\psi_2'(z)| < \infty$$

and

$$\widetilde{F_3} = \sup_{z\in\mathbb{D}}(1-|z|^2)\log\frac{2}{1-|z|}|\psi_2(z)\varphi'(z))| < \infty$$

Since $r_j f_{r_j}' \to f'$ uniformly on compact subsets of $\mathbb{D}$ as $j \to \infty$, so

$$M_1 \le \widetilde{F_1} = \sup_{z\in\mathbb{D}}(1-|z|^2)\log\frac{2}{1-|z|}|\psi_1'(z)| = 0,$$

$$M_3 \le \widetilde{F_2} = \sup_{z\in\mathbb{D}}(1-|z|^2)\log\frac{2}{1-|z|}|\psi_1(z)\varphi'(z)+\psi_2'(z)| = 0,$$

and

$$M_5 \le \widetilde{F_2} = \sup_{z\in\mathbb{D}}(1-|z|^2)\log\frac{2}{1-|z|}|\psi_2(z)\varphi'(z))| = 0,$$

Next we consider M_2. We have $M_2 \le \limsup_{j\to\infty}(Q_1+Q_2)$, where

$$Q_1 = \sup_{|\varphi(z)|>r_N}(1-|z|^2)\log\frac{2}{1-|z|}|(f(\varphi(z))||\psi_1'(z)|,$$

and

$$Q_2 = \sup_{|\varphi(z)|>r_N}(1-|z|^2)\log\frac{2}{1-|z|}r_j|(f(\varphi(z))|\psi_1'(z)|.$$

Using the fact that $\|f\|_{Lip\alpha} \leq 1$ and (1.2), we obtain

$$\begin{aligned} Q_1 &= \sup_{|\varphi(z)|>r_N} (1-|z|^2)\log\frac{2}{1-|z|}|(f(\varphi(z))||\psi_1'(z)| \\ &\times \frac{(1-|\varphi(z)|^2)^{1-\alpha}}{(j-\alpha)\overline{\varphi(z)}}\frac{(j-\alpha)\overline{\varphi(z)}}{(1-|\varphi(z)|^2)^{1-\alpha}} \\ &\preceq \frac{(j-\alpha)\|f\|_{\mathrm{Lip}_\alpha}}{r_N}\sup_{|\varphi(z)|>r_N}(1-|z|^2)\log\frac{2}{1-|z|}|\psi_1'(z)|\frac{(j-\alpha)\overline{\varphi(z)}}{(1-|\varphi(z)|^2)^{1-\alpha}} \\ &\preceq \sup_{|\varphi(z)|>r_N}(1-|z|^2)\log\frac{2}{1-|z|}|\psi_1'(z)|\frac{(j-\alpha)\overline{\varphi(z)}}{(1-|\varphi(z)|^2)^{1-\alpha}} \\ &\preceq \sup_{|a|>r_N}\|T_{\psi_1,\psi_2,\varphi}(f_{a,j})\|_{\mathcal{B}_{\log}} \qquad j=1,2. \end{aligned}$$

Taking the limit as $N\to\infty$, we obtain

$$\limsup_{j\to\infty} Q_1 \leq \limsup_{|a|\to\infty}\|T_{\psi_1,\psi_2,\varphi}(f_{a,j})\|_{\mathcal{B}_{\log}}.$$

Similarly,

$$\limsup_{j\to\infty} Q_2 \leq \limsup_{|a|\to\infty}\|T_{\psi_1,\psi_2,\varphi}(f_{a,j})\|_{\mathcal{B}_{\log}}.$$

Hence, we get $M_2 \preceq \max\{A_1, A_2\}$. Similarly, it can be shown that $M_4 \preceq \max\{A_1, A_2\}$ and $M_6 \preceq \max\{A_1, A_2\}$. This completes the proof of the theorem. □

Acknowledgment. This article is extracted from the final report of the research project "The Stevic-Sharma operator" which has been implemented from the research credits of Tabriz University.

References

[1] E. Abbasi and H. Vaezi, *Estimates of essential norm of generalized weighted composition operators from Bloch type spaces to nth weighted-type spaces*, Math. Slovaca **70** (1), 71-80, 2020.

[2] E. Abbasi, H. Vaezi and S. Li, *Essential norm of weighted composition operators from H^∞ to nth weighted-type spaces*, Mediterr. J. Math. **16**, Article no:133, 2019.

[3] J. Arazy, *Multipliers of Bloch functions*, University of Haifa Mathematics Publications Series **54**, 1982.

[4] K.R.M. Attele, *Toeplitz and Hankel operators on Bergman one space*, Hokkaido Math. J. **21** (2), 279-293, 1992.

[5] R.E. Castillo, D.D. Clahane, J.F.F. Lopez and J.C.R. Fernndez, *Composition operators from logarithmic Bloch spaces to weighted Bloch spaces*, Appl. Math. Comput. **219** (12), 6692-6706, 2013.

[6] F. Colonna and S. Li, *Weighted composition operators from the Lipschitz space into the Zygmund space*, MIA, Math. Inequal. Appl., Article ID 48478, 2007.

[7] C.C. Cowen and B. MacCluer, *Composition operators on spaces of analytic functions*, Stud. Adv. Math., CRC Press, Boca Raton, 1995.

[8] P.L. Duren, *Theory of H^p spaces*, Academic press, New York, 1970.

[9] P.L. Duren and B. W. Romberg, *Shileds, A.L., Linear functionals on H^p spaces with* $0 < p < 1$. J. Reine Angew. Math. **238**, 32-60, 1969.

[10] P. Galanopoulos, *On blog to Q_p log pullbacks*, J. Math. Anal. Appl. **337** (1), 712-725, 2008.

[11] A.J. Garcia Ortiz and J.C. Ramos-Fernàndez, *Composition operators from logarithmic Bloch spaces to Bloch-type spaces*, Georgian Math. J. **20** (4), 671-686, 2013.

[12] G.H. Hardy and J.E. Littlewood, *Some properties of fractional integrals II*, Math. Z. **34**, 403-439, 1932.

[13] M. Hassanlou, H. Vaezi and M. Wang, *weighted composition operators on weak valued Bergman spaces and Hardy spaces*, Banach J. Math. Anal. **9** (2), 35-43, 2015.

[14] S. Li, E. Abbasi and H. Vaezi, *Weighted composition operators from Bloch-type spaces to nth weighted-type spacese*, Ann. Polon. Math. **124** (1), 93-107, 2020.

[15] B. Maccluer and R. Zhao, *Essential norm of weighted composition operators between Bloch-type spaces*, Rocky M. J. Math. **33** (4), 1437-1458, 2003.

[16] A. Petrov, *Reverse estimates in logarithmic Bloch spaces*, Arch. Math. (Basel) **100** (6), 551-560, 2013.

[17] J. Shapiro, *Composition Operators and alassical function theory*, Springer-Verlag, New York, 1993.

[18] S. Stević, A.K. Sharma and A. Bhat, *Products of multiplication, composition and differentiation operators on weighted Bergman space*, Appl. Math. Comput. **217** (20), 8115-8125, 2011.

[19] M. Tjani, *Compact composition operators on some Mobius invariant Banach spaces*, Ph.D. thesis, Michigan State University, 1996.

[20] H. Vaezi and S. Houdfar, *Weighted composition operators between Besov-type spaces*, Hacet. J. Math. Stat. **49** (1), 78-86, 2020.

[21] H. Vaezi and S. Houdfar, *Composition and weighted composition operators from Bloch-type to Besov-type spaces*, Math. Reports **22** (3-4), 297-308, 2020.

[22] S. Ye, *Multipliers and cyclic vectors on the weighted Bloch type space*, Math. J. Okayama Univ. **48**, 135-143, 2006

[23] R. Yoneda, *The composition operators on weighted Bloch space*, Arch. Math. (Basel) **78** (4), 310-317, 2002.

[24] F. Zhang and Y. Liu, *Products of multiplication, composition and differentiation operators from mixed-norm spaces to weighted-type spaces*, Taiwan. J. Math. **18** (6), 1927-1940, 2014.

[25] K. Zhu, *Bloch type spaces of analytic functions*, Rocky Mountain J. Math. **23**, 1143-1177, 1993.