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Research Article

Statistical ρ-commutative algebras

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Abstract

In this article, we study Codazzi-couples of an arbitrary connection ∇ with a nondegenerate 2-form ω, an isomorphism L on the space of derivation of ρ-commutative algebra A, which the important examples of isomorphism L are almost complex and almost para-complex structures, a metric g that (g, ω, L) form a compatible triple. We study a statistical structure on ρ-commutative algebras by the classical manner on Riemannian manifolds. Then by recalling the notions of almost (para-)Kähler ρ-commutative algebras, we generalized the notion of Codazzi-(para-)Kähler ρ-commutative algebra as a (para-)Kähler (or Fedosov) ρ-commutative algebra which is at the same time statistical and moreover define the holomorphic ρ-commutative algebras.

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1. Introduction

The notion of statistical manifolds was introduced by S. L. Lauritzen [12] in 1980s and created a new field research that is called information geometry. Information geometry is a greatly absorbing limb of science, considered as a combination of differential geometry and statistics, actually it make some objects of differential geometry such as metric, connection and geodesics combine to statistical ones. Statistical structures to information geometers are characterized by the Codazzi-couple of (∇, g) [1, 17, 19].

In 1930 Schouten and Dontzing introduced the concept of complex structure and a Hermitian metric in a differentiable manifold and called it a complex manifold. In 1933 [11], Kähler add a new notion named Kählerian structure on a complex manifold. Later, H. Furuhuta [10] defined and studied a structure on the statistical manifolds which is considered as a Kähler structure with a certain conditions that is called the holomorphic statistical manifold. Statistical manifolds with almost complex structures (with almost contact structures) and its statistical submersions are studied by Tanako in [20, 21]. As a special case, Hessian manifolds (for which ∇ is flat but not Levi-Civita) are the affine analogue of Kähler manifolds, see [4, 18]. Also, interaction of Codazzi-Couples with (Para-)Kähler Geometry was raised by Teng and Zhang in [9].

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The concept of non-commutative geometry was formed by replacing the algebra of smooth functions on a smooth manifold with an abstract associative algebra. In non-commutative geometry, the element that plays a key role is the extension of the concept of differential forms to manifolds. (see [7, 8, 13, 14, 16], for instance). ρ-commutative algebras are a key element in non-commutative geometry because by replacing this algebra with the algebra of smooth functions on a manifold, many geometric objects can be generalized to these algebras. Bonggarts and Pijls introduced ρ-commutative algebras at the first time (see [3]) and differential structures and ρ-derivations were defined on these algebras. A G-graded algebra A with a given cocycle ρ will be called *ρ-commutative* iff $[,]_\rho = 0$ i.e., $fg = \rho(|f|, |g|)gf$ for all homogeneous elements f and g in A (see [2] for more details). These types of algebras are interesting objects and have been in the focus of several authors, including C. Ciupala and F. Ngakeu who studied and defined other geometric objects on these algebras such as linear connections, differential forms, metrics, curvature, Ricci tensor (see [5, 6, 15]). These algebras have also been mentioned with titles such as almost commutative algebras, graded commutative algebras, colour algebras and Γ-graded algebras.

The aim of this paper is the generalization of the geometry objects on ρ-commutative algebras and discuss about some notions such as statistical structure, Codazzi-(para-)Kähler ρ-commutative algebras, Kähler statistical structure and holomorphic statistical structure.

This paper is arranged as follows. Section 2 is devoted to a summary of some previous developments on ρ-commutative algebras. The core of the paper is contained in Section 3 where we investigate the interaction of an arbitrary connection ∇ with three geometric structures on A, namely, a nondegenerate 2- form ω, an isomorphism $L : \rho-DerA \longrightarrow \rho-DerA$ and a metric g and we introduce the notion of a statistical ρ-commutative algebras as a triple (A, ∇, g) where A is a ρ-commutative algebra and ∇ is a torsion free connection such that (∇, g) is Codazzi-couple. The study of statistical curvature and its properties is also included in this section. Also, we introduce g-conjugation ∇^*, ω-conjugation $\nabla^\dagger$, L-gauge transformation ∇^L of ∇ and investigate the Codazzi-couple of (∇^*, g), (∇^L, L). We show that $(id, *, \dagger, L)$ acts as the 4-element Klein group. At we deduce that Codazzi-couple of ∇ with both g and L gives rise to a (para-)Kähler ρ-commutative algebra. Section 4 is fully devoted to Codazzi-(para-)Kähler ρ-commutative algebras. We have tried to collect some of the definitions and formulas in almost complex ρ-commutative algebras or Kähler ρ-commutative algebras, then we introduce the Kähler statistical ρ-commutative algebras. In Section 5, holomorphic statistical ρ-commutative algebras are introduced and it is shown that Kähler statistical ρ-commutative algebra (A, ∇, g, J) is a holomorphic statistical ρ-commutative algebra.

2. Statistical ρ-commutative algebras

In this section we proceed with the study of statistical ρ-commutative algebra. Let (A, g) be a metric ρ-commutative algebra and ∇ be a connection on A. We denote by $\widehat{\nabla}$ the Levi-Civita connection of g.

Let $\rho\text{-}DerA = \bigoplus_{a \in G} \rho\text{-}DerA_a$ be the ρ-Lie algebra of ρ-derivations of a ρ-commutative algebra A and $L : \rho\text{-}DerA \longrightarrow \rho\text{-}DerA$ be an isomorphism. Almost complex structures and almost para-complex structures are the most important examples of the isomorphism on ρ-$DerA$ (refer to [2] for an introduction to the almost complex structures and almost para-complex structures, Kähler and para-Kähler ρ-commutative algebra and related concepts of them). We denote by J and K almost complex structures and almost para-complex structures, respectively, and use L when these two structures can be treated in a unified way. We say that g is compatible with L (either J or K) if $g(LX, LY) = g(X, Y)$ or $g(X, LY) + g(LX, Y) = 0$, for any $X, Y \in \rho\text{-}DerA$. We define 2-covariant tensor ω by $\omega(X, Y) = g(LX, Y)$. One check that ω satisfies $\omega(X, LY) + \omega(LX, Y) = 0$, for any

$X, Y \in \rho\text{-}DerA$. If we start with ω, we can define g by $g(X,Y) = \omega(L^{-1}X, Y)$. The three structures (g, ω, L) form a compatible triple when each structure can be specified by the two others. We define a new connection ∇^L by L-gauge transformation as follows

$$\nabla^L_X Y = L^{-1}(\nabla_X(LY)). \tag{2.1}$$

Definition 2.1. (∇, L) is called Codazzi-couple if the following condition holds

$$\rho(X,Y)(\nabla_X L)Y = (\nabla_Y L)X, \quad \forall X, Y \in Hg(\rho\text{-}DerA). \tag{2.2}$$

Proposition 2.2. *The following assertions are equivalent*

i) (∇, L) *is Codazzi-couple.*
ii) *The torsions of* ∇ *and* ∇^L *are equal.*
iii) (∇^L, L^{-1}) *is Codazzi-couple.*

Proof. Let (i) holds. Since (∇, L) is Codazzi-couple, then $\rho(X,Y)(\nabla_X L)Y = (\nabla_Y L)X$. Hence

$$\begin{aligned}
0 &= L^{-1}(\rho(X,Y)(\nabla_X L)Y - \rho(X,Y)\rho(Y,X)(\nabla_Y L)X) \\
&= L^{-1}(\nabla_X(LY) - L\nabla_X Y - \rho(X,Y)(\nabla_Y(LX) - L\nabla_Y X)) \\
&= L^{-1}\nabla_X(LY) - \rho(X,Y)L^{-1}(\nabla_Y(LX)) - \nabla_X Y + \rho(X,Y)\nabla_Y X - [X,Y] + [X,Y] \\
&= \nabla^L_X Y - \rho(X,Y)\nabla^L_Y X - [X,Y] - (\nabla_X Y - \rho(X,Y)\nabla_Y X - [X,Y]) \\
&= T^L(X,Y) - T(X,Y).
\end{aligned}$$

So $T^L(X,Y) = T(X,Y)$. Therefore (i) and (ii) are equivalent. The rest is proved in a similar way. □

Corollary 2.3. *Consider isomorphism* L *(either* J *or* K*). Then*

i) $\nabla^L = \nabla^{L^{-1}}$ *(i.e* $(\nabla^L)^L = \nabla$*)),*
ii) (∇, L) *is Codazzi-couple if and only if* (∇^L, L) *is Codazzi-couple.*

Definition 2.4. The pair (∇, g) consisting of a connection ∇ and a metric g is called Codazzi-couple if the following identity holds

$$\rho(X, Y+Z)(\nabla_X g)(Y,Z) = \rho(Y, X+Z)\rho(X,Y)(\nabla_Y g)(X,Z) = (\nabla_Z g)(X,Y), \tag{2.3}$$

for any $X, Y, Z \in Hg(\rho\text{-}DerA)$. Set $C(X,Y,Z) = \rho(X, Y+Z)(\nabla_X g)(Y,Z)$, where C is a 3-covariant-tensor that is called cubic form associated to the pair (∇, g). Clearly C is totally symmetric in all of it's indices, that is

$$\begin{aligned}
C(X,Y,Z) &= \rho(X,Y)C(Y,X,Z); \\
C(X,Y,Z) &= \rho(Y,Z)C(X,Y,Z); \\
C(X,Y,Z) &= \rho(X+Y,Z)C(Z,X,Y); \\
C(X,Y,Z) &= \rho(X,Y)\rho(X+Y,Z)C(Z,Y,X).
\end{aligned}$$

Definition 2.5. The pair (∇, g) is called a statistical structure on A if it is Codazzi-couple and the connection ∇ is torsion free. If (∇, g) is a statistical structure on A then the triple (A, ∇, g) is called statistical ρ-commutative algebra.

Example 2.6. Consider the extended hyperplane $A^2_q := \langle 1, x, y, x^{-1}, y^{-1}, xy = qyx \rangle$. It is a $\mathbb{Z} \times \mathbb{Z}$-graded ρ-commutative algebra, with

$$\rho(n, n') = q^{\sum_{j,k=1}^{n} n_j n'_k \alpha_{jk}},$$

where $\alpha_{jk} = 1$ if $j < k$, 0 if $j = k$ and -1 if $j > k$. $\rho\text{-}DerA^q_2$ is the A^q_2-bi-module generated by $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$, and $\Omega^1(A)$ is generated by dx, dy, such that $dx_j(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial x_i}(x_j) = \delta_{i,j}$, $|\frac{\partial}{\partial x_i}| = -|x_i|$, $|x_i^{-1}| = -x_i$ and $|dx_i| = |x_i|$, where $x_1 = x$ and $x_2 = y$. Now we consider

$$g = dx \otimes dx g_{11} + (dx \otimes dy + q dy \otimes dx) g_{12} + dy \otimes dy g_{22},$$

where $g_{11} = 0, g_{12} = x^{-1}y^{-1}, g_{22} = 0$. Thus we can see g is a homogeneous metric on A_q^2 (of degree $(0,0)$) if and only if $D = -x^{-2}y^{-2}$ is invertible. Considering $\tilde{g}_{mk} := \rho(x_m, x_k)g_{mk}$, we have

$$(\tilde{g}_{mk}) = \begin{pmatrix} 0 & qx^{-1}y^{-1} \\ x^{-1}y^{-1} & 0 \end{pmatrix},$$

and

$$(\tilde{g}^{mk}) = (\tilde{g}_{mk})^{-1} = \begin{pmatrix} 0 & q^{-1}xy \\ q^{-2}xy & 0 \end{pmatrix}.$$

Now, we compute the ρ-Christoffel coefficient $\hat{\Gamma}^t_{ij}$ of G-degree $|\hat{\Gamma}^t_{ij}| = |x_t| - |x_i| - |x_j|$ of Levi-Civita connection. By the ρ-symmetry property of g, we have

$$g_{21} = g(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) = \rho(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})g(\frac{\partial}{\partial y}, \frac{\partial}{\partial x}) = qg_{12},$$

and, by torsion-free condition we can find $\hat{\Gamma}^t_{ij} = \rho(x_i, x_j)\hat{\Gamma}^t_{ji}$. So, we have

$$\hat{\Gamma}^2_{11} = \hat{\Gamma}^2_{21} = \hat{\Gamma}^2_{12} = \hat{\Gamma}^1_{22} = \hat{\Gamma}^1_{12} = \hat{\Gamma}^2_{12} = 0.$$

We have also $\hat{\Gamma}^2_{22} = -q^2y^{-1}, \hat{\Gamma}^1_{11} = -x^{-1}$. Now, we find all connections ∇ such that (∇, g) is a statistical structure. (2.3) for any $X, Y, Z \in Hg(\rho\text{-}DerA_q^2)$ gives us

$$\begin{aligned}
&(1) \quad -x^{-2}y^{-1} - \Gamma^2_{12} - q^2\Gamma^1_{11}x^{-1}y^{-1} = -2q\Gamma^2_{21}x^{-1}y^{-1} \\
&\qquad = -x^{-2}y^{-1} - \Gamma^1_{11}x^{-1}y^{-1} - \Gamma^2_{12}x^{-1}y^{-1}, \\
&(2) \quad -2q\Gamma^1_{12}x^{-1}y^{-1} = -q^3x^{-1}y^{-2} - q^{-2}\Gamma^1_{22}x^{-1}y^{-1} - \Gamma^2_{22}x^{-1}y^{-1}, \\
&(3) \quad -qx^{-1}y^{-2} - \Gamma^2_{22}x^{-1}y^{-1} - \Gamma^1_{21}x^{-1}y^{-1} = -2q^{-1}\Gamma^1_{12}x^{-1}y^{-1}, \\
&(4) \quad -qx^{-2}y^{-1} - q\Gamma^1_{11}x^{-1}y^{-1} - q\Gamma^2_{12}x^{-1}y^{-1} = -2q^2\Gamma^2_{21}x^{-1}y^{-1}, \\
&(5) \quad -2\Gamma^2_{21}x^{-1}y^{-1} = -q^{-1}x^{-2}y^{-1} - q^{-1}\Gamma^2_{12}x^{-1}y^{-1} - q\Gamma^1_{11}x^{-1}y^{-1}, \\
&(6) \quad -q^2x^{-1}y^{-2} - q\Gamma^1_{21}x^{-1}y^{-1} - q^{-1}\Gamma^2_{22}x^{-1}y^{-1} = -2\Gamma^1_{12}x^{-1}y^{-1} \\
&\qquad = -q^2x^{-1}y^{-2} - q\Gamma^2_{22}x^{-1}y^{-1} - q\Gamma^1_{21}x^{-1}y^{-1}.
\end{aligned}$$

The torsion-free condition gives us $\Gamma^t_{ij} = \rho(x_i, x_j)\Gamma^t_{ji}$. Using the formula (1), we get $x^{-1} + \Gamma^1_{11} = \Gamma^2_{12}$ and $x^{-1} + q^2\Gamma^1_{11} = \Gamma^2_{12}$, this equations give us $q^2 = 1$ and then $q = \pm 1$. Let us to assume that $q = -1$ and the other equalities imply

$$\begin{aligned}
&\Gamma^1_{11} = \Gamma^2_{12} - x^{-1}, \quad \Gamma^2_{22} = -\Gamma^1_{12} - y^{-1}. \\
&\Gamma^1_{22} = -y^{-1} - 2\Gamma^1_{12} - \Gamma^2_{22} = -y^{-1} - 2\Gamma^1_{12} + \Gamma^1_{12} + y^{-1} = -\Gamma^1_{12}.
\end{aligned}$$

For any connection ∇, we denote the g-conjugate of ∇ by ∇^* and define it for any $X, Y, Z \in Hg(\rho\text{-}DerA)$ by the following formula

$$X \cdot g(Y, Z) = g(\nabla_X Y, Z) + \rho(X, Y)g(Y, \nabla^*_X Z). \tag{2.4}$$

It is easy to check that ∇^* is involutive, i.e., $(\nabla^*)^* = \nabla$.

Lemma 2.7. ∇^* *is a connection on* A *and moreover* (∇^*, g) *is a statistical structure.*

Proof. It is easy to see that ∇^* is a connection. We show (∇^*, g) is a statistical structure. For this, we have

$$\begin{aligned}
g(Z, \nabla^*_X Y - \rho(X, Y)\nabla^*_Y X - [X, Y]) &= \rho(Z, X + Y)(\nabla_X g)(Y, Z) \\
&\quad - \rho(X + Z, Y)\rho(X, Z)(\nabla_Y g)(X, Z) \\
&= \rho(Z, X + Y)((\nabla_X g)(Y, Z) \\
&\quad - \rho(X, Y)(\nabla_Y g)(X, Z)) = 0.
\end{aligned}$$

So, $T^*(X,Y) = \nabla^*_X Y - \rho(X,Y)\nabla^*_Y X - [X,Y] = 0$, that is ∇^* is torsion free. Now, we show that (∇^*, g) is Codazzi-couple. For this we have

$$\begin{aligned}\rho(X,Y+Z)(\nabla^*_X g)(Y,Z) - \rho(Y,X+Z)\rho(X,Y)(\nabla^*_Y g)(X,Z) &= X\cdot g(Y,Z) - g(\nabla^*_X Y, Z)\\ &- \rho(X,Y)g(Y,\nabla^*_X Z) - \rho(X,Y)Y\cdot g(X,Z)\\ &+ \rho(X,Y)g(\nabla^*_Y X, Z) + g(X,\nabla^*_Y Z).\end{aligned}$$

From (2.4), we obtain

$$\begin{aligned}\rho(X,Y+Z)(\nabla^*_X g)(Y,Z) - \rho(Y,X+Z)\rho(X,Y)(\nabla^*_Y g)(X,Z) &= -\rho(X,Y+Z)(\nabla_X g)(Y,Z)\\ &\quad + \rho(Y,X+Z)\rho(X,Y)(\nabla_Y g)(X,Z)\\ &= 0.\end{aligned}$$

□

Using (2.3) and (2.4), we get

$$C(X,Y,Z) = \rho(X,Y)g(Y,(\nabla^* - \nabla)_X Z). \tag{2.5}$$

We can also find $C^*(X,Y,Z) = -C(X,Y,Z)$. So $C^*(X,Y,Z) = C(X,Y,Z) = 0$ if $\nabla = \nabla^*$, that is ∇ is g-self-conjugate. A short calculation shows that

$$\rho(Y,X)C(X,Y,Z) - \rho(X+Y,Z)C(Z,Y,X) = g(Y,(T^* - T)(X,Z)). \tag{2.6}$$

So, $C(X,Y,Z) = \rho(X,Y)\rho(X+Y,Z)C(Z,Y,X)$ if and only if $T^* = T$. Let us briefly summarize the above discussion by the following proposition:

Proposition 2.8. *The following assertions are equivalent*

i) (∇, g) *is Codazzi-couple;*
ii) (∇^*, g) *is Codazzi-couple;*
iii) C *is totally symmetric;*
iv) C^* *is totally symmetric;*
v) $T^* = T$.

Let us construct a family of connections $\{\nabla^{(\alpha)}\}_{\alpha\in\mathbb{R}}$ by the connections ∇ and ∇^* as follows

$$\nabla^{(\alpha)} = \frac{1+\alpha}{2}\nabla + \frac{1-\alpha}{2}\nabla^*, \tag{2.7}$$

with $\nabla^{(1)} = \nabla$ and $\nabla^{(-1)} = \nabla^*$. This connections is called *α-connections*.

Remark 2.9. For α-connections we have

i) $(\nabla^{(\alpha)})^* = \nabla^{(-\alpha)}$,
ii) $C^\alpha(X,Y,Z) = \alpha C(X,Y,Z)$,
iii) $g(\nabla^{(\alpha)}_X Y, Z) = g(\nabla^{(0)}_X Y, Z) - \frac{\alpha}{2}\rho(X,Z)C(X,Y,Z)$.

Note that, for the connections ∇ and ∇^*, $\widehat{\nabla} = \frac{1}{2}(\nabla + \nabla^*)$ is a Levi-civita connection.

Let (∇, g) be a statistical structure on A. We define on ρ-$DerA$ a linear map K : ρ-$DerA \longrightarrow End(\rho\text{-}DerA)$ by

$$K_X Y = \nabla_X Y - \widehat{\nabla}_X Y, \tag{2.8}$$

for any $X, Y \in Hg(\rho\text{-}DerA)$. One can easily see that the linear map K satisfies the following conditions

$$K_X Y = \rho(X,Y)K_Y X, \quad g(K_X Y, Z) = \rho(X,Y)g(Y, K_X Z),. \tag{2.9}$$

We have also $K = \frac{1}{2}(\nabla - \nabla^*) = \widehat{\nabla} - \nabla^*$.

Remark 2.10. For a metric g on A, if a linear map $K : \rho\text{-}DerA \longrightarrow End(\rho\text{-}DerA)$ satisfies (2.9), then the pair $(\nabla := \widehat{\nabla} + K, g)$ is a statistical structure on A.

Definition 2.11. Let (∇, g) be a statistical structure on A. The statistical curvature tensor of (∇, g) is the map $\mathfrak{R} : \rho\text{-}DerA \times \rho\text{-}DerA \longrightarrow End(\rho\text{-}DerA)$ defined by

$$\mathfrak{R}(X,Y)Z = \frac{1}{2}\{R(X,Y)Z - R^*(X,Y)Z\},$$

where R and R^* denote the curvature tensor of ∇ and ∇^*, respectively.

Definition 2.12. A statistical ρ-commutative algebra (A, ∇, g) is said to be of constant statistical curvature $c \in \mathbb{R}$ if, for any $X, Y, Z \in \rho\text{-}DerA$ the equality $\mathfrak{R}(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y\}$ holds.

Lemma 2.13. *Let (∇, g) be a statistical structure on A and $\mathfrak{R}$ be the statistical curvature tensor of (∇, g). Then for any $X, Y, Z \in Hg(\rho\text{-}DerA)$, we have*

$$\widehat{R}(X,Y)Z = \mathfrak{R}(X,Y)Z - [K_X, K_Y]Z,$$

where $\widehat{R}$ is denoted for the curvature tensor of $\widehat{\nabla}$ and $K = \widehat{\nabla} - \nabla^*$.

Proof. By definition of $\widehat{R}$, we get

$$\begin{aligned}\widehat{R}(X,Y)Z &= \widehat{\nabla}_X\widehat{\nabla}_Y Z - \rho(X,Y)\widehat{\nabla}_Y\widehat{\nabla}_X Z - \widehat{\nabla}_{[X,Y]}Z \\ &= \frac{1}{2}S(X,Y)Z - \frac{1}{4}\{\nabla_{[X,Y]}Z + \nabla^*_{[X,Y]}Z\} \\ &\quad - \frac{1}{4}\rho(X,Y)\{\nabla_Y\nabla^*_X Z + \nabla^*_Y\nabla_X Z\} + \frac{1}{4}\{\nabla_X\nabla^*_Y Z + \nabla^*_X\nabla_Y Z\}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}[K_X, K_Y]Z = K_X K_Y Z - \rho(X,Y)K_Y K_X Z &= \frac{1}{2}S(X,Y)Z + \frac{1}{4}\{\nabla_{[X,Y]}Z + \nabla^*_{[X,Y]}Z\} \\ &+ \frac{1}{4}\rho(X,Y)\{\nabla_Y\nabla^*_X Z + \nabla^*_Y\nabla_X Z\} - \frac{1}{4}\{\nabla_X\nabla^*_Y Z + \nabla^*_X\nabla_Y Z\}.\end{aligned}$$

Therefore

$$\begin{aligned}S(X,Y)Z - [K_X, K_Y]Z &= \frac{1}{2}S(X,Y)Z - \frac{1}{4}\{\nabla_{[X,Y]}Z + \nabla^*_{[X,Y]}Z\} \\ &\quad - \frac{1}{4}\rho(X,Y)\{\nabla_Y\nabla^*_X Z + \nabla^*_Y\nabla_X Z\} + \frac{1}{4}\{\nabla_X\nabla^*_Y Z + \nabla^*_X\nabla_Y Z\} \\ &= \widehat{R}(X,Y)Z.\end{aligned}$$

□

Lemma 2.14. *Let (∇, g) be a statistical structure on A. Then for any $X, Y, Z, W \in Hg(\rho\text{-}DerA)$, we have*

1) $R(X,Y,Z,W) = -\rho(X,Y)R(Y,X,Z,W)$,
2) $R^*(X,Y,Z,W) = -\rho(X,Y)R^*(Y,X,Z,W)$,
3) $R(X,Y,Z,W) = -\rho(Z,W)R^*(X,Y,W,Z)$,
4) $\rho(Z,X)R(X,Y)Z + \rho(X,Y)R(Y,Z)X + \rho(Y,Z)R(Z,X)Y = 0$,
5) $\rho(Z,X)R^*(X,Y)Z + \rho(X,Y)R^*(Y,Z)X + \rho(Y,Z)R^*(Z,X)Y = 0$,

where $R(X,Y,Z,W) = g(R(X,Y)Z, W)$.

Proof. We show the assertion (1) and the other equalities are just as easy to prove.

$$\begin{aligned} R(X,Y,Z,W) = g(R(X,Y)Z,W) &= g(\nabla_X\nabla_Y Z,W) \\ &\rho(X,Y)g(\nabla_Y\nabla_X Z,W) - g(\nabla_{[X,Y]}Z,W) \\ &= -\rho(X,Y)\{-\rho(Y,X)g(\nabla_X\nabla_Y Z,W) \\ &\quad + g((\nabla_Y\nabla_X Z,W) + \rho(Y,X)g(\nabla_{[X,Y]}Z,W)\} \\ &= -\rho(X,Y)R(Y,X,Z,W). \end{aligned}$$

□

For the Levi-Civita connection $\widehat{\nabla}$ and its curvature $\widehat{R}$, the above lemma is expressed in the following:

(a) $\rho(X,Y)\widehat{R}_{YZ}X + \rho(Y,Z)\widehat{R}_{ZX}Y + \rho(Z,X)\widehat{R}_{XY}Z = 0$ $(Bianchi1)$,
(b) $\widehat{R}(X,Y,V,W) = -\rho(X,Y)\widehat{R}(Y,X,V,W) = -\rho(V,W)\widehat{R}(X,Y,W,V)$,
(c) $\widehat{R}(X,Y,V,W) = \rho(X+Y,V+W)\widehat{R}(V,W,X,Y)$,

where $\widehat{R}(X,Y,V,W) := g(\widehat{R}_{XY}V,W)$, for any $X,Y,Z,V,W \in Hg(\rho\text{-}DerA)$.

Lemma 2.15. *Let $\mathfrak{R}$ be the statistical curvature tensor of (∇,g). Then for any $X,Y,Z,W \in Hg(\rho\text{-}DerA)$, we have*

1) $\mathfrak{R}(X,Y,Z,W) = -\rho(X,Y)\mathfrak{R}(Y,X,Z,W)$,
2) $\mathfrak{R}(X,Y,Z,W) = -\rho(Z,W)\mathfrak{R}(X,Y,W,Z)$,
3) $\rho(Z,X)\mathfrak{R}(X,Y)Z + \rho(X,Y)\mathfrak{R}(Y,Z)X + \rho(Y,Z)\mathfrak{R}(Z,X)Y = 0$,

where $\mathfrak{R}(X,Y,Z,W) = g(\mathfrak{R}(X,Y)Z,W)$.

Proof. It is easy to show the above assertions by direct calculations. □

2.1. Codazzi-couple of ∇ with ω

Consider 2-form ω by $\omega(X,Y) = g(LX,Y)$. Let us define 3-covariant-tensor Γ by

$$\Gamma(X,Y,Z) = \rho(X,Y+Z)(\nabla_X\omega)(Y,Z).$$

It is clear that $\Gamma(X,Y,Z) = -\rho(Y,Z)\Gamma(X,Z,Y)$.

Definition 2.16. The pair (∇,ω) is Codazzi-couple if the following identity hold

$$\rho(X,Y+Z)(\nabla_X\omega)(Y,Z) = \rho(Y,X+Z)\rho(X,Y)(\nabla_Y\omega)(X,Z) = (\nabla_Z\omega)(X,Y). \quad (2.10)$$

The relation (2.10) says that $\Gamma(X,Y,Z) = \rho(X,Y)\Gamma(Y,X,Z) = \rho(X+Y,Z)\Gamma(Z,X,Y)$. So

$$\begin{aligned} \Gamma(X,Y,Z) &= \rho(X+Y,Z)\Gamma(Z,X,Y) = -\rho(X+Y,Z)\rho(X,Y)\Gamma(Z,Y,X) \\ &= -\rho(X,Y+Z)\Gamma(Y,Z,X) = \rho(X,Y+Z)\rho(Y+Z,X)\rho(Y,Z)\Gamma(X,Z,Y) \\ &= -\rho(Y,Z)\rho(Z,Y)\Gamma(X,Y,Z). \end{aligned}$$

Thus $2\Gamma(X,Y,Z) = 0$. Therefore $\Gamma(X,Y,Z) = 0$, that is $\nabla\omega = 0$.

Corollary 2.17. *The Codazzi-couple of (∇,ω) follows that $\nabla\omega = 0$. If $\nabla\omega = 0$ and ∇ is torsion-free, then $d\omega = 0$.*

For any connection ∇, we define the ω-conjugate connection $\nabla^\dagger$ by

$$X\cdot\omega(Y,Z) = \omega(\nabla_X Y,Z) + \rho(X,Y)\omega(Y,\nabla^\dagger_X Z). \quad (2.11)$$

Lemma 2.18. *For the ω-conjugate connection $\nabla^\dagger$, we have*

i) $\nabla^\dagger$ *is a connection.*
ii) $(\nabla^\dagger)^\dagger = \nabla$.
iii) $\Gamma^\dagger(X,Y,Z) = -\Gamma(X,Y,Z)$.

Proof. (i) We check just one of the property of the connection. We show that $\nabla^{\dagger}_{fX}Y = f\nabla^{\dagger}_{X}Y$. For this, we have

$$\begin{aligned}\rho(f+X,Y)\omega(Y,\nabla^{\dagger}_{fX}Y) &= fX\cdot\omega(Y,Z) - f\omega(\nabla_X Y,Z)\\ &= f\cdot\{X\cdot\omega(Y,Z) - \omega(\nabla_X Y,Z)\}\\ &= \rho(X,Y)f\omega(Y,\nabla^{\dagger}_{X}Z)\\ &= \rho(f+X,Y)\omega(Y,f\nabla^{\dagger}_{X}Z).\end{aligned}$$

So, for all $Y\in Hg(\rho\text{-}Der A)$ we have

$$\rho(f+X,Y)\omega(Y,\nabla^{\dagger}_{fX}Y - f\nabla^{\dagger}_{X}Z) = 0.$$

Since ω is non-degenerate, therefore

$$\nabla^{\dagger}_{fX}Y = f\nabla^{\dagger}_{X}Z.$$

(ii) We have

$$X\cdot\omega(Y,Z) = \omega(\nabla^{\dagger}_{X}Y,Z) + \rho(X,Y)\omega(Y,(\nabla^{\dagger})^{\dagger}_{X}Z).$$

Therefore

$$\begin{aligned}\rho(X,Y)\omega(Y,(\nabla^{\dagger})^{\dagger}_{X}Z) &= X\cdot\omega(Y,Z) - \omega(\nabla^{\dagger}_{X}Y,Z)\\ &= -\rho(Y,Z)X\cdot\omega(Z,Y) + \rho(X+Y,Z)\omega(Z,\nabla^{\dagger}_{X}Y)\\ &= -\rho(Y,Z)\omega(\nabla_X Z,Y) = \rho(X,Y)\omega(Y,\nabla_X Z).\end{aligned}$$

Since ω is non-degenerate, then $(\nabla^{\dagger})^{\dagger} = \nabla$.
(iii) This assertion can be proved directly by (2.11). We have

$$\begin{aligned}\Gamma^{\dagger}(X,Y,Z) &= \rho(X,Y+Z)(\nabla^{\dagger}_{X}\omega)(Y,Z) = X\cdot\omega(Y,Z) - \omega(\nabla^{\dagger}_{X}Y,Z) - \rho(X,Y)\omega(Y,\nabla_X Z)\\ &= -X\cdot\omega(Y,Z) + \omega(\nabla_X Y,Z) + \rho(X,Y)\omega(Y,\nabla_X Z)\\ &= -\rho(X,Y+Z)(\nabla_X\omega)(Y,Z)\\ &= -\Gamma(X,Y,Z).\end{aligned}$$

□

Proposition 2.19. *Consider $\omega,\nabla,\nabla^{\dagger}$ respectively as an any ρ-skew-symmetric 2-form, an arbitrary connection and ω-conjugate of ∇. Then the following assertions are equivalent:*

i) $\nabla\omega = 0$.
ii) $\nabla = \nabla^{\dagger}$.
iii) $T = T^{\dagger}$.

In the following theorem, we show the relationship between $\nabla,\nabla^*,\nabla^{\dagger}$ and ∇^L, where denote respectively an arbitrary connection, g-conjugation, ω-conjugation and L-gauge transformation of ∇.

Theorem 2.20. *Let (g,ω,L) be a compatible triple. Then, we can realize $(id,*,\dagger,L)$ as a 4-element Klein group action on the space of connections, that is*

$$\begin{aligned}&(\nabla^*)^* = (\nabla^{\dagger})^{\dagger} = (\nabla^L)^L = \nabla;\\ &\nabla^{\dagger} = (\nabla^*)^L = (\nabla^L)^*;\\ &\nabla^* = (\nabla^{\dagger})^L = (\nabla^L)^{\dagger};\\ &\nabla^L = (\nabla^*)^{\dagger} = (\nabla^{\dagger})^*.\end{aligned}$$

Proof. By (2.11) and this fact that $(\nabla^\dagger)^\dagger = \nabla$, we have

$$\begin{aligned}\omega(\nabla^\dagger_X Y, Z) &= X \cdot \omega(Y,Z) - \rho(X,Y)\omega(Y, \nabla_X Z)\\ &= X \cdot g(LY, Z) - \rho(X,Y) g(LY, \nabla_X Z)\\ &= g(\nabla^*_X(LY), Z) = g(L(\nabla^*)^L_X Y, Z)\\ &= \omega((\nabla^*)^L_X Y, Z).\end{aligned}$$

Since ω is non-degenerate, then

$$\nabla^\dagger = (\nabla^*)^L. \tag{2.12}$$

If L-gauge transformation is applied to both sides of (2.12), we have

$$(\nabla^\dagger)^L = \nabla^*. \tag{2.13}$$

The last equation holds because L is involutive. Substituting ∇^* into ∇ in (2.13), we arrive at the following formula

$$(\nabla^*)^\dagger = \nabla^L. \tag{2.14}$$

□

From Proposition 2.19 and Theorem 2.13, we deduce that $\nabla\omega = 0$ if and only if $\nabla^* = \nabla^L$.
Actually, we can write $\nabla\omega = 0$ if and only if

$$\nabla^*_X Y = \nabla_X Y + \rho(X,Y)L^{-1}((\nabla_X L)Y) = \nabla_X Y + \rho(X,Y)L((\nabla_X L^{-1})Y).$$

Because by $(\nabla^L)^L = \nabla$, we have

$$L^{-1}((\nabla_X L)Y) = L((\nabla_X L^{-1})Y).$$

Definition 2.21. The Nijenhuis tensor associated with L is defined by the following relation

$$N_L(X,Y) = L[X, LY] + L[LX, Y] - [LX, LY] - L^2[X,Y]. \tag{2.15}$$

When $N_L = 0$, the operator L is said to be integrable.

Definition 2.22. An isomorphism $L : \rho\text{-}DerA \longrightarrow \rho\text{-}DerA$ is said to be a quadratic operator if there exists $\alpha, \beta \in \mathbb{C}$ such that $\alpha+\beta$ and $\alpha\beta$ are real numbers and

$$L^2 - (\alpha+\beta)L + \alpha\beta \cdot id = 0.$$

The important examples of quadratic operators are complex structure J and para-complex structure K.

Lemma 2.23. *Let (∇, L) be the Codazzi-couple, where L is a quadratic operator. Then*

$$N_L(X,Y) = L^2 T(X,Y) - LT(X, LY) - LT(X,Y) - T(LX, LY). \tag{2.16}$$

Proof. Since (∇, L) is Codazzi-couple, then

$$\rho(X,Y)(\nabla_X L)Y = (\nabla_Y L)X.$$

Thus

$$\nabla_X(LY) - L\nabla_X Y - \rho(X,Y)\nabla_Y(LX) + \rho(X,Y)L\nabla_Y X = 0.$$

So

$$\nabla_X(LY) - \rho(X,Y)\nabla_Y(LX) - L[X,Y] = LT(X,Y). \tag{2.17}$$

Substitute X and Y by LY and LX respectively, we have

$$\nabla_{LY}(L^2X) - \rho(Y,X)\nabla_{LX}(L^2Y) - L[LY, LX] = LT(LY, LX).$$

L is a quadratic operator, then $L^2 - (\alpha+\beta)L + \beta \cdot id = 0$. Thus, we get

$$(\alpha+\beta-L)[LY, LX] - \alpha\beta(\nabla_{LY}X - \rho(Y,X)\nabla_{LX}Y) = (L-\alpha-\beta)T(LY, LX). \tag{2.18}$$

Using (2.17) and (2.18), we get the following relations:

$$\alpha\beta L(\nabla_X(LY) - L\nabla_X Y - \rho(X,Y)\nabla_Y(LX) + \rho(X,Y)L\nabla_Y X) = 0$$

and

$$L(\nabla_X(LY) - \rho(X,Y)\nabla_Y(LX) - L[X,Y]) = L^2 T(X,Y).$$

Summing the above two equations we obtain

$$\alpha\beta(N_L(X,Y)) = \alpha\beta\{L^2T(X,Y) - LT(X,LY) - LT(LX,Y) + T(LX,LY)\}.$$

□

Remark 2.24. From the above lemma, we can deduce that the quadratic operator L, which is Codazzi-couple with a connection ∇, is integrable if ∇ is torsion-free. The converse of this corollary does not hold, because if ∇ is torsion-free and (∇, L) is Codazzi-couple, then $N_L = 0$ if $(\nabla_{LX}L)Y = L(\nabla_X L)Y$. But from $(\nabla_X L^2) = 0$, we get

$$(\nabla_X L)(LY) = L(\nabla_Y L)X. \tag{2.19}$$

From Remark 2.24 and Proposition 2.2, we have the following proposition

Proposition 2.25. *If there exists a torsion-free connection ∇ such that ∇^L is torsion-free, then the quadratic operator L is integrable.*

3. Codazzi-(Para-)Kähler ρ-commutative algebras

In this section, we introduce the notion of holomorphic statistical structure on A and give some lemma and propositions to find some results.

Definition 3.1. [2] An almost complex connection on A is a ρ-linear connection ∇ such that $\nabla J = 0$.

Theorem 3.2. *[2] Let (A, J, g) be an almost Hermitian ρ-commutative algebra and $\widehat{\nabla}$ be Levi-Civita connection. Then we have*

$$2g(Z,(\widehat{\nabla}_X J)Y) = d\omega(JZ,JY,X) - d\omega(Z,Y,X) + \rho(Z,Y)\rho(Y+Z,X)g(JX,N_J(Y,Z)),$$
$$\forall X,Y,Z \in Hg(\rho\text{-}Der A).$$

Moreover, $\widehat{\nabla}$ is the almost complex connection if and only if $N_J = 0$ and Kähler form ω is closed (i.e., $d\omega = 0$).

Lemma 3.3. *If ∇ is a torsion-free connection, then for all $X, Y, Z \in Hg(\rho\text{-}Der A)$ we have*

$$d\omega(X,Y,Z) = \rho(X,Y+Z)\nabla_X\omega(Y,Z) + \rho(X+Y,Z)\nabla_Y\omega(Z,X) + \nabla_Z\omega(X,Y). \tag{3.1}$$

Proof. The proof of this lemma is straightforward. □

Theorem 3.4. *Consider the following assumptions*

i) *(∇, g) is Codazzi-couple, where ∇ is a torsion-free connection and g is a metric.*
ii) *(∇, L) is Codazzi-couple, where L (either J or K) is compatible with g.*

Then, (A, g, L) is a (Para)-Kähler ρ-commutative algebra.

Proof. Remark 2.24 follows that $N_J = 0$, thus it is enough to show that $d\omega = 0$. For this we have

$$\begin{aligned}\rho(X,Y+Z)(\nabla_X\omega)(Y,Z) &= X\cdot\omega(Y,Z) - \omega(\nabla_X Y,Z) - \rho(X,Y)\omega(Y,\nabla_X Z)\\ &= X\cdot g(LY,Z) - g(L\nabla_X Y,Z) - \rho(X,Y)g(LY,\nabla_X Z)\\ &= \rho(X,Y+Z)(\nabla_X g)(LY,Z) + g(\nabla_X(LY),Z) - g(L\nabla_X Y,Z)\\ &= \rho(X,Y+Z)(\nabla_X g)(LY,Z) + \rho(X,Y)g((\nabla_X L)Y,Z)\\ &= C(X,LY,Z) + \rho(X,Y)g((\nabla_X L)Y,Z).\end{aligned} \tag{3.2}$$

By Lemma 3.3 and (3.2), we get

$$\begin{aligned} d\omega(X,Y,Z) = & \, C(X,LY,Z) + \rho(X,Y+Z)C(Y,LZ,X) + \rho(X+Y,Z)C(Z,LX,Y) \quad (3.3) \\ & + \rho(X,Y)g((\nabla_X L)Y,Z) + \rho(X,Y+Z)\rho(Y,Z)g((\nabla_Y L)Z,X) \quad (3.4) \\ & + \rho(Y,Z)g((\nabla_Z L)X,Y), \end{aligned}$$

and

$$\begin{aligned} d\omega(Z,Y,X) = & \, C(Z,LY,X) + \rho(Z,X+Y)C(Y,LZ,X) + \rho(Y+Z,X)C(X,LZ,Y) \\ & + \rho(Z,Y)g((\nabla_Z L)Y,X) + \rho(Z,Y+X)\rho(Y,X)g((\nabla_Y L)X,Z) \\ & + \rho(Y,X)g((\nabla_X L)Z,Y). \end{aligned}$$

Since C is totally ρ-symmetric, we get

$$d\omega(X,Y,Z) = \rho(X+Y,Z)\rho(X,Y)d\omega(Z,Y,X).$$

On the other hand, $d\omega(X,Y,Z) = -\rho(X,Y)d\omega(Y,X,Z) = -\rho(Y,Z)d\omega(X,Z,Y)$. So

$$\begin{aligned} d\omega(X,Y,Z) &= \rho(X+Y,Z)\rho(X,Y)d\omega(Z,Y,X) = -\rho(X+Y,Z)d\omega(Z,X,Y) \\ &= \rho(Y,Z)d\omega(X,Z,Y) = -d\omega(X,Y,Z). \end{aligned}$$

Therefore $d\omega = 0$. □

Theorem 3.5. *Let ∇ be a torsion-free connection, g is a metric, L be either J or K and (g,ω,L) a compatible triple. Then any two of the following statements imply the third*

i) (∇, g) *is Codazzi-couple.*
ii) (∇, L) *is Codazzi-couple.*
iii) $\nabla\omega = 0$.

Proof. From (i) and (ii) to (iii): Setting $\alpha(X,Y,Z) = \rho(X,Y)g((\nabla_X L)Y,Z)$. (2.19) follows that

$$\alpha(X,LY,Z) = \alpha(X,Y,LZ). \tag{3.5}$$

By (3.2), we get

$$\rho(X,Y+Z)(\nabla_X\omega)(Y,Z) = C(X,LY,Z) + \alpha(X,Y,Z).$$

On the other hand $\rho(X,Y+Z)(\nabla_X\omega)(Y,Z) = -\rho(X,Y+Z)\rho(Y,Z)(\nabla_X\omega)(Z,Y)$. Thus

$$C(X,LY,Z) + \alpha(X,Y,Z) + \rho(Y,Z)C(X,LZ,Y) + \rho(Y,Z)\alpha(X,Z,Y) = 0. \tag{3.6}$$

Codazzi-couple of ∇ and L implies that

$$\alpha(X,Y,Z) = \rho(X,Y)\alpha(Y,X,Z). \tag{3.7}$$

By invoking (3.3), (3.6), (3.7) and totally ρ-symmetry of C, we deduce that

$$d\omega(X,Y,Z) = \rho(X+Y,Z)C(Z,LX,Y) + \rho(X+Y,Z)\rho(X,Y)\alpha(Y,Z,X). \tag{3.8}$$

On the other hand, by Theorem 3.4, (3.8) translates to

$$0 = \rho(X+Y,Z)C(Z,LX,Y) + \rho(X+Y,Z)\rho(X,Y)\alpha(Y,Z,X). \tag{3.9}$$

Substituting X with LX, we have

$$\rho(Y,X+Z)(\nabla_Y\omega)(LZ,X) = 0,$$

for any arbitrary ρ-derivations. Therefore $\nabla\omega = 0$.
From (i) and (iii) to (ii): Statement (iii) implies that

$$C(X,LY,Z) + \alpha(X,Y,Z) = 0. \tag{3.10}$$

Since $\alpha(X,LY,Z) = \alpha(X,Y,LZ)$, then by (3.10) we get

$$C(X,L^2Y,Z) = C(X,LY,LZ).$$

By (i), C is total symmetric, so

$$C(X, LY, LZ) = \rho(X,Y)C(Y, LX, LZ),$$

which, in terms of A

$$A(X, Y, LZ) = \rho(X,Y)A(Y, X, LZ).$$

Thus

$$\rho(X,Y)g((\nabla_X L)Y, LZ) = \rho(X,Y)\rho(Y,X)g((\nabla_Y L)X, LZ).$$

Therefore $\rho(X,Y)(\nabla_X L)Y = (\nabla_Y L)X$, which is (ii).
From (ii) and (iii) to (i): (3.5) and (3.7) give us

$$\alpha(X, LY, Z) = \rho(X,Y)\alpha(Y, LX, Z). \tag{3.11}$$

Then by statement (iii), we get

$$C(X, L^2Y, Z) = \rho(X,Y)C(Y, L^2X, Z).$$

Therefore

$$C(X,Y,Z) = \rho(X,Y)C(Y,X,Z).$$

□

Definition 3.6. Let A be a ρ-commutative algebra and ∇ be a connection on A.

i) A non-degenerate 2-form ω on A is called a symplectic form if $d\omega = 0$. In this case, (A, ω) is said to be a symplectic ρ-commutative algebra.
ii) If $\nabla\omega = 0$, then ∇ is called a symplectic connection.
iii) The triple (A, ω, ∇) is called a Fedesov ρ-commutative algebra if ω is a symplectic form and ∇ is a symplectic connection.

According to Theorem 3.4, we have a (para-)Kähler ρ-commutative algebra (A, g, ∇, L, ρ), where (A, g, ∇, ρ) is an arbitrary statistical ρ-commutative algebra and L ((para-)complex structure) is compatible with g such that ∇ and L are Codazzi-couple. Furthermore, Theorem 3.5 implies that if ∇ is a symplectic connection, then $(A, \nabla, \omega, \rho)$ is a Fedosov ρ-commutative algebra. Theorem 3.5 also says that, we have a (para-)Kähler ρ-commutative algebra (A, ∇, ω, L), where (A, ∇, ω) is any Fedosov ρ-commutative algebra and L is a (para-)complex structure compatible with ω such that ∇ and L are Codazzi-couple. Actually, if we consider statistical ρ-commutative algebra (A, ∇, g), Fedosov ρ-commutative algebra (A, ∇, ω) and Codazzi-couple of ∇ and L, then any two of these statements imply the third.

Let ∇ be a torsion-free connection on A and ∇^*, $\nabla^\dagger$, and ∇^L be g-conjugate, ω-conjugate and L-gauge transformation of a connection ∇, respectively. Then, by Propositions 2.2, 2.8 and 2.19, we have the following theorem.

Theorem 3.7. *Let (g, ω, L) be a compatible triple. A is (para-)Kähler ρ-commutative algebra if any two of the following three statements are true:*

i) ∇^* *is torsion-free.*
ii) $\nabla^\dagger$ *is torsion-free.*
iii) ∇^L *is torsion-free.*

Definition 3.8. Let $(A, g, J(K), \rho)$ be an almost (Para-)Kähler ρ-commutative algebra with a connection ∇. If ∇ is Codazzi-couple to both g and $J(K)$, then $(A, g, J(K), \rho)$ is called an almost Codazzi-(Para-) Kähler ρ-commutative algebra. Furthermore, if ∇ is torsion-free, then by 2.24, $J(K)$ is automatically integrable and $d\omega = 0$. In this case, $(A, g, J(K), \nabla, \rho)$ is said to be Codazzi-(Para-) Kähler ρ-commutative algebra.

Definition 3.9. Let (A, ∇, g) be a statistical ρ-commutative algebra. If (A, J, g) is a (an almost) (Para-) Hermitian ρ-commutative algebra, then (A, ∇, g, J) is called a (an almost) (Para-) Hermitian statistical ρ-commutative algebra. If (A, J, g) is a (an almost) (Para-) Kähler ρ-commutative algebra, then (A, ∇, g, J) is called a (an almost) (Para-) Kähler statistical ρ-commutative algebra.

By the above definition, Definition 3.8 can be implied in other words, that is Codazzi-(para-)Kähler ρ-commutative algebra is a (para-)Kähler (or Fedosov) ρ-commutative algebra which is at the same time statistical.

Definition 3.10. A Quadruple (g, ω, L, ∇) consisting of two non-degenerate 2-forms g and ω, which are ρ-symmetric and ρ-skew-symmetric respectively, an isomorphism L, which is either J or K and a torsion-free connection ∇ is called a compatible quadruple on ρ-commutative algebra A, if the following conditions hold

i) $\omega(X, Y) = g(LX, Y)$.
ii) $g(LX, Y) + g(X, LY) = 0$.
iii) $\omega(LX, Y) = \omega(LY, X)$.
iv) $\rho(X, Y)(\nabla_X L)Y = (\nabla_Y L)X$.
v) $\rho(X, Y + Z)(\nabla_X g)(Y, Z) = \rho(Y, X + Z)\rho(X, Y)(\nabla_Y g)(X, Z)$.
vi) $\rho(X, Y + Z)(\nabla_X \omega)(Y, Z) = 0$.

Proposition 3.11. *Consider the following conditions*

a) (g, L, ∇) *satisfy (ii), (iv) and (v),*
b) (ω, L, ∇) *satisfy (iii), (iv) and (vi),*
c) (g, ω, ∇) *satisfy (v) and (vi).*

Then a ρ-commutative algebras A is a Codazzi-(Para-) Käler ρ-commutative algebras if and only if any of the above conditions holds.

4. Holomorphic statistical ρ-commutative algebras

In this section we introduce the notion of holomorphic statistical structure on A and give some lemmas and propositions to find some results.

Lemma 4.1. *Let (A, ∇, g, J) be an almost Hermitian statistical ρ-commutative algebra. Then we have*

$$g((\nabla_X J)Y, Z) = -\rho(X, Y)g(Y, (\nabla^*_X J)Z), \quad \forall X, Y, Z \in Hg(\rho\text{-}DerA). \tag{4.1}$$

Proof. Using $\rho(X, Y)(\nabla_X J)Y = \nabla_X(JY) - J\nabla_X Y$, we get

$$\begin{aligned} g((\nabla_X J)Y, Z) &= \rho(Y, X)g(\nabla_X(JY), Z) - \rho(Y, X)g(J\nabla_X Y, Z) \\ &= \rho(Y, X)X \cdot g(JY, Z) - g(JY, \nabla^*_X Z) \\ &\quad + \rho(Y, X)X \cdot g(Y, JZ) - g(Y, \nabla^*_X(JZ)) \\ &= -\rho(X, Y)g(Y, (\nabla^*_X J)Z). \end{aligned}$$

□

Proposition 4.2. *Let (A, ∇, g, J) be an almost Hermitian statistical ρ-commutative algebra. The covariant derivatives $\nabla J, \nabla^* J$ of J with respect to the torsion free connections ∇ and ∇^* are given by*

$$\begin{aligned} 2g(Z, (\nabla_X J)Y) &= 2g(Z, (K_X J)Y) + d\omega(JZ, JY, X) - d\omega(Z, Y, X) \\ &\quad + \rho(Z, Y)\rho(Y + Z, X)g(JX, N_J(Y, Z)), \\ 2g(Z, (\nabla^*_X J)Y) &= -2g(Z, (K_X J)Y) + d\omega(JZ, JY, X) - d\omega(Z, Y, X) \\ &\quad + \rho(Z, Y)\rho(Y + Z, X)g(JX, N_J(Y, Z)), \end{aligned}$$

for any $X, Y, Z \in Hg(\rho\text{-}DerA)$.

Proof. The proof can be completed by Theorem 3.2 and $K_XY = \nabla_XY - \widehat{\nabla}_XY = \widehat{\nabla}_XY - \nabla^*_XY$. □

Corollary 4.3. *Let (A, ∇, g, J) be an almost Kähler statistical ρ-commutative algebra. Then*

$$2g(Z,(\nabla_XJ)Y) = 2g(Z,(K_XJ)Y) + \rho(Z,Y)\rho(Y+Z,X)g(JX,N_J(Y,Z)),$$
$$2g(Z,(\nabla^*_XJ)Y) = -2g(Z,(K_XJ)Y) + \rho(Z,Y)\rho(Y+Z,X)g(JX,N_J(Y,Z)),$$

for any $X,Y,Z \in Hg(\rho\text{-}DerA)$.

Corollary 4.4. *Let (A, ∇, g, J) be a Kähler statistical ρ-commutative algebra. Then*

$$2(Z,(\nabla_XJ)Y) = g(Z,(K_XJ)Y),$$
$$2(Z,(\nabla^*_XJ)Y) = -g(Z,(K_XJ)Y),$$

for any $X,Y,Z \in Hg(\rho\text{-}DerA)$.

Lemma 4.5. *Let (A, ∇, g, J) be a Kähler statistical ρ-commutative algebra. Then for any $X, Y \in Hg(\rho\text{-}DerA)$, we have*

$$\nabla_X(JY) = J\nabla^*_XY.$$

Proof. We have

$$\begin{aligned}0 &= (\nabla_X\omega)(JY,Z) = X\cdot g(JY,Z) - \omega(\nabla_X(JY),Z) - \rho(X,Y)\omega(JY,\nabla_XZ)\\ &= -g(\nabla^*_XY,Z) + g(\nabla_X(JY),JZ) = -g(J\nabla^*_XY,JZ) + g(\nabla_X(JY),JZ)\\ &= g(\nabla_X(JY) - J\nabla^*_XY,JZ).\end{aligned}$$

So $\nabla_X(JY) = J\nabla^*_XY$. □

Definition 4.6. A holomorphic statistical structure on A is a triple (∇, g, J) such that

i) (∇, g) is a statistical structure on A,
ii) (g, J) is a Kähler structure on A,
iii) $K_X(JY) + JK_XY = 0 \quad \forall X, Y \in \rho\text{-}DerA$.

The ρ-commutative algebra A equipped with this structure is called holomorphic statistical ρ-commutative algebra.

Theorem 4.7. *Let (A, ∇, g, J) be a Kähler statistical ρ-commutative algebra. Then (A, ∇, g, J) is a holomorphic statistical ρ-commutative algebra.*

Proof. It is enough to show that $K_X(JY) + JK_XY = 0$ for any $X, Y \in Hg(\rho\text{-}DerA)$. Since $\nabla\omega = 0$, we have

$$\begin{aligned}0 = (\nabla_X\omega)(Y,Z) &= X\cdot\omega(Y,Z) - \omega(\nabla_XY,Z) - \rho(X,Y)\omega(Y,\nabla_XZ)\\ &= g(\widehat{\nabla}_X(JY),Z) + g(K_XY,JZ) + g(\widehat{\nabla}_XY,JZ) - \rho(X,Y)g(JY,K_XZ)\\ &= \rho(X,Y)g((\widehat{\nabla}_XJ)Y,Z) + g(K_XY,JZ) - \rho(X,Y)g(JY,K_XZ).\end{aligned}$$

By Theorem 3.2, $\widehat{\nabla}$ is an almost complex connection. So, we conclude that

$$0 = (\nabla_X\omega)(Y,Z) = g(K_XY,JZ) - \rho(X,Y)g(JY,K_XZ) = -\{g(JK_XY + K_X(JY),Z)\}.$$

Therefore, $JK_XY + K_X(JY) = 0$. □

Example 4.8. Let us go back to the extended hyperplane $A^2_q := \langle 1, x, y, x^{-1}, y^{-1}, xy = qyx\rangle$ given by Example 2.6. In [2], the present authors obtained all complex structures of degree zero that have the following expression

$$J(\frac{\partial}{\partial x}) = -i\frac{\partial}{\partial x}, \quad J(\frac{\partial}{\partial y}) = i\frac{\partial}{\partial y},$$
$$J(\frac{\partial}{\partial x}) = i\frac{\partial}{\partial x}, \quad J(\frac{\partial}{\partial y}) = -i\frac{\partial}{\partial y}.$$

As seen, for this structures $d\omega = 0$, $N_J = 0$ and all Hermitian metrics are as follows

$$g = (dx \otimes dy + q dy \otimes dx) g_{12}.$$

Moreover, in Example 2.6 we obtain all statistical connections on A_q^2 with respect to metric g with components $g_{11} = g_{22} = 0$, $g_{12} = x^{-1}y^{-1}$, which actually the components of Hermitian metric can have the same expression. In this example, we intend to investigate which connections admits a holomorphic structure. For this, it is enough to check the property $JK_XY + K_X(JY) = 0$ for any $X, Y \in \rho\text{-}DerA_q^2$. Let $J(\frac{\partial}{\partial x}) = i\frac{\partial}{\partial x}$, $J(\frac{\partial}{\partial y}) = -i\frac{\partial}{\partial y}$ and consider the following cases:

Case 1: If $X = \frac{\partial}{\partial x}$ and $Y = \frac{\partial}{\partial y}$, we have

$$\begin{aligned} K_{\frac{\partial}{\partial x}}(J(\frac{\partial}{\partial y})) + JK_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y} &= -i\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y} + i\widehat{\nabla}_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y} + J(\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y} - \widehat{\nabla}_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y}) \\ &= -i(\Gamma_{12}\frac{\partial}{\partial x} + \Gamma_{12}^2\frac{\partial}{\partial y}) + i(\hat{\Gamma}_{12}^1\frac{\partial}{\partial x} + \hat{\Gamma}_{12}^2\frac{\partial}{\partial y}) \\ &\quad + J(\Gamma_{12}\frac{\partial}{\partial x} + \Gamma_{12}^2\frac{\partial}{\partial y}) - J(\hat{\Gamma}_{12}^1\frac{\partial}{\partial x} + \hat{\Gamma}_{12}^2\frac{\partial}{\partial y}) \\ &= -i\Gamma_{12}\frac{\partial}{\partial x} - i\Gamma_{12}^2\frac{\partial}{\partial y} + i\Gamma_{12}^1\frac{\partial}{\partial x} - i\Gamma_{12}^2\frac{\partial}{\partial y} \\ &= -2i\Gamma_{12}^2\frac{\partial}{\partial y}. \end{aligned}$$

Case 2: If $X = \frac{\partial}{\partial y}$ and $Y = \frac{\partial}{\partial x}$, in the similar way, we have

$$K_{\frac{\partial}{\partial y}}(J(\frac{\partial}{\partial x})) + JK_{\frac{\partial}{\partial y}}\frac{\partial}{\partial x} = 2i\Gamma_{21}^1\frac{\partial}{\partial x}.$$

Case 3: If $X = \frac{\partial}{\partial x}$ and $Y = \frac{\partial}{\partial x}$, we have

$$K_{\frac{\partial}{\partial x}}(J(\frac{\partial}{\partial x})) + JK_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x} = 2i\Gamma_{11}^1\frac{\partial}{\partial x} - 2i\Gamma_{11}^2\frac{\partial}{\partial y}.$$

Case 4: If $X = \frac{\partial}{\partial y}$ and $Y = \frac{\partial}{\partial y}$, we get

$$K_{\frac{\partial}{\partial y}}(J(\frac{\partial}{\partial y})) + JK_{\frac{\partial}{\partial y}}\frac{\partial}{\partial y} = -2i\Gamma_{22}^2\frac{\partial}{\partial y} + 2i\hat{\Gamma}_{22}^2\frac{\partial}{\partial y}.$$

So $JK_XY + K_X(JY) = 0$ if and only if

$$\Gamma_{11}^1 = -x^{-1}, \;\; \Gamma_{22}^2 = -y^{-1}, \;\; \Gamma_{11}^2 = -x^{-1}, \;\; \Gamma_{12}^2 = \Gamma_{21}^2 = 0, \;\; \Gamma_{12}^1 = \Gamma_{21}^1 = 0, \;\; \Gamma_{22}^1 = 0.$$

Therefore, there exists only one connection on $\rho\text{-}DerA_q^2$ with the above components which the triple (∇, g, J) is a holomorphic statistical structure.

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