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Clairaut Pointwise Slant Submersions from Locally Product Riemannian Manifolds

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(Dedicated to the memory of Prof. Dr. Krishan Lal DUGGAL (1929 - 2022))

ABSTRACT

In this paper, we consider pointwise slant submersions from locally product Riemannian manifolds. We first give a necessary and sufficient condition for a curve on the total manifold to be a geodesic and then focus investigate new Clairaut conditions for considered submersion. In a main theorem, we find a new necessary and sufficient condition for a pointwise slant submersion to be Clairaut in case of its total manifold is locally product Riemannian manifold. Finally, we present an illustrative example for this kind of submersion which satisfies Clairaut condition.

Keywords: *Riemannian submersion; Clairaut pointwise slant submersion; locally product Riemannian manifold.*

AMS Subject Classification (2020): *Primary: 53C12 ; Secondary: 53C15, 53C20, 53C55.*

1. Introduction

O'Neill [20] and Gray [10] introduced Riemannian submersion between two Riemannian manifolds for the first time. In differential geometry, to equate geometric structures described on the above mentioned manifolds, Riemannian submersions are utilized broadly as differential maps. Riemannian submersions have important application areas in medical imaging, in robotics theory and Kaluza-Klein theory and many more. Afterwards, Watson [38] introduced almost Hermitian submersions and then Şahin [28] presented the concept of anti-invariant submersions and Lagrangian submersion from almost Hermitian manifolds onto Riemannian manifolds and this concept studied in [21, 32, 11, 5, 26, 15]. We refer interested readers to [31] and references therein for current progress and applications of Riemannian submersions. Şahin [30] generalized anti-invariant submersions, showing the advantages of study the geometry of the total manifold of semi invariant submersions and the same idea investigated by Ozdemir et al. [21] and [14]. Most of the studies related to Riemannian, almost Hermitian or contact Riemannian submersions can be found in the book [9]. Then, some researchers studies some different types of Riemannian submersions such as generic submersion [1, 7, 23, 27], slant submersion [29, 17, 12, 16], semi-slant submersion [22], pointwise slant submersion [4, 8, 18], hemi-slant submersion [33], conformal semi-slant submersion [2] and pointwise semi-slant submersions [24]. Pointwise slant submersions from locally product Riemannian manifolds are natural generalizations of anti-invariant submersions from locally product Riemannian manifolds which were studied in [36].

In the investigation of geodesic upon a surface of revolution, a well known Clairaut's theorem [6] says that for any geodesic ς on the revolution surface $\overline{M}_L$ the product $r\sin\alpha$ is constant along ς, where $\alpha(s)$ be the angle between $\varsigma(s)$ and the meridian curve through $\varsigma(s)$, $s \in J$. He also introduced and studied the theory of Riemannian submersions which satisfy a generalization of Clairaut's theorem. Then by following this study, Clairaut submersions have been studied different kinds of structures. Allison [3] presented Lorentzian Clairaut submersions. Lee et al. [19] considered Clairaut anti-invariant submersions with total Kaehler manifolds. Clairaut anti-invariant submersions whose total manifolds are Sasakian and Kenmotsu were given by Tastan and Gerdan [34] and in [35], the authors also investigated Clairaut anti-invariant submersions from cosymplectic manifolds. Clairaut anti-invariant submersions whose total manifold is paracosymplectic manifold are given in [13] with characterization theorems.

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In [31], Şahin investigated Clairaut conditions for pointwise slant submersions from a Kaehler manifold onto a Riemannian manifold and the author studied pointwise slant submersions by providing a consequence which defines the geodesics on the total space of this type submersions.

In this paper, we consider Clairaut pointwise slant submersions from a locally product Riemannian manifold (l.p.R manifold) onto a Riemannian manifold. In Section 2, we give some expressions that we will need in the next subsequent section. In Section 3, we investigate pointwise slant submersions by providing a consequent which defines the geodesics on the total space of these types of submersions. We also give a non-trivial example of the Clairaut pointwise slant submersions whose total manifolds are locally product Riemannian.

2. Preliminaries

In this section, we give the definitions and terminology used throughout this paper. We recall some necessary facts and formulas from the theory of Riemannian manifold.

2.1. Locally product Riemannian manifolds

In this section, we give brief information for locally product Riemannian manifolds.

Let $\overline{M}_L$ be a $(m+n)$-dimensional smooth manifold given a tensor P of type $(1,1)$ such that

$$P^2 = I, \ (P \neq \pm I), \tag{2.1}$$

where I is the identity morphism of tangent space $T_p\overline{M}_L$ at $p \in \overline{M}_L$. If $\overline{M}_L$ is equipped with the structure P, then $(\overline{M}_L, P)$ is an almost product manifold. If an almost product manifold $\left(\overline{M}_L, P\right)$ admits a Riemannian metric $g_{\bar{M}_L}$ such that

$$g_{\bar{M}_L}(P\overline{V_1}, P\overline{V_2}) = g_{\bar{M}_L}(\overline{V_1}, \overline{V_2}) \text{ or } g_{\bar{M}_L}(P\overline{V_1}, \overline{V_2}) = g_{\bar{M}_L}(\overline{V_1}, P\overline{V_2}), \tag{2.2}$$

where $\overline{V_1}, \overline{V_2} \in T\overline{M}_L$, then we say that $\overline{M}_L$ is an almost product Riemannian manifold. An almost product Riemannian manifold $\overline{M}_L$ is called a l.p.R manifold if

$$(\overline{\nabla}_{\overline{V_1}} P)\overline{V_2} = 0, \tag{2.3}$$

where $\overline{V_1}, \overline{V_2} \in T\overline{M}_L$ and $\overline{\nabla}$ is the Riemannian connection on $\overline{M}_L$ [37].

2.2. Riemannian submersions

In this section, we recall the fundamental definitions and notions of a Riemannian submersion

Let be a surjective mapping $\varphi : \overline{M}_L \to \overline{N}_R$ between two Riemannian manifolds $\left(\overline{M}_L, g_{\bar{M}_L}\right)$ and $\left(\overline{N}_R, g_{\bar{N}_R}\right)$ such that $\dim(\overline{M}_L) > \dim\left(\overline{N}_R\right)$, is called a Riemannian submersion if it satisfies the following conditions:

(i). The fibers $\varphi^{-1}(a), a \in \overline{N}_R$, are $r-$dimensional Riemannian submanifolds of $\overline{M}_L$, where $r = \dim(\overline{M}_L) - \dim\left(\overline{N}_R\right)$.

In which case, for a vector field X_1 on $\overline{M}_L$, if it is always tangent to fibers then it is called vertical and if it is always orthogonal to fibers then it is called horizontal. If a vector field X_1 on $\overline{M}_L$ is horizontal and $\varphi-$related to a vector field X_{1*} on $\overline{N}_R$, then it is called basic, i.e., for all $a \in \overline{N}_R$, $\varphi_* X_{1a} = X_{1*\varphi_*(a)}$, where φ_* is the derivative map of φ.

(ii). φ_{*q} preserves the length of the horizontal vectors.

In which case, we get $g_{\bar{M}_L}(X_1, Y_1) = g_{\bar{N}_R}(\varphi_* X_1, \varphi_* Y_1)$, for all $q \in \overline{M}_L$ and for any horizontal vectors $X_1, Y_1 \in (\ker \varphi_*)^{\perp}$ at q,

A Riemannian submersion $\varphi : \overline{M}_L \to \overline{N}_R$ specifies two $(1,2)$ types of tensor fields $\mathcal{T}$ and $\mathcal{A}$ on $\overline{M}_L$, by the following formulas [20]:

$$\mathcal{T}(E,G) = \mathcal{T}_E G = h\overline{\nabla}_{vE} vG + v\overline{\nabla}_{vE} hG, \tag{2.4}$$

$$\mathcal{A}(E,G) = \mathcal{A}_E G = v\overline{\nabla}_{hE} hG + h\overline{\nabla}_{hE} vG, \tag{2.5}$$

for all $E, G \in \chi(\overline{M}_L)$, where h and v denote the horizontal and vertical projections, respectively. It is easy to see that $\mathcal{A}_E$ and $\mathcal{T}_E$ are skewsymmetric operators.

Let X_1, X_2 be horizontal and V_1, V_2 be vertical vector fields on $\overline{M}_L$, then we get

$$\mathcal{A}_{X_1} X_2 = -\mathcal{A}_{X_2} X_1 = \frac{1}{2} v\left[X_1, X_2\right], \tag{2.6}$$

$$\mathcal{T}_{V_1}V_2 = \mathcal{T}_{V_2}V_1. \tag{2.7}$$

From (2.4) and (2.5), we get

$$\overline{\nabla}_{V_1}V_2 = \mathcal{T}_{V_1}V_2 + \hat{\nabla}_{V_1}V_2, \tag{2.8}$$

$$\overline{\nabla}_{V_1}X_1 = \mathcal{T}_{V_1}X_1 + h\overline{\nabla}_{V_1}X_1, \tag{2.9}$$

$$\overline{\nabla}_{X_1}V_1 = \mathcal{A}_{X_1}V_1 + v\overline{\nabla}_{X_1}V_1, \tag{2.10}$$

$$\overline{\nabla}_{X_1}X_2 = \mathcal{A}_{X_1}X_2 + h\overline{\nabla}_{X_1}X_2, \tag{2.11}$$

for any $X_1, X_2 \in \Gamma(\ker\varphi_*)^{\perp}$ and $V_1, V_2 \in \Gamma(\ker\varphi_*)$. Also, if X_1 is basic then $h\overline{\nabla}_{V_1}X_1 = h\overline{\nabla}_{X_1}V_1 = \mathcal{A}_{X_1}V_1$. We observe that the horizontal distribution is totally geodesic if and only if $\mathcal{A} \equiv 0$. From above equation, we can also see that on the fibers, $\mathcal{T}$ take actions as the second fundamental form.

Let $\varphi : \overline{M}_L \to \overline{N}_R$ be a surjective mapping between two Riemannian manifolds $(\overline{M}_L, g_{\bar{M}_L})$ and $(\overline{N}_R, g_{\bar{N}_R})$. Then for $E, G \in \Gamma\left(T\overline{M}_L\right)$ the second fundamental form of φ is described as

$$\left(\overline{\nabla}\varphi_*\right)(E,G) = \overline{\nabla}^{\varphi}_{E}\varphi_* G - \varphi_*\left(\overline{\nabla}_E G\right), \tag{2.12}$$

where $\overline{\nabla}$ is the Riemannian connection and $\overline{\nabla}^{\varphi}$ is the pull-back connection. From [18], the second fundamental form is well-known to be symmetric. Besides, φ is called totally geodesic if $\left(\overline{\nabla}\varphi_*\right)(E,G) = 0$ for all $E, G \in \Gamma\left(T\overline{M}_L\right)$.

The fibers of φ is called totally umbilical if

$$\mathcal{T}_{V_1}V_2 = g_{\bar{M}_L}(V_1, V_2)\,H, \tag{2.13}$$

for any $V_1, V_2 \in \Gamma(\ker\varphi_*)$, here H is the mean curvature vector field of the fiber of φ [21].

2.3. Pointwise slant submersion

In this section, we present results on the geometry of pointwise slant submersions from locally product Riemannian manifolds.

Definition 2.1. [25] Let $\varphi : (\overline{M}_L, g_{\bar{M}_L}, P) \to (\overline{N}_R, g_{\bar{N}_R})$ be a Riemannian submersion. Where $(\overline{M}_L, g_{\bar{M}_L}, P)$ is an almost product Riemannian manifold and $(\overline{N}_R, g_{\bar{N}_R})$ is a Riemannian manifold. If the Wirtinger angle $\alpha(V)$ between PV and the space $(\ker\varphi_*)_q$ is independent of the choice of the nonzero vector $V \in \Gamma(\ker\varphi_*)$,at each given point $q \in \overline{M}_L$, then φ is called a pointwise slant submersion. The angle α is called the slant function of the pointwise slant submersion.

Definition 2.2. [25] If the slant function $\alpha = \frac{\pi}{2}$ at q of the pointwise slant submersion, then a point q in a pointwise slant submersion is called totally real. In the same way, if the slant function of the pointwise slant submersion $\alpha = 0$ at q, then a point q is called a complex point. If the slant function of the pointwise slant submersion is neither a totally real nor a complex Riemannian submersion, then a pointwise slant submersion is said to be proper.

Remark 2.1. [18] If the slant function α of the pointwise slant submersion is globally constant, then a pointwise slant submersion is called slant, means that, α is also independent of the choice of the point on $\overline{M}_L$. In this state, the constant angle α is called the slant angle of the slant submersion. If every point of $\overline{M}_L$ is a totally real point, then a pointwise slant submersion φ is called totally real.

We can say that slant submersions, anti-invariant and invariant submersions can be given as examples of pointwise slant submersions. We will give an example for proper pointwise slant submersions.

Example 2.1. Let R^5 be the standard Euclidean space with the standard metric $g_{\bar{M}_L}$. Suppose that P_1 and P_2 are the almost product Riemannian structures on R^5 such that

$$\left\{\begin{array}{c} P_1(\frac{\partial}{\partial x_1}) = -\frac{\partial}{\partial x_2}, P_1(\frac{\partial}{\partial x_2}) = -\frac{\partial}{\partial x_1}, P_1(\frac{\partial}{\partial x_3}) = -\frac{\partial}{\partial x_4}, \\ P_1(\frac{\partial}{\partial x_4}) = -\frac{\partial}{\partial x_3}, P_1(\frac{\partial}{\partial x_5}) = \frac{\partial}{\partial x_5} \end{array}\right\},$$

$$\left\{\begin{array}{c} P_2(\frac{\partial}{\partial x_1}) = \frac{\partial}{\partial x_2}, P_2(\frac{\partial}{\partial x_2}) = \frac{\partial}{\partial x_1}, P_2(\frac{\partial}{\partial x_3}) = \frac{\partial}{\partial x_4}, \\ P_2(\frac{\partial}{\partial x_4}) = \frac{\partial}{\partial x_3}, P_2(\frac{\partial}{\partial x_5}) = \frac{\partial}{\partial x_5} \end{array}\right\}.$$

Now, we define product structure P_α on R^5 by

$$P_\alpha = \sin\alpha P_1 + \cos\alpha P_2.$$

Then we say that $(R^5, P_\alpha, g_{\bar{M}_L})$ is an almost product Riemannian manifold. Define a map $\varphi : R^5 \to R^2$ by

$$\varphi(x_1, x_2, x_3, x_4, x_5) = (\frac{x_1 + x_4}{\sqrt{2}}, \frac{x_5}{\sqrt{2}}).$$

Then we have

$$\ker\varphi_* = span\left\{V_1 = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_4}, V_2 = \frac{\partial}{\partial x_2}, V_3 = \frac{\partial}{\partial x_3}\right\},$$

$$(\ker\varphi_*)^\perp = span\left\{X_1 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_4}, X_2 = \sqrt{2}\frac{\partial}{\partial x_5},\right\}.$$

φ is a Riemannian submersion. Moreover, φ is a pointwise slant submersion with slant function α such that $\alpha = \cos^{-1}(\frac{\sin\alpha - \cos\alpha}{\sqrt{2}})$.

Let φ be a pointwise slant submersion from l.p.R manifold $(\overline{M}_L, g_{\bar{M}_L}, P)$ onto a Riemannian manifold $(\overline{N}_R, g_{\bar{N}_R})$. For any $V_1 \in \Gamma(\ker\varphi_*)$, we set

$$PV_1 = \phi V_1 + \omega V_1, \tag{2.14}$$

where $\phi V_1 \in \Gamma(\ker\varphi_*)$ and $\omega V_1 \in \Gamma(\ker\varphi_*)^\perp$. Also, for $X_1 \in \Gamma(\ker\varphi_*)^\perp$ we write,

$$PX_1 = BX_1 + CX_1 \tag{2.15}$$

where $BX_1 \in \Gamma(\ker\varphi_*)$ and $CX_1 \in \Gamma(\ker\varphi_*)^\perp$. We can denote $(\ker\varphi_*)^\perp$ such as

$$(\ker\varphi_*)^\perp = \omega(\ker\varphi_*)\perp\eta,$$

where η indicate the orthogonal complementary distribution to $\omega(\ker\varphi_*)$ in $(\ker\varphi_*)^\perp$.

Theorem 2.1. *[25] Let $(\overline{M}_L, g_{\bar{M}_L}, P)$ be a l.p.R manifold and $(\overline{N}_R, g_{\bar{N}_R})$ a Riemannian manifold. A Riemannian submersion $\varphi : (\overline{M}_L, g_{\bar{M}_L}, P) \to (\overline{N}_R, g_{\bar{N}_R})$ is a pointwise slant submersion if and only if there exists a slant function α such that for $V_1 \in \Gamma(\ker\varphi_*)$*

$$\phi^2 = \cos^2\alpha V_1. \tag{2.16}$$

Lemma 2.1. *[25]Let $\varphi : (\overline{M}_L, g_{\bar{M}_L}, P) \to (\overline{N}_R, g_{\bar{N}_R})$ be a pointwise slant submersion from l.p.R manifold onto a Riemannian manifold . Then, for any $V_1, V_2 \in \Gamma(\ker\varphi_*)$ we have*

$$\begin{aligned} g_{\bar{M}_L}(\phi V_1, \phi V_2) &= \cos^2\alpha g_{\bar{M}_L}(V_1, V_2), \\ g_{\bar{M}_L}(\omega V_1, \omega V_2) &= \sin^2\alpha g_{\bar{M}_L}(V_1, V_2). \end{aligned} \tag{2.17}$$

3. Clairaut pointwise slant submersions from locally product Riemannian manifolds

In this section, we give a new necessary and sufficient condition for a pointwise slant submersion to be Clairaut in case of its total manifold is locally product Riemannian manifold. Finally, we present an illustrative example for this kind of submersion which satisfies Clairaut condition.

Definition 3.1. [6] Let $\varphi : (\overline{M}_L, g_{\bar{M}_L}) \to (\overline{N}_R, g_{\bar{N}_R})$ be a Riemannian submersion and ς a geodesic on $\overline{M}_L$. If there exists a positive function r on $\overline{M}_L$, such that the function $(r \circ \varsigma)\sin\alpha$ is constant, then φ is called a Clairaut submersion. Here $\alpha(s)$ is the angle between the horizontal space at $\dot{\varsigma}(s)$ and $\varsigma(s)$, for any $s \in J$.

In [6], Bishop introduced Clairaut submersion and he obtained the necessary and sufficient condition for a Riemannian submersion to be a Clairaut submersion as follows:

Theorem 3.1. *[6] Let $\varphi : (\overline{M}_L, g_{\bar{M}_L}) \to (\overline{N}_R, g_{\bar{N}_R})$ be a Riemannian submersion between two Riemannian manifolds with connected fibers. Then, φ is a Clairaut submersion with the function $r = e^\beta$ if each fiber is totally umbilical and has the mean curvature vector field $H = -grad\beta$, here according to $g_{\bar{M}_L}$, $grad\beta$ is the gradient of the function β.*

Herein, after giving several supporting results, we state new Clairaut conditions for pointwise slant submersions.

Theorem 3.2. *Let $\varphi : (\overline{M}_L, g_{\bar{M}_L}) \to (\overline{N}_R, g_{\bar{N}_R})$ be a pointwise slant submersion from a l.p.R manifold onto a Riemannian manifold. If $\varsigma : I_2 \subset R \to \overline{M}_L$ is a regular curve, then ς is a geodesic if and only if the following equations hold:*

$$\begin{aligned} 0 &= -2\sin 2\alpha\dot{\varsigma}(\alpha(s))V_1 + \cos^2\alpha\left\{\hat{\nabla}_{V_1}V_1 + v\overline{\nabla}_{X_1}V_1\right\} \\ &\quad +(T_{V_1} + A_{X_1})\omega\phi V_1 + Bh\left\{\overline{\nabla}_{V_1}\omega V_1 + \overline{\nabla}_{X_1}\omega V_1\right\} \\ &\quad +\phi((T_{V_1} + A_{X_1})\omega V_1) + T_{V_1}X_1, \end{aligned} \tag{3.1}$$

$$\begin{aligned} 0 &= \cos^2\alpha\left\{T_{V_1}V_1 + A_{X_1}V_1\right\} \\ &\quad +h\left\{\overline{\nabla}_{V_1}\omega\phi V_1 + \overline{\nabla}_{X_1}\omega\phi V_1 + \overline{\nabla}_{V_1}X_1 + \overline{\nabla}_{X_1}X_1\right\} \\ &\quad +Ch\left\{\overline{\nabla}_{V_1}\omega V_1 + \overline{\nabla}_{X_1}\omega V_1\right\} + \omega((T_{V_1} + A_{X_1})\omega V_1) + A_{X_1}X_1. \end{aligned} \tag{3.2}$$

where $V_1(s)$ and $X_1(s)$ denote the vertical and horizontal parts of the tangent vector field $\dot{\varsigma}(s)$ of $\varsigma(s)$, respectively.

Proof. Let $\varsigma : I_2 \subset R \to \overline{M}_L$ be a regular curve and $V_1(s)$ and $X_1(s)$ are the vertical and horizontal parts of the tangent vector field $\dot{\varsigma}(s)$ of $\varsigma(s)$, respectively. Since $\overline{M}_L$ is a locally product manifold, we get

$$\begin{aligned} \overline{\nabla}_{\dot{\varsigma}(s)}\dot{\varsigma}(s) &= P(\overline{\nabla}_{\dot{\varsigma}(s)}P\varsigma(s)) \\ &= P(\overline{\nabla}_{\dot{\varsigma}(s)}P(V_1(s) + X_1(s)) \\ &= P\overline{\nabla}_{\dot{\varsigma}(s)}PV_1(s) + P\overline{\nabla}_{\dot{\varsigma}(s)}PX_1(s)). \end{aligned}$$

Using (2.14) and (2.3), we can write

$$\overline{\nabla}_{\dot{\varsigma}(s)}\dot{\varsigma}(s) = P(\overline{\nabla}_{\dot{\varsigma}(s)}(\phi V_1 + \omega V_1)(s) + \overline{\nabla}_{\dot{\varsigma}(s)}X_1(s).$$

Again using (2.14) and (2.3), we get

$$\begin{aligned} \overline{\nabla}_{\dot{\varsigma}(s)}\dot{\varsigma}(s) &= \overline{\nabla}_{\dot{\varsigma}(s)}P\phi V_1 + \overline{\nabla}_{\dot{\varsigma}(s)}P\omega V_1 + \overline{\nabla}_{\dot{\varsigma}(s)}X_1 \\ &= \overline{\nabla}_{\dot{\varsigma}(s)}\phi^2 V_1 + \overline{\nabla}_{\dot{\varsigma}(s)}\omega\phi V_1 + P\overline{\nabla}_{\dot{\varsigma}(s)}\omega V_1 + \overline{\nabla}_{\dot{\varsigma}(s)}X_1. \end{aligned}$$

By Theorem 2.1, we have

$$\overline{\nabla}_{\dot{\varsigma}(s)}\dot{\varsigma}(s) = \overline{\nabla}_{\dot{\varsigma}(s)}\cos^2\alpha V_1 + \overline{\nabla}_{\dot{\varsigma}(s)}\omega\phi V_1 + P\overline{\nabla}_{\dot{\varsigma}(s)}\omega V_1 + \overline{\nabla}_{\dot{\varsigma}(s)}X_1.$$

On the other hand, from the definition of covariant derivative and after some calculations, we obtain

$$\begin{aligned} \overline{\nabla}_{\dot{\varsigma}(s)}\dot{\varsigma}(s) &= -2\cos\alpha\sin\alpha\dot{\varsigma}(\alpha(s))V_1 + \cos^2\alpha\left\{\overline{\nabla}_{V_1}V_1 + \overline{\nabla}_{X_1}V_1\right\} \\ &\quad +\overline{\nabla}_{V_1}\omega\phi V_1 + \overline{\nabla}_{X_1}\omega\phi V_1 + P\overline{\nabla}_{V_1}\omega V_1 + P\overline{\nabla}_{X_1}\omega V_1 + \overline{\nabla}_{V_1}X_1 + \overline{\nabla}_{X_1}X_1. \end{aligned}$$

Using (2.14), (2.15) and (2.8)-(2.11), we get

$$\begin{aligned} \overline{\nabla}_{\dot{\varsigma}(s)}\dot{\varsigma}(s) &= -\sin 2\alpha\dot{\varsigma}(\alpha(s))V_1 + \cos^2\alpha\left\{T_{V_1}V_1 + \hat{\nabla}_{V_1}V_1 + A_{X_1}V_1 + v\overline{\nabla}_{X_1}V_1\right\} \\ &\quad +T_{V_1}\omega\phi V_1 + h\overline{\nabla}_{V_1}\omega\phi V_1 + A_{X_1}\omega\phi V_1 + h\overline{\nabla}_{X_1}\omega\phi V_1 \\ &\quad +\overline{\nabla}_{V_1}(B\omega V_1 + C\omega V_1) + \overline{\nabla}_{X_1}(B\omega V_1 + C\omega V_1) + \overline{\nabla}_{V_1}X_1 + \overline{\nabla}_{X_1}X_1 \\ &= -\sin 2\alpha\dot{\varsigma}(\alpha(s))V_1 + \cos^2\alpha\left\{T_{V_1}V_1 + \hat{\nabla}_{V_1}V_1 + A_{X_1}V_1 + v\overline{\nabla}_{X_1}V_1\right\} \\ &\quad +h\left\{\overline{\nabla}_{V_1}\omega\phi V_1 + \overline{\nabla}_{X_1}\omega\phi V_1 + \overline{\nabla}_{V_1}X_1 + \overline{\nabla}_{X_1}X_1\right\} \\ &\quad +(T_{V_1} + A_{X_1})\omega\phi V_1 + Bh\left\{\overline{\nabla}_{V_1}\omega V_1 + \overline{\nabla}_{X_1}\omega V_1\right\} \\ &\quad +Ch\left\{\overline{\nabla}_{V_1}\omega V_1 + \overline{\nabla}_{X_1}\omega V_1\right\} + BT_{V_1}\omega V_1 + BA_{X_1}\omega V_1 \\ &\quad +CT_{V_1}\omega V_1 + CA_{X_1}\omega V_1 + T_{V_1}X_1 + A_{X_1}X_1 \\ &= -\sin 2\alpha\dot{\varsigma}(\alpha(s))V_1 + \cos^2\alpha\left\{T_{V_1}V_1 + \hat{\nabla}_{V_1}V_1 + A_{X_1}V_1 + v\overline{\nabla}_{X_1}V_1\right\} \\ &\quad +h\left\{\overline{\nabla}_{V_1}\omega\phi V_1 + \overline{\nabla}_{X_1}\omega\phi V_1 + \overline{\nabla}_{V_1}X_1 + \overline{\nabla}_{X_1}X_1\right\} + (T_{V_1} + A_{X_1})\omega\phi V_1 \\ &\quad +Bh\left\{\overline{\nabla}_{V_1}\omega V_1 + \overline{\nabla}_{X_1}\omega V_1\right\} + Ch\left\{\overline{\nabla}_{V_1}\omega V_1 + \overline{\nabla}_{X_1}\omega V_1\right\} \\ &\quad +\phi((T_{V_1} + A_{X_1})\omega V_1) + \omega((T_{V_1} + A_{X_1})\omega V_1) + T_{V_1}X_1 + A_{X_1}X_1. \end{aligned}$$

If we take into account the horizontal and vertical parts of this equation, then we acquire

$$\begin{aligned} h\overline{\nabla}_{\dot{\varsigma}(s)}\dot{\varsigma}(s) &= \cos^2\alpha\{T_{V_1}V_1 + A_{X_1}V_1\} \\ &+h\left\{\overline{\nabla}_{V_1}\omega\phi V_1 + \overline{\nabla}_{X_1}\omega\phi V_1 + \overline{\nabla}_{V_1}X_1 + \overline{\nabla}_{X_1}X_1\right\} \\ &+Ch\left\{\overline{\nabla}_{V_1}\omega V_1 + \overline{\nabla}_{X_1}\omega V_1\right\} + \omega((T_{V_1} + A_{X_1})\omega V_1) \\ &+A_{X_1}X_1. \end{aligned}$$

and

$$\begin{aligned} v\overline{\nabla}_{\dot{\varsigma}(s)}\dot{\varsigma}(s) &= -\sin 2\alpha\dot{\varsigma}(\alpha(s))V_1 + \cos^2\alpha\left\{\hat{\nabla}_{V_1}V_1 + v\overline{\nabla}_{X_1}V_1\right\} \\ &+(T_{V_1} + A_{X_1})\omega\phi V_1 + Bh\left\{\overline{\nabla}_{V_1}\omega V_1 + \overline{\nabla}_{X_1}\omega V_1\right\} \\ &+\phi((T_{V_1} + A_{X_1})\omega V_1) + T_{V_1}X_1, \end{aligned}$$

Now, ς is a geodesic if and only if $\overline{\nabla}_{\dot{\varsigma}}\dot{\varsigma} = 0$, then (3.1) and (3.2) come from the last equations. □

Theorem 3.3. *Let φ be a pointwise slant submersion from a l.p.R manifold $(\overline{M}_L, g_{\bar{M}_L}, P)$ onto a Riemannian manifold $(\overline{N}_R, g_{\bar{N}_R})$. Let $\varsigma : I_2 \subset R \to \overline{M}_L$ be a regular curve, then φ is a Clairaut submersion with the function $r = e^\beta$ if and only if the following equation hold:*

$$\begin{aligned} &-\sec^2\alpha\left\{\begin{array}{c} g_{\bar{M}_L}((T_{V_1} + A_{X_1})\omega\phi V_1) + Bh\left\{\overline{\nabla}_{V_1}\omega V_1 + \overline{\nabla}_{X_1}\omega V_1\right\} \\ +\phi((T_{V_1} + A_{X_1})\omega V_1) + T_{V_1}X_1, V_1(s)) \end{array}\right\} \\ = &\left\{2\tan\alpha\dot{\varsigma}(\alpha(s)) - g_{\bar{M}_L}(grad\beta, \varsigma(s))\right\} g_{\bar{M}_L}(V_1, V_1) \end{aligned}$$

where $V_1(s)$ and $X_1(s)$ denote the vertical and horizontal parts of the tangent vector field $\dot{\varsigma}(s)$ of $\varsigma(s)$, respectively.

Proof. Let $\varsigma(s)$ be a geodesic on $\overline{M}_L$, $V_1(s) = v\varsigma(s)$ and $X_1(s) = h\varsigma(s)$. Let $\sqrt{k}$ be constant speed of ς on $\overline{M}_L$ that is, $k = g_{\bar{M}_L}(\varsigma(s), \varsigma(s)) = \|\varsigma(s)\|^2$. Thence we conclude that,

$$g_{\bar{M}_L}(V_1(s), V_1(s)) = k\sin^2\rho(s) \tag{3.3}$$

$$g_{\bar{M}_L}(X_1(s), X_1(s)) = k\cos^2\rho(s). \tag{3.4}$$

Differentiating (3.3), we have

$$\frac{d}{ds}g_{\bar{M}_L}(V_1(s), V_1(s)) = 2g_{\bar{M}}(\overline{\nabla}_{\varsigma(s)}V_1(s), V_1(s)) = 2k\sin\rho(s)\cos\rho(s)\frac{d\rho}{ds}.$$

So, it follows that

$$g_{\bar{M}_L}(\overline{\nabla}_{\varsigma(s)}V_1(s), V_1(s)) = k\sin\rho(s)\cos\rho(s)\frac{d\rho}{ds}.$$

From (2.8)-(2.11), we have

$$\begin{aligned} k\sin\rho(s)\cos\rho(s)\frac{d\rho}{ds} &= g_{\bar{M}_L}(\overline{\nabla}_{\varsigma(s)}V_1(s), V_1(s)) \\ &= g_{\bar{M}_L}(\overline{\nabla}_{V_1(s)}V_1(s) + \overline{\nabla}_{X_1(s)}V_1(s), V_1(s)) \\ &= g_{\bar{M}_L}(T_{V_1}V_1 + A_{X_1}V_1 + \hat{\nabla}_{V_1}V_1 + v\overline{\nabla}_{X_1}V_1, V_1(s)) \\ &= g_{\bar{M}_L}(\hat{\nabla}_{V_1}V_1 + v\overline{\nabla}_{X_1}V_1, V_1(s)). \end{aligned} \tag{3.5}$$

On the other hand, from (3.1), we have

$$\begin{aligned} \cos^2\alpha\left\{\hat{\nabla}_{V_1}V_1 + v\overline{\nabla}_{X_1}V_1\right\} &= 2\sin 2\alpha\varsigma(\alpha(s))V_1 - (T_{V_1} + A_{X_1})\omega\phi V_1 \\ &-Bh\left\{\overline{\nabla}_{V_1}\omega V_1 + \overline{\nabla}_{X_1}\omega V_1\right\} \\ &-\phi((T_{V_1} + A_{X_1})\omega V_1) - T_{V_1}X_1. \end{aligned}$$

If this equation is substituted in (3.5), then we have

$$\begin{aligned} &g_{\bar{M}_L}(\sin 2\alpha\varsigma(\alpha(s))V_1 - (T_{V_1} + A_{X_1})\omega\phi V_1) \\ &-Bh\left\{\overline{\nabla}_{V_1}\omega V_1 + \overline{\nabla}_{X_1}\omega V_1\right\} - \phi((T_{V_1} + A_{X_1})\omega V_1) - T_{V_1}X_1, V_1(s)) \\ = &\ k\cos^2\alpha\sin\rho(s)\cos\rho(s)\frac{d\rho}{ds}. \end{aligned}$$

Now, φ is a Clairaut pointwise slant submersion with $r = e^{\beta}$ if and only if $\frac{d}{ds}((r \circ \varsigma)\sin\rho(s)) = 0$. Therefore

$$\frac{d}{ds}(\left(e^{\beta} \circ \varsigma\right)\sin\rho(s)) = 0 \Longleftrightarrow \left(e^{\beta} \circ \varsigma\right)(\frac{d\beta}{ds}\varsigma(s)\sin\rho(s) + \cos\rho(s)\frac{d\rho}{ds}) = 0.$$

Since r is a positive function, then

$$\frac{d\beta}{ds}\varsigma(s)\sin\rho + \cos\rho\frac{d\rho}{ds} = 0.$$

By multiplying this with non-zero factor $k\sin\rho$, then we have

$$-\frac{d\beta}{ds}\varsigma(s)k\sin^2\rho = k\cos\rho\sin\rho\frac{d\rho}{ds}. \tag{3.6}$$

Since the right-hand sides of equations (3.3) and (3.6) are equal, then we have

$$\begin{aligned} & \sin 2\alpha\varsigma(\alpha(s))g_{\bar{M}_L}(V_1, V_1) - \cos^2\alpha\frac{d\beta}{ds}\varsigma(s)g_{\bar{M}_L}(V_1, V_1) \\ = & -g_{\bar{M}_L}((T_{V_1} + A_{X_1})\omega\phi V_1) + Bh\left\{\overline{\nabla}_{V_1}\omega V_1 + \overline{\nabla}_{X_1}\omega V_1\right\} \\ & +\phi((T_{V_1} + A_{X_1})\omega V_1) + T_{V_1}X_1, V_1(s)). \end{aligned}$$

By multiplying this with non-zero factor $\sec^2\alpha$ and since $\frac{d\beta}{ds}\varsigma(s) = \varsigma[\beta](s) = g_{\bar{M}_L}(grad\beta, \varsigma(s)) = g_{\bar{M}_L}(grad\beta, V_1(s) + X_1(s) = g_{\bar{M}_L}(grad\beta, X_1)$, we obtain

$$\begin{aligned} & \left\{2\tan\alpha\varsigma(\alpha(s)) - g_{\bar{M}_L}(grad\beta, \varsigma(s))\right\}g_1(V_1, V_1) \\ = & -\sec^2\alpha\left\{\begin{array}{c} g_{\bar{M}_L}((T_{V_1} + A_{X_1})\omega\phi V_1) + Bh\left\{\overline{\nabla}_{V_1}\omega V_1 + \overline{\nabla}_{X_1}\omega V_1\right\} \\ +\phi((T_{V_1} + A_{X_1})\omega V_1) + T_{V_1}X_1, V_1(s)) \end{array}\right\}. \end{aligned}$$

Hence the theorem is proved. □

Let $\varphi : (\overline{M}_L, g_{\bar{M}_L}, P) \to (\overline{N}_R, g_{\bar{N}_R})$ be a pointwise slant submersion with slant function α. φ is a slant submersion if the function α is constant. So, we can give the following results, which are not difficult to prove.

Theorem 3.4. *Let $\varphi : (\overline{M}_L, g_{\bar{M}_L}, P) \to (\overline{N}_R, g_{\bar{N}_R})$ be a slant submersion from a l.p.R manifold onto a Riemannian manifold. If $\varsigma : I_2 \subset R \to \overline{M}_L$ is a regular curve, then ς is a geodesic if and only if the following equations hold:*

$$\begin{aligned} 0 = & \cos^2\alpha\left\{\hat{\nabla}_{V_1}V_1 + v\overline{\nabla}_{X_1}V_1\right\} + (T_{V_1} + A_{X_1})\omega\phi V_1 \\ & +Bh\left\{\overline{\nabla}_{V_1}\omega V_1 + \overline{\nabla}_{X_1}\omega V_1\right\} + \phi((T_{V_1} + A_{X_1})\omega V_1) + T_{V_1}X_1, \end{aligned} \tag{3.7}$$

$$\begin{aligned} 0 = & \cos^2\alpha\left\{T_{V_1}V_1 + A_{X_1}V_1\right\} \\ & +h\left\{\overline{\nabla}_{V_1}\omega\phi V_1 + \overline{\nabla}_{X_1}\omega\phi V_1 + \overline{\nabla}_{V_1}X_1 + \overline{\nabla}_{X_1}X_1\right\} \\ & +Ch\left\{\overline{\nabla}_{V_1}\omega V_1 + \overline{\nabla}_{X_1}\omega V_1\right\} + \omega((T_{V_1} + A_{X_1})\omega V_1) + A_{X_1}X_1, \end{aligned} \tag{3.8}$$

where $V_1(s)$ and $X_1(s)$ denote the vertical and horizontal parts of the tangent vector field $\dot{\varsigma}(s)$ of $\varsigma(s)$, respectively.

Corollary 3.1. *Let $\varphi : (\overline{M}_L, g_{\bar{M}_L}, P) \to (\overline{N}_R, g_{\bar{N}_R})$ be an anti-invariant submersion with $\alpha = \frac{\pi}{2}$. In this case*

$$\begin{aligned} 0 = & (T_{V_1} + A_{X_1})\omega\phi V_1 + Bh\left\{\overline{\nabla}_{V_1}\omega V_1 + \overline{\nabla}_{X_1}\omega V_1\right\} \\ & +\phi((T_{V_1} + A_{X_1})\omega V_1) + T_{V_1}X_1, \end{aligned}$$

$$\begin{aligned} 0 = & h\left\{\overline{\nabla}_{V_1}\omega\phi V_1 + \overline{\nabla}_{X_1}\omega\phi V_1 + \overline{\nabla}_{V_1}X_1 + \overline{\nabla}_{X_1}X_1\right\} \\ & +Ch\left\{\overline{\nabla}_{V_1}\omega V_1 + \overline{\nabla}_{X_1}\omega V_1\right\} + \omega((T_{V_1} + A_{X_1})\omega V_1) + A_{X_1}X_1, \end{aligned}$$

the equalities satisfy if and only if every fibre is totally geodesic.

Theorem 3.5. *Let $\varphi : (\overline{M}_L, g_{\bar{M}_L}, P) \to (\overline{N}_R, g_{\bar{N}_R})$ be a slant submersion from a l.p.R manifold onto a Riemannian manifold and let $\varsigma : I_2 \subset R \to \overline{M}_L$ be a regular curve, then φ is a Clairaut submersion with the function $r = e^{\beta}$ if and only if the following equation hold:*

$$\begin{aligned} & \sec^2\alpha \left\{ \begin{array}{c} g_{\bar{M}_L}((T_{V_1} + A_{X_1})\omega\phi V_1) + Bh\left\{\overline{\nabla}_{V_1}\omega V_1 + \overline{\nabla}_{X_1}\omega V_1\right\} \\ +\phi((T_{V_1} + A_{X_1})\omega V_1) + T_{V_1}X_1, V_1(s)) \end{array} \right\} \\ = \; & g_{\bar{M}_L}(grad\beta, \varsigma(s)) g_{\bar{M}_L}(V_1, V_1) \end{aligned}$$

where $V_1(s)$ and $X_1(s)$ denote the vertical and horizontal parts of the tangent vector field $\dot{\varsigma}(s)$ of $\varsigma(s)$, respectively.

Theorem 3.6. *Let φ be a Clairaut pointwise slant submersion from a l.p.R manifold $(\overline{M}_L, g_{\bar{M}_L}, P)$ onto a Riemannian manifold $(\overline{N}_R, g_{\bar{N}_R})$ with totally umbilical fibers with the function $r = e^{\beta}$. Then, $grad\beta \in \Gamma(\eta)$.*

Proof. For any $V \in \Gamma(\ker\varphi_*)$, from (2.13) and (2.17) we have

$$\begin{aligned} T_{\phi V}\phi V &= -g_{\bar{M}_L}(\phi V, \phi V) grad\beta \\ &= -\cos^2\alpha g_1(V, V) grad\beta. \end{aligned}$$

From (2.8), we get

$$\overline{\nabla}_{\phi V}\phi V - \hat{\nabla}_{\phi V}\phi V = -\cos^2\alpha g_{\bar{M}_L}(V, V) grad\beta.$$

From (2.14) and again from (2.8), we have

$$P(T_{\phi V}V + \hat{\nabla}_{\phi V}V) - T_{\phi V}\omega V - h\overline{\nabla}_{\phi V}\omega V - \hat{\nabla}_{\phi V}\phi V = -\cos^2\alpha g_{\bar{M}_L}(V, V) grad\beta.$$

Since fibers are totally umbilical, then we get

$$P\hat{\nabla}_{\phi V}V - h\overline{\nabla}_{\phi V}\omega V - \hat{\nabla}_{\phi V}\phi V = -\cos^2\alpha g_{\bar{M}_L}(V, V) grad\beta.$$

If we take the inner product of both sides with ωV, then get

$$\begin{aligned} & g_{\bar{M}_L}(P\hat{\nabla}_{\phi V}V, \omega V) - g_{\bar{M}_L}(h\overline{\nabla}_{\phi V}\omega V, \omega V) - g_{\bar{M}_L}(\hat{\nabla}_{\phi V}\phi V, \omega V) \\ = \; & -\cos^2\alpha g_{\bar{M}_L}(V, V) g_{\bar{M}_L}(grad\beta, \omega V). \end{aligned}$$

From (2.8), (2.14) and after some easy calculations, we have

$$\cos^2\alpha g_{\bar{M}_L}(V, V) g_{\bar{M}_L}(grad\beta, \omega V) = 0.$$

So, Since φ is a pointwise slant submersion, then $grad\beta \in \Gamma(\eta)$. □

Theorem 3.7. *Let φ be a Clairaut pointwise slant submersion from a l.p.R manifold $(\overline{M}_L, g_{\bar{M}_L}, P)$ onto a Riemannian manifold $(\overline{N}_R, g_{\bar{N}_R})$ with the function $r = e^{\beta}$. Then*

$$-\sin\alpha V(\alpha) + g_{\bar{M}_L}(V, T_{V^*}\omega V) = g_{\bar{M}_L}(V, T_V\omega V^* + \hat{\nabla}_V\phi V^*), \tag{3.9}$$

for any vertical unit vector field V^*. Furthermore, if φ has totally umbilical fibers then $\overline{\nabla}_V V \in \Gamma((\ker\varphi_*)^{\perp})$.

Proof. Let $V \in \Gamma(\ker\varphi_*)$ be any unit vertical vector field. From (2.16), there exists a unit vertical vector field V^* such that $\phi V = (\cos\alpha)V^*$. For $V \in \Gamma(\ker\varphi_*)$, Using (2.1), (2.2), (2.3) and (2.8)-(2.11) we get

$$\begin{aligned} \overline{\nabla}_V PV &= \overline{\nabla}_V(\cos\alpha V^* + \omega V) \\ &= \overline{\nabla}_V \cos\alpha V^* + \overline{\nabla}_V\omega V \\ &= -\sin\alpha V(\alpha)V^* + \cos\alpha\overline{\nabla}_V V^* + \overline{\nabla}_V\omega V \\ &= -\sin\alpha V(\alpha)V^* + \cos\alpha(T_V V^* + \hat{\nabla}_V V^*) + T_V\omega V + h\overline{\nabla}_V\omega V. \end{aligned}$$

On the other hand, the l.p.R manifold property $(\overline{\nabla}_V P)V = 0$,

$$\begin{aligned} (\overline{\nabla}_V PV) &= P(\overline{\nabla}_V V) = P(T_V V + \hat{\nabla}_V V) \\ &= BT_V V + CT_V V + \phi\hat{\nabla}_V V + \omega\hat{\nabla}_V V. \end{aligned}$$

Comparing the vertical components of the resulting equation, we get

$$-\sin\alpha V(\alpha)V^* + \cos\alpha\hat{\nabla}_V V^* + T_V\omega V = BT_V V + \phi\hat{\nabla}_V V.$$

Thus, if we take the inner product with V^*,we have

$$\begin{aligned} & -\sin\alpha V(\alpha) g_{\bar{M}_L}(V^*, V^*) + g_{\bar{M}_L}(\cos\alpha\hat{\nabla}_V V^*, V^*) + g_{\bar{M}_L}(T_V\omega V, V^*) \\ = \; & g_1(BT_V V, V^*) + g_{\bar{M}_L}(\phi\hat{\nabla}_V V, V^*). \end{aligned}$$

Using the facts that V^* is unit vertical vector field and the tensor T is skew-symmetric and substituting $\phi V = \cos\alpha V^*$ in last expression, we get

$$-\sin\alpha V(\alpha) + g_{\bar{M}_L}(T_V V^*, \omega V) = g_1(T_V V, \omega V^*) + g_{\bar{M}_L}(\hat{\nabla}_V V, \phi V^*).$$

Using (2.7) in the above equation, we obtain (3.9). □

From Theorem 3.7, we can give the following result.

Corollary 3.2. *Let φ be a Clairaut pointwise slant submersion from a l.p.R manifold $(\overline{M}_L, g_{\bar{M}_L}, P)$ onto a Riemannian manifold $(\overline{N}_R, g_{\bar{N}_R})$ with the function $r = e^{\beta}$. Then, φ is a slant submersion if and only if*

$$g_{\bar{M}_L}(T_V V^*, \omega V) = g_{\bar{M}_L}(T_V V, \omega V^*) + g_{\bar{M}_L}(\hat{\nabla}_V V, \phi V^*). \tag{3.10}$$

holds.

Corollary 3.3. *Let φ be a Clairaut slant submersion from a l.p.R manifold $(\overline{M}_L, g_{\bar{M}_L}, P)$ onto a Riemannian manifold $(\overline{N}_R, g_{\bar{N}_R})$. From Theorem 3.6, we know that $grad\beta \in \Gamma(\eta)$, then using (2.13) in (3.10), we obtain*

$$T_{\phi V}\omega V = T_V\omega\phi V.$$

Now, we give an example of a Clairaut pointwise slant submersion from a l.p.R manifold $(\overline{M}_L, g_{\bar{M}_L}, P)$ onto a Riemannian manifold $(\overline{N}_R, g_{\bar{N}_R})$ with the function $r = e^{\beta}$.

Example 3.1. Let $\overline{M}_L$ be an Euclidean space given by

$$\overline{M}_L = \left\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : (x_2, x_3, x_4, x_5) \neq 0, x_1 \neq 0\right\}.$$

We define the Riemannian metric $g_{\bar{M}_L}$ on $\overline{M}_L$ by

$$g_{\bar{M}_L} = e^{-2x_1}dx_1^2 + e^{-2x_1}dx_2^2 + 2e^{-2x_1}dx_3^2 + e^{-2x_1}dx_4^2 + e^{-2x_1}dx_5^2.$$

We take the almost product Riemannian structure $(P_\alpha, g_{\bar{M}_L})$ on $\overline{M}_L$ given in Example 2.1. Let $\overline{N}_R = \left\{(v_1, v_2, v_3) \in \mathbb{R}^3\right\}$ be a Riemannian manifold with Riemannian metric $g_{\bar{N}_R}$ on $\overline{N}_R$ given by

$$g_{\bar{N}_R} = e^{-2x_1}dv_1^2 + e^{-2x_1}dv_2^2 + 2e^{-2x_1}dv_3^2.$$

A $P-$basis can be given by

$$\left\{e_1 = e^{x_1}\frac{\partial}{\partial x_1}, e_2 = e^{x_1}\frac{\partial}{\partial x_2}, e_3 = e^{x_1}\frac{\partial}{\partial x_3}, e_4 = e^{x_1}\frac{\partial}{\partial x_4}, e_5 = e^{x_1}\frac{\partial}{\partial x_5}\right\},$$

on $T_q\overline{M}_L$ and

$$\left\{e_1^* = \frac{\partial}{\partial y_1}, e_2^* = \frac{\partial}{\partial y_2}, e_3^* = \frac{\partial}{\partial y_3}\right\},$$

on $T_{\varphi(q)}\overline{N}_R$ for all $q \in \overline{M}_L$. Now, we define a map $\varphi : (\overline{M}_L, P, g_{\bar{M}_L}) \to (\overline{N}_R, g_{\bar{N}_R})$ by

$$\varphi(x_1, x_2, x_3, x_4) = (\frac{x_2 - x_3}{\sqrt{2}}, \frac{x_4 - x_5}{\sqrt{2}}, x_1).$$

Then, we have

$$\ker\varphi_* = span\left\{V_1 = e_2 + e_3, V_2 = e_4 + e_5\right\},$$

$$(\ker\varphi_*)^\perp = span\left\{X_1 = e_2 - e_3, X_2 = e_4 - e_5, X_3 = e_1\right\}.$$

Hence it is easy to see that

$$g_{\bar{M}_L}(X_i, X_i) = g_{\bar{N}_R}(\varphi_*(X_i), \varphi_*(X_i)) = 2,$$

for $i = 1, 2, 3$.Thus φ is a Riemannian submersion. Moreover, φ is a pointwise slant submersion with slant function α such that $\alpha = \cos^{-1}(\frac{\cos\alpha - \sin\alpha}{2\sqrt{1-\sin 2\alpha}})$. Now, we will find smooth function β on $\overline{M}_L$ satisfying $T_V V = -g_{\bar{M}_L}(V, V) grad\beta$, for all $V \in \Gamma(\ker\varphi_*)$. We can simply calculate that

$$\begin{aligned}
\Gamma^1_{22} &= 1, \Gamma^2_{22} = \Gamma^3_{22} = \Gamma^4_{22} = 0,\\
\Gamma^1_{33} &= 2, \Gamma^2_{33} = \Gamma^3_{33} = \Gamma^4_{33} = 0,\\
\Gamma^1_{44} &= 1, \Gamma^2_{44} = \Gamma^3_{44} = \Gamma^4_{44} = 0,\\
\Gamma^1_{55} &= 1, \Gamma^2_{44} = \Gamma^3_{44} = \Gamma^4_{44} = 0,
\end{aligned}$$

other connection coefficients are zero. And

$$\begin{aligned}
\overline{\nabla}_{e_2} e_2 &= e^{2x_1}\frac{\partial}{\partial x_1}, \overline{\nabla}_{e_2} e_3 = 0, \overline{\nabla}_{e_3} e_2 = 0, \overline{\nabla}_{e_3} e_3 = 2e^{2x_1}\frac{\partial}{\partial x_1},\\
\overline{\nabla}_{e_4} e_4 &= e^{2x_1}\frac{\partial}{\partial x_1}, \overline{\nabla}_{e_4} e_5 = 0, \overline{\nabla}_{e_5} e_4 = 0, \overline{\nabla}_{e_5} e_5 = e^{2x_1}\frac{\partial}{\partial x_1},\\
\overline{\nabla}_{e_2} e_4 &= 0, \overline{\nabla}_{e_2} e_5 = 0, \overline{\nabla}_{e_3} e_4 = 0, \overline{\nabla}_{e_4} e_5 = 0,\\
\overline{\nabla}_{e_4} e_2 &= 0, \overline{\nabla}_{e_4} e_3 = 0, \overline{\nabla}_{e_5} e_2 = 0, \overline{\nabla}_{e_5} e_3 = 0.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\overline{\nabla}_{V_1} V_1 &= 3e^{2x_1}\frac{\partial}{\partial x_1}, \overline{\nabla}_{V_2} V_2 = 2e^{2x_1}\frac{\partial}{\partial x_1},\\
\overline{\nabla}_{V_1} V_2 &= \overline{\nabla}_{V_2} V_1 = 0.
\end{aligned}$$

Now, if we take $V = \lambda_1 V_1 + \lambda_2 V_2$ for $\lambda_1, \lambda_2 \in \mathbb{R}$ then

$$T_V V = \lambda_1^2 T_{V_1} V_1 + \lambda_2^2 T_{V_2} V_2 + 2\lambda_1\lambda_2 T_{V_1} V_2.$$

From (2.8)-(2.10), by direct calculations, we have

$$T_V V = (3\lambda_1^2 + 2\lambda_2^2) e^{2x_1}\frac{\partial}{\partial x_1}.$$

Since $V = \lambda_1 V_1 + \lambda_2 V_2$, then by direct calculations, we obtain

$$g_{\bar{M}_L}(V, V) = (3\lambda_1^2 + 2\lambda_2^2).$$

Moreover, for any smooth function β on $\mathbb{R}^5$, the gradient of β with respect to the metric $g_{\bar{M}_L}$ is given by

$$\begin{aligned}
\nabla\beta &= \sum_{i,j=1}^{5} g^{ij}_{\bar{M}_L}\frac{\partial\beta}{\partial x_i}\frac{\partial}{\partial x_j}\\
&= e^{2x_1}\frac{\partial\beta}{\partial x_1}\frac{\partial}{\partial x_1} + e^{2x_1}\frac{\partial\beta}{\partial x_2}\frac{\partial}{\partial x_2} + \frac{1}{2}e^{2x_1}\frac{\partial\beta}{\partial x_3}\frac{\partial}{\partial x_3}\\
&\quad + e^{2x_1}\frac{\partial\beta}{\partial x_4}\frac{\partial}{\partial x_4} + e^{2x_1}\frac{\partial\beta}{\partial x_5}\frac{\partial}{\partial x_5}.
\end{aligned}$$

Hence $\nabla\beta = -e^{2x_1}\frac{\partial}{\partial x_1}$ for the function $\beta = -x_1$. Therefore it is easy to see that $T_V V = -g_{\bar{M}_L}(V, V) grad\beta$. Hence φ is a Clairaut pointwise slant submersion.

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