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Weakly Poor Modules

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Abstract

In this paper, weakly poor modules are introduced as modules whose injectivity domains are contained in the class of all copure-split modules. This notion gives a generalization of both poor modules and copure-injectively poor modules. Properties involving weakly poor modules as well as examples that show the relations between weakly poor modules, poor modules, impecunious modules and copure-injectively poor modules are given. Rings over which every module is weakly poor are right CDS. A ring over which there is a cyclic projective weakly poor module is proved to be weakly poor. Moreover, the characterizations of weakly poor abelian groups is given. It states that an abelian group A is weakly poor if and only if A is impecunious if and only if for every prime integer p, A has a direct summand isomorphic to $\mathbb{Z}_{p^n}$ for some positive integer n. Consequently, an example of a weakly poor abelian group which is neither poor nor copure-injectively poor is given so that the generalization defined is proper.

Keywords: *Copure-injective modules; Copure-split modules; (weakly) poor modules; CDS rings.*
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1. Introduction

Throughout this paper, R will denote an associative ring with identity, and modules will be unital right R-modules, unless otherwise stated. As usual, the category of right R-modules is denoted by $Mod-R$. For a module M, $E(M)$, $Z(M)$ and $T(M)$ denote the the injective hull, singular submodule and torsion submodule of M, respectively.

Some new studies in module theory have focused on to approach to the injectivity from the point of relative notions. The injectivity domain $\mathfrak{In}^{-1}(A)$ for a module A, is the class of all modules B such that A is B-injective, where A is said to be B-injective if for every monomorphism $f: C \to B$ and homomorphism $g: C \to A$, there exists a homomorphism $h: B \to A$ such that $hf = g$ (see [2]). It is clear that for every R-module A, $\mathfrak{In}^{-1}(A)$ contains all semisimple R-modules. In [3], poor modules are introduced as an opposite to the concept of injective modules. An R-module A is called poor if $\mathfrak{In}^{-1}(A)$ is the class of all semisimple R-modules. Over an arbitrary ring, poor modules exist and the structure of poor modules over the ring of integers can be found in [4].

An R-module A is said to be cofinitely generated if $E(A) = E(S_1) \oplus E(S_2) \oplus ... \oplus E(S_n)$, where $S_1, S_2, ..., S_n$ are simple R-modules (see [17]). If an R-module A is isomorphic to $\prod\{E(S_\alpha)|S_\alpha$is a simple right R-module, $\alpha \in I\}$, where I is some index set, then A is called a cofree module (see [9]). A right R-module A is said to be cofinitely related if there is an exact sequence $0 \to A \to B \to C \to 0$ of R-modules with B cofinitely generated, cofree and C cofinitely generated (see [9]). As a dual notion of purity, by using cofinitely related modules, the notion of copurity is introduced in [10]. An exact sequence of R-modules $0 \to A \to B \to C \to 0$ is called a copure exact sequence if every cofinitely related right R-module is injective relative to this sequence. A right R-module A is called (co)pure-split in case every (co)pure submodule of A is a direct summand of A.

Following idea on pure-injectivity profile of [8], in [16], copure-injectivity profile of a ring is introduced. For two modules A and B, A is called B-copure-injective if for every copure monomorphism $f: C \to B$ and a homomorphism $g: C \to A$, there exists a homomorphism $h: B \to A$ such that $hf = g$. A is copure-injective if it is injective with respect to every copure exact sequences (see [11]). The copure-injectivity domain $\mathfrak{CPI}^{-1}(A)$ of A is the class of modules B such that A is B-copure-injective. In [16], copure-injectively-poor (shortly copi-poor) modules introduced as modules with minimal copure-injectivity domain and studied properties of copi-poor modules.

So far, copurity, copure-injectivity and their relative notions are studied by many authors (see, [1, 10, 11, 13, 16]).

The modules whose injectivity domain is contained in some classes of modules has been studied in recent years by many papers. For example, in [6], modules whose injectivity domain is contained in the class of all modules with zero radical are studied as working-class modules. In [5], modules whose injectivity domain is contained in the class of all pure-split modules are introduced as impecunious modules. It is clear that any semisimple module is pure-split with zero radical.

Another property that not only semisimple modules admit, in general, is copure-split modules. With this idea in mind, we expand the restriction in definition of poor modules and consider modules with their injectivity domain contained in the class of all copure-split. We call

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a module A is weakly poor, if $\mathfrak{In}^{-1}(A)$ contained in the class of all copure-split R-modules, equivalently if whenever A is B-injective for an R-module A, then B is copure-split. Every semisimple module is copure-split and every B-injective R-module is B-copure-injective for an R-module B. Therefore, this definition gives a generalization of both poor modules and copi-poor modules. Indeed, it is shown in the last section that this generalization is proper (see Example 3.3). The existence of weakly poor modules for arbitrary rings is guaranteed by that of poor modules.

Some properties of weakly poor modules along with some examples are given in Sec. 2. We give some special rings such that the notions poor, copi-poor and weakly poor coincide. Rings over which every module is weakly poor is shown to be right CDS so that the conditions "every R-module is copi-poor" and "every R-module is weakly poor" are equivalent for a ring R. The existence of projective weakly poor modules is characterized and shown to be equivalent to the ring being weakly poor as a module over itself.

Let A be a group, P denote the set of prime integers and $p,q \in P$. A group A is called a p-group if the order of every nonzero element of A is p^n for some $n \in \mathbb{N}$. The set $T_p(A) = \{n \in A | p^k m = 0$ for some $k \in \mathbb{N}\}$, which is called p-primary component of A, is a subgroup of A. For the torsion subgroup $T(A)$ of A, we have $T(A) = \bigoplus_{p\in P} T_p(A)$. A is said to be torsion if $T(A) = A$. A is called a bounded group if $nA = 0$ for some nonzero integer n.

In section 3, We study the weakly poor abelian groups in details. In [5], it is proved that R is a right pure-semisimple ring if and only if every R-module is impecunious. Since commutative pure-semisimple rings are CDS (see [7]), a weakly poor module need not be impecunious and an impecunious module need not be weakly poor, in general. We first consider the commutative co-noetherian rings over which weakly poor modules and impecunious modules coincide. Since commutative co-noetherian rings are noetherian, over the ring of integers $\mathbb{Z}$, weakly poor modules and impecunious modules coincide. By using this result, for an abelian group A, it is proven that the following are equivalent: (1) A is weakly poor; (2) A is impecunious; (3) for every prime integer p, A has a direct summand isomorphic to $\mathbb{Z}_{p^n}$ for some positive integer n; (4) The reduced part of A isweakly poor; (5) The torsion subgroup $T(A)$ of A is weakly poor. Consequently, an example of a weakly poor abelian group which is neither poor nor copure-injectively poor is given so that the generalization defined is proper. Moreover, the existence of semisimple projective weakly poor modules is investigated and it is shown that a commutative co-noetherian ring is semisimple artinian if it has a semisimple projective weakly poor module.

2. Weakly poor modules

In this section, we study weakly poor modules that are generalization of poor modules and copi-poor modules.

Definition 2.1. *A module A is called weakly poor if whenever it is B-injective for a module B, then B is copure-split, equivalently if injectivity domain $\mathfrak{In}^{-1}(A)$ of A is contained in the class of all copure-split modules.*

Proposition 1. *Poor modules and copi-poor modules are weakly poor.*

Proof. Let A be a module. If A is B-injective for any module B, then A is B-copure-projective. So, copi-poor modules are also weakly poor. Since every semisimple module is copure-split module, then poor modules are also weakly poor. □

Remark 2.2. *The class of weakly poor modules differs from the classes of poor modules and copi-poor modules, in general.*
(i) Let the module A denote $\mathbb{Z}/4\mathbb{Z}$ as a $\mathbb{Z}/4\mathbb{Z}$-module. Then A is copi-poor and so is weakly poor but not poor by [16, Example 3.1].
(ii) On the other hand, consider the $\mathbb{Z}$-module $B = \prod_{p\in\mathscr{P}} \mathbb{Z}/p\mathbb{Z}$, where $\mathscr{P}$ denotes the set of prime integers. Then B is poor and so weakly poor, but not copi-poor by [16, Example 3.2].

Remark 2.3. *The class of all copure-split modules need not be closed under submodules and factor modules in general. But if for weakly poor module A, B is contained in $\mathfrak{In}^{-1}(A)$, then submodules and factor modules of B are also copure-split.*

Recall that a ring R is a right V-ring if every simple right R-module is injective.

Lemma 2.4. *Over a right V-ring, the classes of poor modules, copi-poor modules and weakly poor modules coincide.*

Proof. Over a right V-ring, since all monomorphisms are copure by [10, Proposition 5], the proof is obvious. □

Lemma 2.5. *If a direct summand of a module B is weakly poor, then so is A.*

Proof. Let B be a weakly poor module. We will prove that $A = B \oplus C$ is weakly poor. Assume $B \oplus C$ is D-injective, then B is D-injective. Since B is weakly poor, D must be copure-split. Thus A is weakly poor. □

Of course the converse of Lemma 2.5 need not be true in general (see Example 3.3). However, the following lemma shows that the converse of Lemma 2.5 is true in a special case.

Lemma 2.6. *B is weakly poor if $E \oplus B$ is weakly poor and E is injective.*

Proof. Assume B is D-injective, then $E \oplus B$ is D-injective. So, D is copure-split. □

By the following proposition, the converse of Lemma 2.5 characterizes the right CDS rings. A ring R is a right CDS ring if and only if every R-module is copure-injective if and only if every R-module is copure-split.

Rings over which every module is poor (resp. copi-poor) are characterized in [3, Remark 2.3] (resp. [16, Theorem 3.1]), those are semisimple (resp. CDS) rings. It turns out that the conditions "every R-module is copi-poor" and "every R-module is weakly poor" are equivalent for a ring R.

Proposition 2. *For a ring R, the following conditions are equivalent:*

(1) R is right CDS.
(2) Every right R-module is weakly poor.

(3) There exists an injective weakly poor right R-module.
(4) 0 is a weakly poor right R-module.
(5) Every weakly poor rightR-module is copure-split.
(6) Every factor of a weakly poor right R-module is weakly poor.
(7) Every summand of a weakly poor right R-module is weakly poor.

Proof. The implications $(1) \Rightarrow (2)$ and $(1) \Rightarrow (5)$ are clear since every R-module is copure-split.
The implications $(2) \Rightarrow (3)$, $(2) \Rightarrow (4)$ and $(2) \Rightarrow (6) \Rightarrow (7)$ are clear.
$(4) \Rightarrow (2)$ It immediately from Remark 2.5.
$(3) \Rightarrow (1)$ Let C be an injective weakly poor right R-module and A a right R-module. Since C is A-injective, A is copure-split. Then R is a right CDS ring.
$(5) \Rightarrow (1)$ Let A be a right R-module. Then for a weakly poor right R-module B, $A \oplus B$ is also weakly poor for any right R-module A by Remark 2.5. So $A \oplus B$ is copure-split by (5), whence A is copure-split by [16, Proposition 2.2]. Thus R is a right CDS ring.
$(7) \Rightarrow (2)$ Let A be a right R-module. Then $A \oplus B$ is a weakly poor module for some weakly poor right R-module B. Hence, A is weakly poor by the hypothesis. □

Remark 2.7. *Every commutative pure-semisimple ring and every commutative uniserial ring (e.g. $\mathbb{Z}_n$ for every $n \in \mathbb{N}$) is CDS by [11, Proposition 19(i)] and [7, Theorem 10.4], respectively. Therefore, every R-module over these rings are weakly poor.*

Proposition 3. *A right R-module A is weakly poor if and only if $\prod_{i\in I} A_i$ is weakly poor where $A_i = A$ for all $i \in I$.*

Proof. The proof is obvious by the fact that A is B-injective if and only if for any index set I with $A_i = A$ for all $i \in I$, $\prod_{i\in I} A_i$ is B-injective for any right R-module B. □

By Remark 2.5 and Proposition 3, weakly poor rings are characterized as follows:

Corollary 1. *For a ring R, the following are equivalent:*

(1) R_R is weakly poor.
(2) Any direct product of copies of R is weakly poor.
(3) Every projective right R-module is weakly poor.
(4) There exists a cyclic projective weakly poor right R-module.

Let $\mathfrak{C}$ be a class of R-modules. Recall that $\mathfrak{C}^{\perp} = \{X : \mathrm{Ext}^1_R(C,X) = 0 \text{ for all } C \in \mathfrak{C}\}$ is called the right orthogonal class of $\mathfrak{C}$, and ${}^{\perp}\mathfrak{C} = \{X : \mathrm{Ext}^1_R(X,C) = 0 \text{ for all } C \in \mathfrak{C}\}$ is called the left orthogonal class of $\mathfrak{C}$.

Proposition 4. *If $\mathfrak{C}$ is a class of modules closed under homomorphic images and there is a module A from $\mathfrak{C}^{\perp}$ that is weakly poor then every module C from $\mathfrak{C}$ is copure-split.*

Proof. Let $C \in \mathfrak{C}$. We will prove that A is C-injective. Let $f : B \to C$ be any monomorphism. For the short exact sequence

$$0 \to B \to C \xrightarrow{\pi} C/B \to 0$$

we have the exact sequence

$$0 \to \mathrm{Hom}(C/B,A) \to \mathrm{Hom}(C,A) \xrightarrow{\pi_*} \mathrm{Hom}(B,A) \to \mathrm{Ext}^1(C/B,A) \to \dots$$

Since $\mathfrak{C}$ is closed under homomorphic images, $C/B \in \mathfrak{C}$, so $\mathrm{Ext}^1(C/B,A) = 0$, whence π_* is an epimorphism. Thus A is C-injective. Since A is weakly poor, C is copure-split. □

Recall that the singular submodule $Z(A)$ of a right R-module A is the set of elements $a \in A$ such that $aI = 0$ for some essential right ideal I of R. A right module A is called *singular* if $Z(A) = A$, and *nonsingular* if $Z(A) = 0$.

Proposition 5. *If a ring R has a nonsingular weakly poor module, then every singular module is copure-split.*

Proof. Let A be a nonsingular weakly poor module and B a singular module. For any submodule C of B, C is singular, and so $Hom_R(C,A) = 0$. This means that A is B-injective, and so B is copure-split.

□

In [16, Lemma 3.1], the authors investigated the property of being copi-poor for modules under factor rings. In order to do the same for weakly poor modules, we need the following result.

Lemma 2.8. *[16, Lemma 3.1] Let I be an ideal of a ring R and A be an R/I-modules. Then A is a copure-split R-module if and only if A is a copure-split R/I-module.*

Corollary 2. *Let I be an ideal of a ring R and A be an R/I-module. If A is a weakly poor R-module, then it is a weakly poor R/I-module.*

Corollary 3. *Let A be a semisimple weakly poor R-module. Then it is a weakly poor R/I-module for every ideal I of R contained in $J(R)$.*

Proposition 6. *Let R be a ring with the property that every R-module is either injective or weakly poor. Then, for every ideal I of R, R/I has the same property.*

Proof. Let A be an R/I-module which is not injective. Then it is not injective as an R-module so that A is a weakly poor R-module by assumption. Hence, A is a weakly poor R/I-module by Corollary 2. □

3. weakly poor abelian groups

In this section, weakly poor modules are compared with impecunious modules and weakly poor abelian groups are investigated.

Remark 3.1. *In [5], it is proved that R is a right pure-semisimple ring if and only if every right R-module is impecunious. Since commutative pure-semisimple rings are CDS (see [7]), a weakly poor module need not be impecunious in general and also an impecunious module need not be weakly poor (see Proposition 2).*

Remark 3.2. *If R is a commutative Von Neumann regular ring, impecunious modules coincide with poor, copi-poor and weakly poor modules by [5, Remark 3.2] and Lemma 2.4.*

A ring R is said to be right co-noetherian if every homomorphic image of a cofinitely generated right R-module is cofinitely generated, equivalently for each simple right R-module S the injective hull $E(S)$ is right Artinian (see [12, Theorem]). Over a commutative noetherian ring, the injective hull of each simple right R-module is right Artinian by [15, Exercise 4.17]. Thus every commutative Noetherian ring is co-noetherian.

By [16], pure-split R-modules need not be copure-split and copure-split R-modules need not be pure-split. This implies that a weakly poor module need not be impecunious and an impecunious module need not be weakly poor. By the following proposition, over a commutative co-noetherian ring, weakly poor modules coincide with impecunious modules.

Proposition 7. *Let R be a commutative co-noetherian ring and A be an R-module. Then the following are equivalent.*

(1) A is weakly poor.
(2) A is impecunious.

Proof. (1) $\Rightarrow$ (2) Let A be a weakly poor R-module. Then injectivity domain $\mathfrak{In}^{-1}(A)$ of A is contained in the class of all copure-split modules. Over a commutative co-noetherian ring R, every copure-split R-module is pure-split by [16, Lemma 3.4]. So, in this case A is impecunious.
(2) $\Rightarrow$ (1) Let A be an impecunious R-module. Then injectivity domain $\mathfrak{In}^{-1}(A)$ of a module A is contained in the class of all pure-split modules. By [16, Lemma 3.3], every pure-split R-module is copure-split over a commutative ring. Therefore in this case A is weakly poor. □

The characterization of impecunious abelian groups is given in [5, Theorem 4.1]. Using Proposition 7, an abelian group A is weakly poor if and only if it is impecunious. As a consequences of [5, Theorem 4.1] and [5, Corollary 4.1], we can give the characterization of weakly poor groups.

Corollary 4. *The following are equivalent for an abelian group A:*

(1) A is weakly poor.
(2) A is impecunious.
(3) For every prime integer p, A has a direct summand isomorphic to $\mathbb{Z}_{p^n}$ for some positive integer n.
(4) For every prime integer p, basic subgroup of $T_p(A)$ is not zero.
(5) The reduced part of A is weakly poor.
(6) $T(A)$ is weakly poor.

As the impecunious abelian groups ($\mathbb{Z}$-modules) are equivalent to the weakly poor groups, it is now possible to give an example of a weakly poor group which is neither poor nor copi-poor.

Example 3.3. *Let $A = \bigoplus_{p\in\mathscr{P}} \mathbb{Z}_{p^2}$. By [5, Example 4.1], A is an impecunious group, and so is weakly poor. However, A is neither poor nor copi-poor by [16, Proposition 3.7] and [5, Example 4.1].*

Remark 3.4. *Let A be a torsion abelian group. If $T_p(B) = 0$ is bounded for all $p \in P$, then A is pure-split by [5, Lemma 4.1], and so is copure-split.*

Remark 3.5. *Note that the abelian group in Example 3.3 is copure-split by Remark 3.4 so that there are copure-split weakly poor abelian groups which are neither poor nor copi-poor. Moreover, for any proper direct summand B of a module A, B is not weakly poor as $T_p(B) = 0$ for some $p \in P$.*

In [16, Corollary 3.7], it is shown that every cofinitely generated abelian group is copure-split but not a copi-poor module. It is clear that for every (co-)finitely generated abelian group A, $T_p(A) \neq 0$ for only finite number of primes p. Thus by Corollary 4, we have the following result.

Corollary 5. *Every cofinitely generated abelian group ($\mathbb{Z}$-modules) is copure-split but not weakly poor.*

Two modules are called orthogonal if they have no nonzero isomorphic submodules [14]. In [3], the existence of projective semisimple poor module A implies that any semisimple module B orthogonal to A is injective. Before asking this question for weakly poor modules, first we need the following lemma.

Lemma 3.6. *Let R be a commutative co-noetherian ring R, and S be a projective semisimple module. Then S is copure submodule of its injective hull $E(S)$.*

Proof. By [5, Proposition 3.4], S is pure in $E(S)$. Since every pure submodule of an R-module is also a copure submodule over a commutative co-noetherian ring by [10, Proposition 12], S is copure in $E(S)$. □

By the following result, we consider the existence of semisimple projective weakly poor modules under the hypothesis of R being commutative co-noetherian.

Corollary 6. *Let R be a commutative co-noetherian ring and A be a projective semisimple weakly poor R-module. Then every semisimple R-module S orthogonal to A is injective.*

Proof. The proof is easy since over a commutative co-noetherian ring R, weakly poor modules coincide with impecunious modules by Remark 7(2). □

A commutative ring is called classical if the injective hull $E(S)$ of all simple modules S are linearly compact (see [18]). Since every pure short exact sequence of R-modules is copure over a commutative classical ring by [11, Corollary 16], we generalize Corollary 6 to commutative classical rings.

Corollary 7. *Let R be a commutative classical ring and A be a projective semisimple weakly poor R-module. Then every semisimple R-module S orthogonal to A is injective.*

As the consequences of Corollaries 6, 7 and [5, Corollary 3.2], existence of a semisimple projective weakly poor R-module gives that the ring itself is a semisimple artinian ring.

Corollary 8. *Let R be a commutative co-noetherian or a commutative classical ring. If there is a semisimple projective weakly poor R-module, then R semisimple artinian.*

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Author's contributions

The author contributed to the writing of this paper. The author read and approved the final manuscript.

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