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Approximate Solutions to Fractional Multi-Dimensional Navier-Stokes Equation Using the (FHPTM)

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Keywords

Fractional derivative in Caputo sence, Laplace transform, FHPTM, Navier Stokes equations, Mittag Leffler function.

Abstract

This work focuses on presenting a reliable method, called fractional homotopy perturbation transform method (FHPTM) to solve nonlinear Naviers Stoks equations with the Caputo type fractional derivatives. The (FHPTM) is a combination of Laplace transform and homotopy perturbation method (He -Laplace method). He's polynomial is used to simplify the nonlinearity which arise in our considered equation. Furthermore, three numerical examples are presented, it is supported by graphs and tables to compare solutions with little computational effort, which confirms the effectiveness and accuracy of the current method.

1. Introduction

The origins of the theory of derivative and integrals of non-integer order go back to the end of the 17^{th} century, when Isaac Newton and Gottfried Wihelm Leibniz developed the foundations of differential and integral calculus. The merit of the first conference is attributed to B. Ross who organized it at the University of New Aven in June 1974 and the conference was intitled "Fractional calculus and its applications". For the first study, another credit is given to KB. Oldham and J. Spanier [1] who published a book in 1974. Fractional calculus has spread widely in recent years. Actually, concrete applications, for example, measuring memory with the order of fractional derivative [2]. There are several definitions of a fractional derivative of order α [3]. The fractional derivative also appears in many fields such as, viscoelasticity [4], magnetohydrodynamic [5], mechanics [6], and other applications [7–10]. Because of many phenomenon in our reality are manifested by differential equations, the focus is on finding exact or approximate solutions to these equations in several methods, such as the tanh method [11], finite difference method [12], Runge–Kutta method [13], etc. Many authors have focused on studying the solutions of fractional partial differential equations (FPDEs) using various methods combined with the Laplace transform. Among these are the homotopy perturbation transform method [14–19], it provides solutions in the form of an infinite series, and the resulting series can converge to a solution in closed form if the exact solution exists. The Navier-Stokes (NS) equation is one of the most important equations. This equation describes many physical things such as ocean currents and fluid flow in tubes [20]. Several procedures have been presented to solve (NS) equation with fractional order, by discrete Adomian decomposition method [21], in [22] by FRDTM, [23] using a new homotopy perturbation transform method, modified Laplace decomposition method in [24], authors of [25], by Adomian decomposition method, the homotopy analysis method in [26], He's variational iteration method in [27], etc. The main objective of this work is to apply the fractional homotopy perturbation transform

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method (FHPTM), to build analytical and approximate solutions for the fractional Navier-Stokes equation (FNS) for an incompressible fluid flow, including the fractional derivative in the sense of Caputo, with $0 < \alpha \leq 1$, as follows [21,22].

$$\begin{cases} U_t^\alpha + UU_x + VU_\eta + WU_z = \rho_0(U_{xx} + U_{\eta\eta} + U_{zz}) - \frac{1}{\rho}p_x, \\ V_t^\alpha + UV_x + VV_\eta + WV_z = \rho_0(V_{xx} + V_{\eta\eta} + V_{zz}) - \frac{1}{\rho}p_\eta, \\ W_t^\alpha + UW_x + VW_\eta + WW_z = \rho_0(W_{xx} + W_{\eta\eta} + W_{zz}) - \frac{1}{\rho}p_z, \end{cases} \tag{1}$$

where (U,V,W), t, p, denote the fluid vector at the point $(x;\eta;z)$, time and the pressure, respectively, ρ is the density, ρ_0 denotes the kinematic viscosity of the flow.

2. Basic procedure of (FHPTM)

This section describes the implementation of (FHPTM) [14–19], we consider a general (FPDEs) with initial conditions of the form

$$D_t^\alpha \Psi(x,t) + R\Psi(x,t) + N\Psi(x,t) = \varphi(x,t), \quad m-1 < \alpha \leq m,\ m \in \mathbb{N}^*, \tag{2}$$

$$\frac{\partial^k \Psi(x,0)}{\partial t^k} = \Psi^{(k)}(x,0), k = 0,1,2,...,m-1,$$

where φ is the source term, N represents the general nonlinear differential operator and R is the linear differential, and $D_t^\alpha \Psi(x,t)$ denotes the Caputo fractional derivative of order α of function Ψ which is defined as

$$D^\alpha \Psi(x,t) = \begin{cases} \Psi^{(n)}(x,t), & \alpha = n, \\ \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \Psi^{(n)}(x,\tau) d\tau, & n-1 < \alpha < n,\ n \in \mathbb{N}. \end{cases} \tag{3}$$

$\Gamma(.)$ indicates the Gamma function define by,

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad Re(z) > 0. \tag{4}$$

The properties of fractional calculus theory due to Liouville and Laplace transform can be found [1,6–9] . Operating the Laplace transform (denoted throughout this paper by $\mathscr{L}$ and its inverse transformation by $\mathscr{L}^{-1}$) on both sides of Eq.(2), gives

$$\mathscr{L}[D_t^\alpha \Psi(x,t)] + \mathscr{L}[R\Psi(x,t)] + \mathscr{L}[N\Psi(x,t)] = \mathscr{L}[\varphi(x,t)]. \tag{5}$$

By using the formula (6) bellow

$$\mathscr{L}[D^\alpha \Psi(x,t)] = s^\alpha \mathscr{L}[\Psi(x,t)] - \sum_{k=0}^{m-1} s^{\alpha-k-1} \Psi^{(k)}(x,0), \quad m-1 < \alpha \leq m, \tag{6}$$

in the above Eq.(5), we have

$$s^\alpha \mathscr{L}[\Psi(x,t)] - \sum_{k=0}^{m-1} s^{\alpha-k-1} \Psi^{(k)}(x,0) + \mathscr{L}[R\Psi(x,t)] + \mathscr{L}[N\Psi(x,t)] = \mathscr{L}[\varphi(x,t)], \tag{7}$$

simplify more

$$\mathscr{L}[\Psi(x,t)] = \sum_{k=0}^{m-1} \frac{\Psi^{(k)}(x,0)}{s^{k+1}} + \frac{1}{s^\alpha}\mathscr{L}[\varphi(x,t)] - \frac{1}{s^\alpha}\mathscr{L}[R\Psi(x,t)] - \frac{1}{s^\alpha}\mathscr{L}[N\Psi(x,t)]. \tag{8}$$

Operating $\mathscr{L}^{-1}$ on the above equation Eq. (8), we get

$$\Psi(x,t) = \Phi(x,t) - \mathscr{L}^{-1}\left[\frac{1}{s^\alpha}\mathscr{L}[R\Psi(x,t)] + \frac{1}{s^\alpha}\mathscr{L}[N\Psi(x,t)]\right], \tag{9}$$

where $\Phi(x,t)$ represents the term arising from the source term and the prescribed initial conditions. Now, by (HPM) technique [28,29]

$$\Psi(x,t) = \sum_{n=0}^{+\infty} p^n \Psi_n(x,t), \tag{10}$$

and

$$N\Psi(x,t)=\sum_{n=0}^{+\infty}p^nH_n(\Psi), \tag{11}$$

where $p\in(0,1)$ is an embedding parameter and H_n is the He's polynomials shown below see [30, 31]

$$H_n(\Psi_0,\cdots,\Psi_n)=\frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{+\infty}(p^i\Psi_i)\right)\right]_{p=0},n=0,1,2,3,\cdots \tag{12}$$

Substituting the values of $\Psi(x,t)$ and $N\Psi(x,t)$ in Eq. (9) we get

$$\sum_{n=0}^{+\infty}p^n\Psi_n(x,t)=\Phi(x,t)-p\mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}\left[R\left(\sum_{n=0}^{+\infty}p^n\Psi_n(x,t)\right)+N\left(\sum_{n=0}^{+\infty}p^n\Psi_n(x,t)\right)\right]\right\}, \tag{13}$$

we obtain the following recurrence relation by equating the terms with identical powers in p.

$$\begin{aligned} p^0 &: \quad \Psi_0(x,t)=\Phi(x,t),\\ p^{n+1} &: \quad \Psi_{n+1}(x,t)=-\mathscr{L}^{-1}\left[\frac{1}{s^\alpha}\mathscr{L}[R\Psi_n(x,t)]+H_n(\Psi)\right],n\geq 0. \end{aligned}$$

(FHPTM) describes the solution of infinite series

$$\Psi(x,t)=\sum_{n=0}^{+\infty}\Psi_n(x,t). \tag{14}$$

3. Numerical results and discussion

Now, we present three examples to illustrate the method and its accuracy.
Example 1: Consider 2-dimensional (FNS) equation with $0<\alpha\leq 1$ [22]:

$$\begin{aligned} U_t^\alpha+UU_x+VU_\eta &= \rho_0\left(U_{xx}+U_{\eta\eta}\right)+q,\\ V_t^\alpha+UV_x+VV_\eta &= \rho_0\left(V_{xx}+V_{\eta\eta}\right)-q, \end{aligned} \tag{15}$$

along with the following initial conditions

$$U(x,\eta,0)=-sin(x+\eta),\ \ V(x,\eta,0)=sin(x+\eta). \tag{16}$$

Operating $\mathscr{L}$ on the above in Eqs. (15), we get

$$\begin{aligned} \mathscr{L}[D^\alpha U(x,\eta,t)]+\mathscr{L}[UU_x+VU_\eta] &= \mathscr{L}[\rho_0\left(U_{xx}+U_{\eta\eta}\right)+q],\\ \mathscr{L}[D^\alpha V(x,\eta,t)]+\mathscr{L}[UV_x+VV_\eta] &= \mathscr{L}[\rho_0\left(V_{xx}+V_{\eta\eta}\right)-q]. \end{aligned} \tag{17}$$

Now, by using the formula (6) for $m=1$, in the above Eqs. (17), we have

$$\begin{aligned} s^\alpha\mathscr{L}[U(x,\eta,t)]-s^{\alpha-1}U(x,\eta,0)+\mathscr{L}[UU_x+VU_\eta] &= \mathscr{L}[\rho_0\left(U_{xx}+U_{\eta\eta}\right)+q],\\ s^\alpha\mathscr{L}[V(x,\eta,t)]-s^{\alpha-1}V(x,\eta,0)+\mathscr{L}[UV_x+VV_\eta] &= \mathscr{L}[\rho_0\left(V_{xx}+V_{\eta\eta}\right)-q]. \end{aligned} \tag{18}$$

We find more detailed

$$\begin{aligned} \mathscr{L}[U(x,\eta,t)] &= \left[\tfrac{-sin(x+\eta)}{s}+\tfrac{q}{s^{\alpha+1}}\right]-\tfrac{1}{s^\alpha}\mathscr{L}[UU_x+VU_\eta]+\tfrac{1}{s^\alpha}\mathscr{L}[\rho_0(\nabla^2U)],\\ \mathscr{L}[V(x,\eta,t)] &= \left[\tfrac{sin(x+\eta)}{s}-\tfrac{q}{s^{\alpha+1}}\right]-\tfrac{1}{s^\alpha}\mathscr{L}[UV_x+VV_\eta]+\tfrac{1}{s^\alpha}\mathscr{L}[\rho_0(\nabla^2V)]. \end{aligned} \tag{19}$$

where $\nabla^2=\frac{\partial}{x^2}+\frac{\partial}{\eta^2}$. Thus, we apply $\mathscr{L}^{-1}$, to the above Eqs. (19), we get

$$\begin{aligned} U(x,\eta,t)] &= -sin(x+\eta)+\tfrac{qt^\alpha}{\Gamma(\alpha+1)}-\mathscr{L}^{-1}\left\{\tfrac{1}{s^\alpha}\left[\mathscr{L}[UU_x+VU_\eta]-\mathscr{L}[\rho_0(\nabla^2U)]\right]\right\},\\ V(x,\eta,t) &= sin(x+\eta)-\tfrac{qt^\alpha}{\Gamma(\alpha+1)}-\mathscr{L}^{-1}\left\{\tfrac{1}{s^\alpha}\left[\mathscr{L}[UV_x+VV_\eta]-\mathscr{L}[\rho_0(\nabla^2V)]\right]\right\}, \end{aligned} \tag{20}$$

by, substituting (10) and (11) in (20), we find

$$\begin{aligned}\sum_{n=0}^{+\infty} p^n U_n &= -sin(x+\eta)+\tfrac{qt^\alpha}{\Gamma(\alpha+1)} - p\mathscr{L}^{-1}\left\{\tfrac{1}{s^\alpha}\left[\mathscr{L}[H_n(U,V)]-\mathscr{L}[\rho_0(\nabla^2\textstyle\sum_{n=0}^{+\infty} p^n U_n)]\right]\right\},\\ \sum_{n=0}^{+\infty} p^n V_n &= sin(x+\eta)-\tfrac{qt^\alpha}{\Gamma(\alpha+1)} - p\mathscr{L}^{-1}\left\{\tfrac{1}{s^\alpha}\left[\mathscr{L}[K_n(U,V)]-\mathscr{L}[\rho_0(\nabla^2\textstyle\sum_{n=0}^{+\infty} p^n V_n)]\right]\right\},\end{aligned} \tag{21}$$

where $H_n(U,V)$, and $K_n(U,V)$ are He's polynomials that represent nonlinear term $UU_x + VU_\eta$ and $UV_x + VV_\eta$, respectively, the first terms for H_n and K_n are given by

$$\begin{aligned} H_0 &= U_0U_{0x} + V_0U_{0\eta}\\ H_1 &= U_0U_{1x} + U_1U_{0x} + V_0U_{1\eta} + V_1U_{0\eta}\\ H_2 &= U_0U_{2x} + U_1U_{1x} + U_2U_{0x} + V_0U_{2\eta} + V_1U_{1\eta} + V_2U_{0\eta}\\ &\vdots \end{aligned}$$

Similarly

$$\begin{aligned} K_0 &= U_0V_{0x} + V_0V_{0\eta}\\ K_1 &= U_0V_{1x} + U_1V_{0x} + V_0V_{1\eta} + V_1V_{0\eta}\\ K_2 &= U_0V_{2x} + U_1V_{1x} + U_2V_{0x} + V_0V_{2\eta} + V_1V_{1\eta} + V_2V_{0\eta}\\ &\vdots \end{aligned}$$

Comparing the coefficient of like powers of p, in (21) we have

$$\begin{aligned} p^0 : U_0(x,\eta,t) &= -sin(x+\eta)+\frac{qt^\alpha}{\Gamma(\alpha+1)},\\ V_0(x,\eta,t) &= sin(x+\eta)-\frac{qt^\alpha}{\Gamma(\alpha+1)}, \end{aligned}$$

$$\begin{aligned} p^1 : U_1(x,\eta,t) &= -\mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\left[\mathscr{L}[H_0]-\mathscr{L}[\rho_0(\nabla^2 U_0)]\right]\right\},\\ &= sin(x+\eta)\frac{2\rho_0 t^\alpha}{\Gamma(\alpha+1)},\\ V_1(x,\eta,t) &= -\mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\left[\mathscr{L}[K_0]-\mathscr{L}[\rho_0(\nabla^2 V_0)]\right]\right\},\\ &= -sin(x+\eta)\frac{2\rho_0 t^\alpha}{\Gamma(\alpha+1)}, \end{aligned}$$

$$\begin{aligned} p^2 : U_2(x,\eta,t) &= -\mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\left[\mathscr{L}[H_1]-\mathscr{L}[\rho_0(\nabla^2 U_1)]\right]\right\},\\ &= -sin(x+\eta)\frac{(2\rho_0)^2 t^{2\alpha}}{\Gamma(2\alpha+1)},\\ V_2(x,\eta,t) &= -\mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\left[\mathscr{L}[K_1]-\mathscr{L}[\rho_0(\nabla^2 V_1)]\right]\right\},\\ &= sin(x+\eta)\frac{(2\rho_0)^2 t^{2\alpha}}{\Gamma(2\alpha+1)}, \end{aligned}$$

$$\begin{aligned} p^3 : U_3(x,\eta,t) &= -\mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\left[\mathscr{L}[H_2]-\mathscr{L}[\rho_0(\nabla^2 U_2)]\right]\right\},\\ &= sin(x+\eta)\frac{(2\rho_0)^3 t^{3\alpha}}{\Gamma(3\alpha+1)},\\ V_3(x,\eta,t) &= -\mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\left[\mathscr{L}[K_2]-\mathscr{L}[\rho_0(\nabla^2 V_2)]\right]\right\},\\ &= -sin(x+\eta)\frac{(2\rho_0)^3 t^{3\alpha}}{\Gamma(3\alpha+1)}, \end{aligned}$$

The solution of Eqs. (15,16) is given by

$$\begin{aligned} U(x,\eta,t) &= -sin(x+\eta)\sum_{k=0}^{+\infty}\frac{(-2\rho_0 t^{\alpha})^k}{\Gamma(k\alpha+1)}+\frac{qt^{\alpha}}{\Gamma(\alpha+1)} \\ &= -\sin(x+\eta)E_{\alpha,1}(-2\rho_0 t^{\alpha})+\frac{qt^{\alpha}}{\Gamma(\alpha+1)} \\ V(x,\eta,t) &= sin(x+\eta)\sum_{k=0}^{+\infty}\frac{(-2\rho_0 t^{\alpha})^k}{\Gamma(k\alpha+1)}-\frac{qt^{\alpha}}{\Gamma(\alpha+1)} \\ &= \sin(x+\eta)E_{\alpha,1}(-2\rho_0 t^{\alpha})-\frac{qt^{\alpha}}{\Gamma(\alpha+1)}. \end{aligned}$$

For $\alpha = 1$ and $q = 0$, the exact solution of Eqs. (15,16) reduces to

$$U(x,\eta,t) = -sin(x+\eta)e^{-2\rho_0 t},\ V(x,\eta,t) = sin(x+\eta)e^{-2\rho_0 t}.$$

The fourth approximate solution is

$$\begin{aligned} &\mathscr{U}_4 = -sin(x+\eta)\left(1-\frac{2\rho_0 t^{\alpha}}{\Gamma(\alpha+1)}+\frac{4\rho_0^2 t^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{8\rho_0^3 t^{3\alpha}}{\Gamma(3\alpha+1)}\right), \\ &\mathscr{V}_4 = -\mathscr{U}_4. \end{aligned}$$

4. Tables and Figures

Table 1: Numerical results for Example 1 when $\rho_0 = 0.5$, for various values of α.

(x,η)	t	Exact U	$\alpha = 0.75$ $\mathscr{U}_4$	Exact V	$\alpha = 0.75$ $\mathscr{V}_4$	$\alpha = 0.9$ $\mathscr{U}_4$	$\alpha = 0.9$ $\mathscr{V}_4$	$\alpha = 1$ $\mathscr{U}_4$	$\alpha = 1$ $\mathscr{V}_4$
$(0.1,0.3)$	0.1	-0.3524	-0.3225	0.3524	0.3225	-0.3420	0.3420	-0.3524	0.3524
	0.2	-0.3188	-0.2848	0.3188	0.2848	-0.3059	0.3059	-0.3188	0.3188
	0.3	-0.2885	-0.2556	0.2885	0.2556	-0.2754	0.2754	-0.2884	0.2884
	0.4	-0.2610	-0.2310	0.2610	0.2310	-0.2487	0.2487	-0.2607	0.2607
	0.5	-0.2362	-0.2089	0.2362	0.2089	-0.2248	0.2248	-0.2353	0.2353
$(0.5,0.4)$	0.1	-0.7088	-0.6487	0.7088	0.6487	-0.6878	0.6878	-0.7088	0.7088
	0.2	-0.6413	-0.5729	0.6413	0.5729	-0.6153	0.6153	-0.6413	0.6413
	0.3	-0.5803	-0.5142	0.5803	0.5142	-0.5540	0.5540	-0.5801	0.5801
	0.4	-0.5251	-0.4646	0.5251	0.4646	-0.5003	0.5003	-0.5243	0.5243
	0.5	-0.4751	-0.4203	0.4751	0.4203	-0.4521	0.4521	-0.4733	0.4733

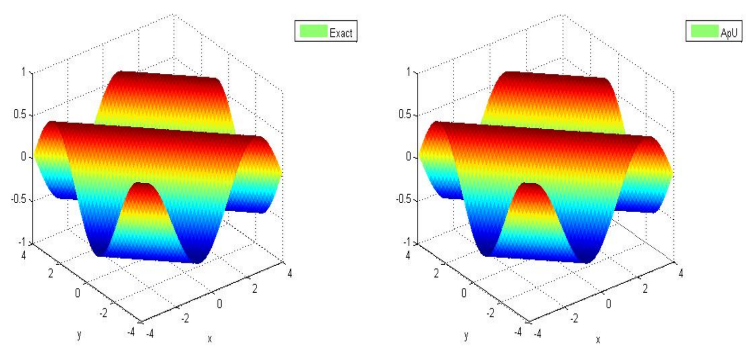

Figure 1: The exact U and $\mathscr{U}_4$ for Example 1. when $t = 0.5,\ \alpha = 1,\ \rho_0 = 0.5$, and $q = 0$.

Table 2: Absolute error in Example 1, when $\rho_0 = 1$, for various values of α.

		$\alpha = 0.75$	$\alpha = 0.9$	$\alpha = 1$
(x,η)	t	$\lvert U-\mathscr{U}_4\rvert = \lvert V-\mathscr{V}_4\rvert$	$\lvert U-\mathscr{U}_4\rvert$	$\lvert U-\mathscr{U}_4\rvert$
$(0.1, 0.3)$	0.1	0.0499	0.0181	0.000025
	0.2	0.0529	0.0203	0.000384
	0.3	0.0567	0.0209	0.001874
	0.4	0.0709	0.0249	0.005711
	0.5	0.1004	0.0359	0.013453
$(0.5, 0.4)$	0.1	0.1004	0.0365	0.000050
	0.2	0.1065	0.0408	0.000773
	0.3	0.1140	0.0420	0.003769
	0.4	0.1425	0.0502	0.011485
	0.5	0.2019	0.0723	0.027061

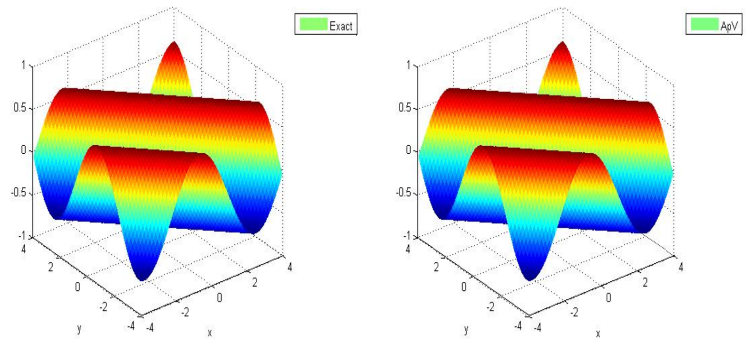

Figure 2: The exact V and $\mathscr{V}_4$ for Example 1. when $t = 0.5,\ \alpha = 1,\ \rho_0 = 0.5$, and $q = 0$.

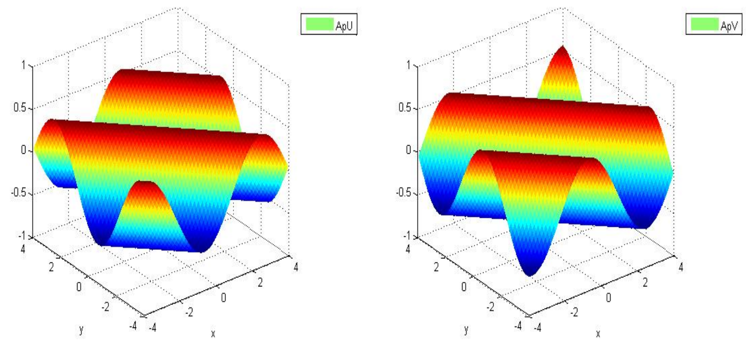

Figure 3: $\mathscr{U}_4$ and $\mathscr{V}_4$ for Example 1., when $t = 0.5,\ \alpha = 0.8,\ \rho_0 = 0.5,$ *and* $q = 0$.

Example 2: Consider 2-dimensional (FNS) Eq. (15) subject to the following initial conditions [22]:

$$U(x,\eta,0) = -e^{x+\eta},\ \ V(x,\eta,0) = e^{x+\eta}, \tag{22}$$

as shown in Example 1, we have

$$\begin{aligned} \textstyle\sum_{n=0}^{+\infty} p^n U_n &= -e^{x+\eta} + \tfrac{qt^\alpha}{\Gamma(\alpha+1)} - p\mathscr{L}^{-1}\left\{\tfrac{1}{s^\alpha}\left[\mathscr{L}[H_n] - \mathscr{L}[\rho_0(\nabla^2 \textstyle\sum_{n=0}^{+\infty} p^n U_n)]\right]\right\}, \\ \textstyle\sum_{n=0}^{+\infty} p^n V_n &= e^{x+\eta} - \tfrac{qt^\alpha}{\Gamma(\alpha+1)} - p\mathscr{L}^{-1}\left\{\tfrac{1}{s^\alpha}\left[\mathscr{L}[K_n] - \mathscr{L}[\rho_0(\nabla^2 \textstyle\sum_{n=0}^{+\infty} p^n V_n)]\right]\right\}. \end{aligned} \tag{23}$$

Then, we get

$$\begin{aligned}
p^0: U_0(x,\eta,t) &= -e^{x+\eta}+\frac{qt^\alpha}{\Gamma(\alpha+1)},\\
V_0(x,\eta,t) &= e^{x+\eta}-\frac{qt^\alpha}{\Gamma(\alpha+1)},\\
p^1: U_1(x,\eta,t) &= -e^{x+\eta}\frac{2\rho_0 t^\alpha}{\Gamma(\alpha+1)},\\
V_1(x,\eta,t) &= e^{x+\eta}\frac{2\rho_0 t^\alpha}{\Gamma(\alpha+1)},\\
p^2: U_2(x,\eta,t) &= -e^{x+\eta}\frac{(2\rho_0)^2 t^{2\alpha}}{\Gamma(2\alpha+1)},\\
V_2(x,\eta,t) &= e^{x+\eta}\frac{(2\rho_0)^2 t^{2\alpha}}{\Gamma(2\alpha+1)},\\
p^3: U_3(x,\eta,t) &= -e^{x+\eta}\frac{(2\rho_0)^3 t^{3\alpha}}{\Gamma(3\alpha+1)},\\
V_3(x,\eta,t) &= e^{x+\eta}\frac{(2\rho_0)^3 t^{3\alpha}}{\Gamma(3\alpha+1)},
\end{aligned}$$

The solution of Eqs.(15,22) is given by

$$\begin{aligned}
U(x,\eta,t) &= -e^{x+\eta}\sum_{k=0}^{+\infty}\frac{(2\rho_0 t^\alpha)^k}{\Gamma(k\alpha+1)}+\frac{qt^\alpha}{\Gamma(\alpha+1)},\\
&= -e^{x+\eta}E_{\alpha,1}(2\rho_0 t^\alpha)+\frac{qt^\alpha}{\Gamma(\alpha+1)}.\\
V(x,\eta,t) &= e^{x+\eta}\sum_{k=0}^{+\infty}\frac{(2\rho_0 t^\alpha)^k}{\Gamma(k\alpha+1)}-\frac{qt^\alpha}{\Gamma(\alpha+1)},\\
&= e^{x+\eta}E_{\alpha,1}(2\rho_0 t^\alpha)-\frac{qt^\alpha}{\Gamma(\alpha+1)}.
\end{aligned}$$

For $\alpha=1$ and $q=0$, the exact solution of Eqs. (15,22) reduces to

$$U(x,\eta,t)=-e^{x+\eta+2\rho_0 t},\quad V(x,\eta,t)=e^{x+\eta+2\rho_0 t}.$$

Example 3: Consider 3-dimensional (FNS) equation [22]:

$$\begin{aligned}
U_t^\alpha+UU_x+VU_\eta+WU_z &= \rho_0\left(U_{xx}+U_{\eta\eta}+U_{zz}\right),\\
V_t^\alpha+UV_x+VV_\eta+WV_z &= \rho_0\left(V_{xx}+V_{\eta\eta}+V_{zz}\right),\\
W_t^\alpha+UW_x+VW_\eta+WW_z &= \rho_0\left(W_{xx}+W_{\eta\eta}+W_{zz}\right),
\end{aligned}\tag{24}$$

along with the following initial conditions

$$U(x,\eta,z,0)=-0.5x+\eta+z, V(x,\eta,z,0)=x-0.5\eta+z, W(x,\eta,z,0)=x+\eta-0.5z.\tag{25}$$

Operating $\mathscr{L}$ on both sides of Eqs.(24), we get

$$\begin{aligned}
\mathscr{L}[D^\alpha U(x,\eta,z,t)]+\mathscr{L}[UU_x+VU_\eta+WU_z] &= \mathscr{L}[\rho_0\left(U_{xx}+U_{\eta\eta}+U_{zz}\right)],\\
\mathscr{L}[D^\alpha V(x,\eta,z,t)]+\mathscr{L}[UV_x+VV_\eta+WV_z] &= \mathscr{L}[\rho_0\left(V_{xx}+V_{\eta\eta}+V_{zz}\right)],\\
\mathscr{L}[D^\alpha W(x,\eta,z,t)]+\mathscr{L}[UW_x+VW_\eta+WW_z] &= \mathscr{L}[\rho_0\left(W_{xx}+W_{\eta\eta}+W_{zz}\right)].
\end{aligned}\tag{26}$$

Using the formula (6) for $m=1$, in the above Eqs.(26), and applying $\mathscr{L}^{-1}$, we have

$$\begin{aligned}
U(x,\eta,z,t) &= -0.5x+\eta+z-\mathscr{L}^{-1}\left\{\tfrac{1}{s^\alpha}\left[\mathscr{L}[UU_x+VU_\eta+WU_z]-\mathscr{L}[\rho_0(\nabla^2 U)]\right]\right\},\\
V(x,\eta,z,t) &= x-0.5\eta+z-\mathscr{L}^{-1}\left\{\tfrac{1}{s^\alpha}\left[\mathscr{L}[UV_x+VV_\eta+WV_z]-\mathscr{L}[\rho_0(\nabla^2 V)]\right]\right\},\\
W(x,\eta,z,t) &= x+\eta-0.5z-\mathscr{L}^{-1}\left\{\tfrac{1}{s^\alpha}\left[\mathscr{L}[UW_x+VW_\eta+WW_z]-\mathscr{L}[\rho_0(\nabla^2 W)]\right]\right\},
\end{aligned}\tag{27}$$

where $\nabla^2 = \frac{\partial}{x^2} + \frac{\partial}{\eta^2} + \frac{\partial}{z^2}$. Assume that with homotopy perturbation method, we get

$$\begin{array}{c} \sum_{n=0}^{+\infty} p^n U_n = -0.5x + \eta + z - p\mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\left[\mathscr{L}[H_n] - \mathscr{L}[\rho_0(\bigtriangledown^2 \sum_{n=0}^{+\infty} p^n U_n)]\right]\right\}, \\ \sum_{n=0}^{+\infty} p^n V_n = x - 0.5\eta + z - p\mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\left[\mathscr{L}[K_n] - \mathscr{L}[\rho_0(\bigtriangledown^2 \sum_{n=0}^{+\infty} p^n V_n)]\right]\right\}, \\ \sum_{n=0}^{+\infty} p^n W_n = x + \eta - 0.5z - p\mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\left[\mathscr{L}[J_n] - \mathscr{L}[\rho_0(\bigtriangledown^2 \sum_{n=0}^{+\infty} p^n W_n)]\right]\right\}, \end{array} \quad (28)$$

where H_n, K_n and J_n are He's polynomials that represent nonlinear term $UU_x + VU_\eta + WU_z$, $UV_x + VV_\eta + WV_z$; and $UW_x + VW_\eta + WW_z$ respectively, and we have a few terms of H_n, K_n and J_n which are given by

$$\begin{aligned} H_0 &= U_0U_{0x} + V_0U_{0\eta} + W_0U_{0z} \\ H_1 &= U_0U_{1x} + U_1U_{0x} + V_0U_{1\eta} + V_1U_{0\eta} + W_0U_{1z} + W_1U_{0z} \\ H_2 &= U_0U_{2x} + U_1U_{1x} + U_2U_{0x} + V_0U_{2\eta} + V_1U_{1\eta} + V_2U_{0\eta} + W_0U_{2z} + W_1U_{1z} + W_2U_{0z} \\ &\vdots \end{aligned}$$

And

$$\begin{aligned} K_0 &= U_0V_{0x} + V_0V_{0\eta} + W_0V_{0z} \\ K_1 &= U_0V_{1x} + U_1V_{0x} + V_0V_{1\eta} + V_1V_{0\eta} + W_0V_{1z} + W_1V_{0z} \\ K_2 &= U_0V_{2x} + U_1V_{1x} + U_2V_{0x} + V_0V_{2\eta} + V_1V_{1\eta} + V_2V_{0\eta} + W_0V_{2z} + W_1V_{1z} + W_2V_{0z} \\ &\vdots \end{aligned}$$

Similarly

$$\begin{aligned} J_0 &= U_0W_{0x} + V_0W_{0\eta} + W_0W_{0z} \\ J_1 &= U_0W_{1x} + U_1W_{0x} + V_0W_{1\eta} + V_1W_{0\eta} + W_0W_{1z} + W_1W_{0z} \\ J_2 &= U_0W_{2x} + U_1W_{1x} + U_2W_{0x} + V_0W_{2\eta} + V_1W_{1\eta} + V_2W_{0\eta} + W_0W_{2z} + W_1W_{1z} + W_2W_{0z} \\ &\vdots \end{aligned}$$

Then, we get

$$\begin{aligned} p^0 : U_0(x,\eta,t) &= -0.5x + \eta + z, \\ V_0(x,\eta,t) &= x - 0.5\eta + z, \\ W_0(x,\eta,t) &= x + \eta - 0.5z, \end{aligned}$$

$$\begin{aligned} p^1 : U_1(x,\eta,t) &= -\mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\left[\mathscr{L}[H_0] - \mathscr{L}[\rho_0(\nabla^2 U_0)]\right]\right\}, \\ &= -\frac{2.25xt^\alpha}{\Gamma(\alpha+1)}, \\ V_1(x,\eta,t) &= -\mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\left[\mathscr{L}[K_0] - \mathscr{L}[\rho_0(\nabla^2 V_0)]\right]\right\}, \\ &= -\frac{2.25\eta t^\alpha}{\Gamma(\alpha+1)}, \\ W_1(x,\eta,t) &= -\mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\left[\mathscr{L}[J_0] - \mathscr{L}[\rho_0(\nabla^2 W_0)]\right]\right\}, \\ &= -\frac{2.25zt^\alpha}{\Gamma(\alpha+1)}, \end{aligned}$$

$$\begin{aligned}
p^2: U_2(x,\eta,t) &= -\mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\left[\mathscr{L}[H_1]-\mathscr{L}[\rho_0(\nabla^2 U_1)]\right]\right\}, \\
&= \frac{2(2.25)t^{2\alpha}}{\Gamma(2\alpha+1)}U_0, \\
V_2(x,\eta,t) &= -\mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\left[\mathscr{L}[K_1]-\mathscr{L}[\rho_0(\nabla^2 V_1)]\right]\right\}, \\
&= \frac{2(2.25)t^{2\alpha}}{\Gamma(2\alpha+1)}V_0, \\
W_2(x,\eta,t) &= -\mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\left[\mathscr{L}[J_1]-\mathscr{L}[\rho_0(\nabla^2 W_1)]\right]\right\}, \\
&= \frac{2(2.25)t^{2\alpha}}{\Gamma(2\alpha+1)}W_0,
\end{aligned}$$

$$\begin{aligned}
p^3: U_3(x,\eta,t) &= -\mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\left[\mathscr{L}[H_2]-\mathscr{L}[\rho_0(\nabla^2 U_2)]\right]\right\}, \\
&= -\frac{(2.25)^2}{\Gamma(3\alpha+1)}\left(\frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}+4\right)xt^{3\alpha}, \\
V_3(x,\eta,t) &= -\mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\left[\mathscr{L}[K_2]-\mathscr{L}[\rho_0(\nabla^2 V_2)]\right]\right\}, \\
&= -\frac{(2.25)^2}{\Gamma(3\alpha+1)}\left(\frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}+4\right)\eta t^{3\alpha}, \\
W_3(x,\eta,t) &= -\mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\left[\mathscr{L}[J_2]-\mathscr{L}[\rho_0(\nabla^2 W_2)]\right]\right\}, \\
&= -\frac{(2.25)^2}{\Gamma(3\alpha+1)}\left(\frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}+4\right)zt^{3\alpha}, \\
&\vdots
\end{aligned}$$

The solution of Eqs. (24,25) in series form is given by

$$\begin{aligned}
U(x,\eta,z,t) &= U_0-\frac{2.25xt^\alpha}{\Gamma(\alpha+1)}+\frac{2(2.25)t^{2\alpha}}{\Gamma(2\alpha+1)}U_0-\frac{(2.25)^2}{\Gamma(3\alpha+1)}\left(\frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}+4\right)xt^{3\alpha}+\cdots, \\
V(x,\eta,z,t) &= V_0-\frac{2.25\eta t^\alpha}{\Gamma(\alpha+1)}+\frac{2(2.25)t^{2\alpha}}{\Gamma(2\alpha+1)}V_0-\frac{(2.25)^2}{\Gamma(3\alpha+1)}\left(\frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}+4\right)\eta t^{3\alpha}+\cdots, \\
W(x,\eta,z,t) &= W_0-\frac{2.25zt^\alpha}{\Gamma(\alpha+1)}+\frac{2(2.25)t^{2\alpha}}{\Gamma(2\alpha+1)}W_0-\frac{(2.25)^2}{\Gamma(3\alpha+1)}\left(\frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}+4\right)zt^{3\alpha}+\cdots.
\end{aligned}$$

Why $\alpha=1$, as in [22]

$$\begin{aligned}
U(x,\eta,z,t) &= (-0.5x+\eta+z-2.25xt)(1+2.25t^2+2.25^2t^4+2.25^3t^6+\cdots) \\
&= \frac{-0.5x+\eta+z-2.25xt}{1-2.25t^2} \\
V(x,\eta,z,t) &= (x-0.5\eta+z-2.25\eta t)(1+2.25t^2+2.25^2t^4+2.25^3t^6+\cdots) \\
&= \frac{x-0.5\eta+z-2.25\eta t}{1-2.25t^2}, \\
W(x,\eta,z,t) &= (x+\eta-0.5z-2.25zt)(1+2.25t^2+2.25^2t^4+2.25^3t^6+\cdots) \\
&= \frac{x+\eta-0.5z-2.25zt}{1-2.25t^2}.
\end{aligned}$$

5. Conclusions

In this paper, (FHPTM) has been successfully and easily applied to obtain approximate solutions for fractional multi-dimensional Navier-Stokes equation. Through the three examples of the studied model and through tables

[1-2] and figures [1-3], it can be seen that this method is effective and accurate for different values of time, space and fractional order.

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No conflict of interest was declared by the authors.

Authorship Contribution Statement

Mohamed Zellal: Conceptualization, Methodology, Validation, Formal Analysis, Writing Original Draft
Kacem Belghaba: Investigation, Resources, Visualization, Supervision.

References

[1] K.B. Oldham and J. Spanier, "The Fractional Calculus," Academic Press, New York, 1974.

[2] M. Du, Z. Wang and H. Hu, "Measuring memory with the order of fractional derivative," Scientific Reports, 3, 3431; 2013.

[3] A.U. Rehman, M.B. Riaz, A. Akgül, S.T. Saeed and D .Baleanu, "Heat and mass transport impact on MHD second grade fluid: A comparative analysis of fractional operators," Heat Transfer, vol. 50, no. 7, pp. 7042-7064, 2021.

[4] R.L. Bagley and P.J. Torvik, "A theoretical basis for the application of fractional calculus to viscoelasticity," J. Rheol., vol. 27, pp. 201–210, 1983.

[5] M. Aleem, M.I. Asjad and A. Akgül, "Heat transfer analysis of magnetohydrodynamic Casson fluid through a porous medium with constant proportional Caputo derivative," Heat Transfer, vol. 50, no. 7, pp. 6444-6464, 2021.

[6] A. Carpinteri and F. Mainardi, "Fractals and Fractional Calculus in Continuum Mechanics," Springer Verlag, Wien, New York, 1997.

[7] K.S. Miller and B. Ross, "An Introduction to the Fractional Calculus and Fractional Differential Equations," Wiley, New York, 1993.

[8] I. Podlubny, "Fractional Differential Equations, " Academic Press, New York, 1999.

[9] A. Kilbas, H.M. Srivastava and J.J. Trujillo, "Theory and Application of Fractional Differential Equations," Amsterdam: Elsevier, 2006.

[10] A. Akgül, "A novel method for a fractional derivative with non-local and non-singular kernel," Chaos, Solitons & Fractals, vol. 114, pp. 478-482, 2018.

[11] M. Inc, E.G. Fan, "Extended tanh-function method for finding travelling wave solutions of some nonlinear partial differential equations," Z. Naturforsch, vol. A60, pp. 7-16, 2005.

[12] K. Pandey , L. Verma, Amit and K. Verma, "On a finite difference scheme for Burgers' equation," Applied Mathematics and Computation, vol. 215, pp. 2206–2214, 2009.

[13] M. Zahid, M.I. Asjad, S. Hussain and A. Akgül, "Nonlinear magnetohydrodynamic flow of nanofluids across a porous matrix over an extending sheet with mass transpiration and bioconvection," Heat Transfer, vol. 50, no. 8, pp. 7588-7603, 2021.

[14] K. S. Brajesh and K. Pramod, "Homotopy perturbation transform method for solving fractional partial differential equations with proportional delay," Sociedad Española de Matemática Aplicada, doi: 10.1007/s40324-017-0117-1, 2017.

[15] Y. Khan, N. Faraz, S. Kumar and A.Yildirim, "A coupling method of homotopy perturbation and Laplace transformation for fractional models," U. P. B. Science Bulletin, Series A, vol. 74, no. 1, pp. 57–68, 2012.

[16] S. Kumar, A. Yildirim, Y. Khan and L. Wei, ”A fractional model of the diffusion equation and its analytical solution using Laplace transform,” Sci. Iran. B, vol. 19, no. 4, pp. 1117–1123, 2012.

[17] L. Yanqin, ”Approximate Solutions of Fractional Nonlinear Equations Using Homotopy Perturbation Transformation Method,” Abstract and Applied Analysis, vol. 2012, 2012.

[18] S. Dinkar, S. Prince, and C. Shubha, ”Homotopy Perturbation Transform Method with He's Polynomial for Solution of Coupled Nonlinear Partial Differential Equations,” Nonlinear Engineering, vol. 5, no. 1, pp. 17–23, 2016.

[19] M. Zellal and K. Belghaba, ”Applications of homotopy perturbation transform method for solving Newell-Whitehead-Segel equation,” General Letters in Mathematics, vol. 3, no. 1, pp. 35-46, 2017.

[20] M. El-Shahed and A. Salem, ”On the generalized Navier–Stokes equations,” Appl Math Comput, vol. 156, no. 1, pp. 287–93, 2005.

[21] G.A. Birajdar, ”Numerical solution of time fractional Navier–Stokes equation by discrete Adomian decomposition method,” Nonlinear Eng, vol. 3, no. 1, pp. 21–26, 2014.

[22] K.S. Brajesh and K. Pramod, ”FRDTM for numerical simulation of multi-dimensional, time fractional model of Navier-Stokes equation,” Ain Shams Engineering Journal, vol. 9, no. 4, pp. 827-834, 2018.

[23] D. Kumar, J. Singh and S. Kumar, ”A Fractional Model of Navier-Stokes Equation Arising in Unsteady Flow of a Viscous Fluid,” Journal of the Association of Arab Universities for Basic and Applied Sciences, vol. 17, pp. 14-19, 2015.

[24] S. Kumar, D. Kumar, M. Abbasbandy and M. Rashidi, ”Analytical Solution of Fractional Navier-Stokes Equation by using Modified Laplace Decomposition Method, ” Ain Shams Engineering Journal, vol. 5, no. 2, pp. 569–574, 2014.

[25] S. Momani and Z. Odibat, ”Analytical solution of a time-fractional Navier–Stokes equation by Adomian decomposition method,” Applied Mathematics and Computation, vol. 177, no. 2, pp. 488–494, 2006.

[26] A. A. Ragab, K. M. Hemida, M. S. Mohamed, and Abd El M. A.Salam, ”Solution of Time-Fractional Navier-Stokes Equation by Using Homotopy Analysis Method,” Gen. Math. Notes, vol. 13, no. 2, pp.13-21, 2012.

[27] M. Zellal and K. Belghaba, ”He's variational iteration method for solving multi-dimensional of Navier Stokes equation,” International Journal of Analysis and Applications, vol. 18, no. 5, pp. 724-737, 2020.

[28] J. H. He, ”Homotopy perturbation technique,” Comput. Methods Appl. Mech. Eng, vol. 178, pp. 257–262, 1999.

[29] J. H. He, ”Homotopy perturbation method: a new nonlinear analytical technique,” App. Math. Comp, vol. 135, pp. 73–79, 2003.

[30] A. Ghorbani and J. S. Nadjafi, ” He's homotopy Perturbation method for calculating Adomian polynomials,” International Journal of Nonlinear Sciences and Numerical Simulation, vol. 8, no. 2, pp. 229-232, 2007.

[31] A. Ghorbani, ”Beyond Adomians polynomials: He polynomials,” Chaos Solitons Fractals, vol. 39, no. 3, pp. 1486–1492, 2009.