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Solutions of multiplicative integral equations via the multiplicative power series method

Çarpımsal integral denklemlerin çarpımsal kuvvet serisi yöntemiyle çözümleri

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Solutions of Multiplicative Integral Equations via The Multiplicative Power Series Method

Highlights

- *This paper focuses on the solution of multiplicative integral equations.*
- *The study used the multiplicative power series method to solve the multiplicative integral equations.*
- *The solutions of different types of multiplicative integral equations are supported with numerical examples.*

Aim

In this paper, we will give definitions of types of integral equations in geometric analysis. And solutions of different types of integral equations in the geometric analysis will be examined. These studies will be reinforced with examples.

Design & Methodology

Integral equations have been used since the 18th century for the solution of many problems in applied mathematics, mathematical physics, and engineering. Integral equations have started to be used in the field of engineering and other basic sciences and have gained an important place because they can be used in applications of differential equations.

Originality

Since integral equations have a very wide field of research, it is not possible to establish a theory that covers all integral equations. For this reason, integral equations are examined separately according to their properties. Considering these separate examinations, it is divided into two as Volterra and Fredholm Integral Equations. At the end of the 19th century Volterra and at the beginning of the 20th century I. Fredholm, D. Hilbert, and E. Schmidt contributed to the development of a general theory of integral equations.

Findings

It is not easy to explain some problems in mathematics with known classical concepts and find analytical solutions. For the solution of such problems, geometric (multiplicative) analysis including multiplicative derivative and multiplicative integral has been developed as an alternative. It has been seen that geometric analysis brings ease of application to some problems and in some cases gives more effective and faster results than classical analysis. In this study, the solutions of multiplicative integral equations are investigated by using the definition and properties of multiplicative integral.

Conclusion

In this work, we give the definitions of MFIE of the second kind, MVIE of the second kind, and MVFIE of the second kind. Then, solutions of MIE via the MPSM are studied. We investigate solutions of MFIE with kernel equal to 1, MFIE with kernel depending only on the variable x*, and MFIE with kernel depending only on the variable* t*. Consequently, we investigate the solution of MVIE. At last, we give the solutions of MVFIE with kernels equal to* 1 *and MVFIE with kernels* $K_1(x,t) = K(x,t)$ *and* $K_2(x,t) = 1$.

Declaration of Ethical Standards

The authors of this article declare that the materials and methods used in this study do not require ethical committee permission and/or legal-special permission.

Solutions of Multiplicative Integral Equations via The Multiplicative Power Series Method

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Research Article / Araştırma Makalesi

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ABSTRACT

In this study, definitions of types of multiplicative integral equations are given. And solutions of different types of multiplicative integral equations are investigated using the multiplicative power series method. These are supported by numerical examples.

Keywords: Multiplicative integral equation; multiplicative Fredholm integral equation; multiplicative Volterra integral equation; multiplicative Volterra-Fredholm integral equation; multiplicative power series.

Çarpımsal İntegral Denklemlerin Çarpımsal Kuvvet Serisi Yöntemiyle Çözümleri

ÖZ

Bu çalışmada çarpımsal integral denklem türlerinin tanımları verilmiştir. Ve değişik türdeki çarpımsal integral denklemlerinin çözümleri, çarpımsal kuvvet serisi yöntemi kullanılarak incelenmiştir. Bunlar sayısal örnekler ile desteklenmiştir.

Anahtar Kelimeler: Çarpımsal integral denklem, çarpımsal Fredholm intergral denklem, çarpımsal Volterra integral denklem, çarpımsal Volterra-Fredholm integral denklem, çarpımsal kuvvet serisi.

1. INTRODUCTION

Abel used an integral equation in 1823 to solve an isochrones (tautochrone) problem. In the nineteenth century Joseph Liouville, Fredholm, C. Neumann, and H. Poincare solved physical and mathematical problems via integral equations. At the end of the nineteenth century Volterra and at the beginning of the twentieth century I. Fredholm, D. Hilbert, and E. Schmidt contributed to the development of a general theory of integral equations [1].

Integral equations (IE) provide convenience for solving some problems in mathematics, physics, and engineering. Solution methods and techniques of integral equations have been discussed by many researchers [2-9]. IE can be used in so many fields of science as there is a correspondence between them and differential equations with conditions.

Multiplicative integral equations (MIE) were used by H. Durmaz [10-11] . N. Yalcin and M. Dedeturk give solutions of multiplicative differential equations via the multiplicative differential transform method [12] and via the multiplicative power series method [13] . More details on the multiplicative analysis topic can be found in [14-26]. In this work, we give solutions of MIE via the multiplicative power series method (MPSM)

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2. MULTIPLICATIVE INTEGRAL EQUATIONS AND MULTIPLICATIVE POWER SERIES METHOD

Definition 2.1. The set $\mathbb{R}_{exp} = \{e^x | x \in \mathbb{R}\}$ is called the exponential real numbers.

Definition 2.2. Let $f(x)$ and $K_1(x,t), K_2(x,t)$ be given functions, $\varphi(x)$ is an unknown function and b be a given positive real number. Suppose the equation

$$\varphi(x) = f(x) \cdot \left(* \int_0^x \varphi(t)^{K_1(x,t)dt} \right)^{s_1} \cdot \left(* \int_0^b \varphi(t)^{K_2(x,t)dt} \right)^{s_2}$$

is given.

a) If $s_1 = 1, s_2 = 0$ then the equation is called a multiplicative Volterra integral equation (MVIE) of the second kind

b)If $s_1 = 0, s_2 = 1$ then the equation is called a multiplicative Fredholm integral equation (MFIE) of the second kind.

c) If $s_1 = 1, s_2 = 1$ then the equation is called a multiplicative Volterra-Fredholm integral equation (MVFIE) of the second kind.

Definition 2.3. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{R}_{exp}$, and $x_0 \in \mathbb{R}$ be a fixed point. Then the product

$$\prod_{n=0}^{\infty} (a_n)^{(x-x_0)^n}$$

is called a multiplicative power series.

Definition 2.4. Suppose $f(x)$ is a positive function which has multiplicative derivatives of any order at the point $x_0 \in \mathbb{R}$. Then the multiplicative Taylor series of $f(x)$ at the point x_0 is given by

$$\prod_{n=0}^{\infty}\left(f^{*(n)}(x_0)\right)^{\frac{(x-x_0)^n}{n!}}$$

In this paper, multiplicative Taylor series of $f(x)$ and additive (classical) Taylor series of $K_1(x,t), K_2(x,t)$ are used to find the unknown function $\varphi(x)$ in given multiplicative integral equations as a multiplicative power series

$$\varphi(x)=\prod_{n=0}^{\infty}(a_n)^{x^n}.$$

3. MULTIPLICATIVE FREDHOLM INTEGRAL EQUATION (MFIE)

3.1. Solution of the MFIE with Kernel Equal to 1

Theorem 3.1. Suppose the MFIE

$$\varphi(x) = f(x) * \int_0^b \varphi(t)^{dt}$$

is given of which the limit $b \in \mathbb{R}^+\backslash\{1\}$. Also, let $f(x) = \prod_{n=0}^{\infty}(c_n)^{x^n}$ be the multiplicative power series (MPS) of $f(x)$. Then the multiplicative power series solution

$$\varphi(x) = \prod_{n=0}^{\infty}(a_n)^{x^n}$$

has bases a_n, $n \geq 0$ which can be calculated by the below formulas.

$$a_n = c_n, \qquad n \geq 1$$

$$a_0 = \left[c_0 \cdot \prod_{k=1}^{\infty}(a_k)^{b^{k+1}/(k+1)}\right]^{\frac{1}{(1-b)}}$$

Proof.

$$\varphi(x) = f(x) * \int_0^b \varphi(t)^{dt}$$

$$\prod_{n=0}^{\infty}(a_n)^{x^n} = \prod_{n=0}^{\infty}(c_n)^{x^n} \cdot * \int_0^b \left[\prod_{k=0}^{\infty}(a_k)^{t^k}\right]^{dt}$$

$$\prod_{n=0}^{\infty}(a_n)^{x^n} = \prod_{n=0}^{\infty}(c_n)^{x^n} \cdot \prod_{k=0}^{\infty} * \int_0^b \left[(a_k)^{t^k}\right]^{dt}$$

$$\prod_{n=0}^{\infty}(a_n)^{x^n} = \prod_{n=0}^{\infty}(c_n)^{x^n} \cdot \prod_{k=0}^{\infty}(a_k)^{\left(\frac{t^{k+1}}{k+1}\right)_{t=0}^{b}}$$

$$\prod_{n=0}^{\infty}(a_n)^{x^n} = c_0 \cdot \prod_{n=1}^{\infty}(c_n)^{x^n} \cdot \prod_{k=0}^{\infty}(a_k)^{\frac{b^{k+1}}{k+1}}$$

$$a_0 \cdot \prod_{n=1}^{\infty}(a_n)^{x^n} = \left[c_0 \cdot \prod_{k=0}^{\infty}(a_k)^{\frac{b^{k+1}}{k+1}}\right] \cdot \prod_{n=1}^{\infty}(c_n)^{x^n}$$

From here we have

$a_n = c_n, n \geq 1$

and

$$a_0 = c_0 \cdot \prod_{k=0}^{\infty}(a_k)^{\left(\frac{b^{k+1}}{k+1}\right)}$$

$$a_0 = c_0 \cdot (a_0)^b \cdot \prod_{k=1}^{\infty}(a_k)^{\left(\frac{b^{k+1}}{k+1}\right)}$$

$$(a_0)^{1-b} = c_0 \cdot \prod_{k=1}^{\infty}(a_k)^{\left(\frac{b^{k+1}}{k+1}\right)}$$

Thus we get

$$a_n = c_n, n \geq 1$$

$$a_0 = \left[c_0 \cdot \prod_{k=1}^{\infty}(a_k)^{\left(\frac{b^{k+1}}{k+1}\right)}\right]^{\frac{1}{(1-b)}} \blacksquare$$

3.2. Solution of the MFIE with Kernel Depending Only on the Variable x

Below, we will give the solution of MFIE with the Kernel depending only on the variable x.

Theorem 3.2. Suppose the MFIE

$$\varphi(x) = f(x) \cdot * \int_0^b \varphi(t)^{K(x)dt}$$

$$\varphi(x) = f(x) \cdot \left[* \int_0^b \varphi(t)^{dt}\right]^{K(x)}$$

is given. Also, let $f(x) = \prod_{n=0}^{\infty}(c_n)^{x^n}$ be the multiplicative power series of $f(x)$ and $K(x) = \sum_{n=0}^{\infty} k_n x^n$ be the additive power series of $K(x)$. Here $b > 0$. Then the multiplicative power series solution

$$\varphi(x) = \prod_{n=0}^{\infty}(a_n)^{x^n}$$

has bases a_n, $n \geq 0$ which can be calculated by the below formulas.

$$a_n = \left[\left(\prod_{j=0}^{\infty}(c_j)^{\left(\frac{b^{j+1}}{j+1}\right)}\right)^{1/\left[1-\sum_{j=0}^{\infty} k_j \cdot \frac{b^{j+1}}{j+1}\right]}\right]^{k_n} \cdot c_n$$

Proof.

$$\varphi(x) = f(x) \cdot * \int_0^b \varphi(t)^{K(x)dt}$$

$$\varphi(x) = f(x) \cdot \left[* \int_0^b \varphi(t)^{dt}\right]^{K(x)}$$

Let us say $A =* \int_0^b \varphi(t)^{dt}$, then we have

$$\varphi(x) = f(x) \cdot \left[* \int_0^b \varphi(t)^{dt} \right]^{K(x)}$$

$$\prod_{n=0}^{\infty} (a_n)^{x^n} = \prod_{n=0}^{\infty} (c_n)^{x^n} \cdot A^{K(x)}$$

$$\prod_{n=0}^{\infty} (a_n)^{x^n} = \prod_{n=0}^{\infty} (c_n)^{x^n} \cdot A^{\sum_{n=0}^{\infty} k_n x^n}$$

$$\prod_{n=0}^{\infty} (a_n)^{x^n} = \prod_{n=0}^{\infty} (c_n)^{x^n} \cdot \prod_{n=0}^{\infty} (A^{k_n})^{x^n}$$

and let us define $b_n = A^{k_n}$. Then we can write

$$\prod_{n=0}^{\infty} (a_n)^{x^n} = \prod_{n=0}^{\infty} (c_n)^{x^n} \cdot \prod_{n=0}^{\infty} (b_n)^{x^n}$$

$$\prod_{n=0}^{\infty} (a_n)^{x^n} = \prod_{n=0}^{\infty} (b_n \cdot c_n)^{x^n} \Rightarrow a_n = (b_n \cdot c_n)$$

$$\varphi(t) = \prod_{n=0}^{\infty} (b_n \cdot c_n)^{t^n}$$

Now we will find the value of A

$$A =* \int_0^b \varphi(t)^{dt} \Rightarrow A =* \int_0^b \left[\prod_{n=0}^{\infty} (b_n \cdot c_n)^{t^n} \right]^{dt}$$

$$A = \prod_{n=0}^{\infty} (b_n \cdot c_n)^{\int_0^b t^n dt} \Rightarrow A = \prod_{n=0}^{\infty} (b_n \cdot c_n)^{\frac{b^{n+1}}{(n+1)}}$$

$$A = \prod_{n=0}^{\infty} (A^{k_n} \cdot c_n)^{\frac{b^{n+1}}{(n+1)}}$$

$$A = \left[\prod_{n=0}^{\infty} (A)^{k_n \cdot \frac{b^{n+1}}{(n+1)}} \right] \cdot \left[\prod_{n=0}^{\infty} (c_n)^{\frac{b^{n+1}}{(n+1)}} \right]$$

$$A = \left[(A)^{\sum_{n=0}^{\infty} k_n \cdot \frac{b^{n+1}}{(n+1)}} \right] \cdot \left[\prod_{n=0}^{\infty} (c_n)^{\frac{b^{n+1}}{(n+1)}} \right]$$

Putting the powers of A to the same side, we write

$$(A)^{1 - \sum_{n=0}^{\infty} k_n \cdot \frac{b^{n+1}}{(n+1)}} = \prod_{n=0}^{\infty} (c_n)^{\frac{b^{n+1}}{(n+1)}}$$

$$A = \left[\prod_{n=0}^{\infty} (c_n)^{\frac{b^{n+1}}{(n+1)}} \right]^{\frac{1}{1 - \sum_{n=0}^{\infty} k_n \cdot \frac{b^{n+1}}{(n+1)}}}$$

So we have

$$a_n = b_n \cdot c_n = A^{k_n} \cdot c_n$$

$$a_n = \left[\left(\prod_{j=0}^{\infty} (c_j)^{\left(\frac{b^{j+1}}{j+1}\right)} \right)^{1 / \left[1 - \sum_{j=0}^{\infty} k_j \cdot \frac{b^{j+1}}{j+1} \right]} \right]^{k_n} \cdot c_n \blacksquare$$

3.3. Solution of the MFIE with Kernel Depending Only on the Variable t

Theorem 3.3. Suppose the MFIE

$$\varphi(x) = f(x) \cdot * \int_0^b \varphi(t)^{K(t)dt}$$

is given. Also, let $f(x) = \prod_{n=0}^{\infty} (c_n)^{x^n}$ be the multiplicative power series of the given function $f(x)$ and $K(x) = \sum_{j=0}^{\infty} k_j t^j$ be the additive power series of $K(t)$. Here $b > 0$. Then the multiplicative power series solution

$$\varphi(x) = \prod_{n=0}^{\infty} (a_n)^{x^n}$$

has bases a_n, $n \geq 0$ which can be calculated by the below formulas.

$$\begin{cases} a_0 = \left[c_0 \cdot \prod_{i=1}^{\infty} (a_i)^{\sum_{j=0}^{\infty} \left(k_j \cdot \frac{b^{i+j+1}}{i+j+1} \right)} \right]^{\frac{1}{1 - \sum_{j=0}^{\infty} \left(k_j \cdot \frac{b^{j+1}}{j+1} \right)}}, \\ a_n = c_n, \qquad n \geq 1. \end{cases}$$

Proof.

$$\varphi(x) = f(x) \cdot * \int_0^b \varphi(t)^{K(t)dt}$$

$$\prod_{n=0}^{\infty} (a_n)^{x^n} = \prod_{n=0}^{\infty} (c_n)^{x^n} \cdot * \int_0^b \left[\prod_{i=0}^{\infty} (a_i)^{t^k \cdot \sum_{j=0}^{\infty} k_j \cdot t^j} \right]^{dt}$$

$$\prod_{n=0}^{\infty} (a_n)^{x^n} = \prod_{n=0}^{\infty} (c_n)^{x^n} \cdot \prod_{i=0}^{\infty} (a_i)^{\int_0^b t^k \cdot \sum_{j=0}^{\infty} k_j \cdot t^j dt}$$

$$\prod_{n=0}^{\infty} (a_n)^{x^n} = \prod_{n=0}^{\infty} (c_n)^{x^n} \cdot \prod_{i=0}^{\infty} (a_i)^{\sum_{j=0}^{\infty} \int_0^b k_j \cdot t^{i+j} dt}$$

and we get

$$\prod_{n=0}^{\infty} (a_n)^{x^n} = \prod_{n=0}^{\infty} (c_n)^{x^n} \cdot \prod_{i=0}^{\infty} (a_i)^{\sum_{j=0}^{\infty} \left(k_j \cdot \frac{t^{i+j+1}}{i+j+1} \right)}$$

$$\prod_{n=0}^{\infty} (a_n)^{x^n} = \prod_{n=0}^{\infty} (c_n)^{x^n} \cdot \prod_{i=0}^{\infty} (a_i)^{\sum_{j=0}^{\infty} \left(k_j \cdot \frac{t^{i+j+1}}{i+j+1} \right)_{t=0}^{b}}$$

$$\prod_{n=0}^{\infty} (a_n)^{x^n} = \prod_{n=0}^{\infty} (c_n)^{x^n} \cdot \prod_{i=0}^{\infty} (a_i)^{\sum_{j=0}^{\infty} \left(k_j \cdot \frac{b^{i+j+1}}{i+j+1} \right)}$$

$$a_0 \cdot \prod_{n=1}^{\infty} (a_n)^{x^n} = \left[c_0 \cdot \left(\prod_{i=0}^{\infty} (a_i)^{\sum_{j=0}^{\infty} \left(k_j \cdot \frac{b^{i+j+1}}{i+j+1} \right)} \right) \right] \cdot \prod_{n=1}^{\infty} (c_n)^{x^n}$$

From here we have

$$a_n = c_n, \qquad n \geq 1$$

and

$$a_0 = c_0 \cdot \prod_{i=0}^{\infty} (a_i)^{\sum_{j=0}^{\infty} \left(k_j \cdot \frac{b^{i+j+1}}{i+j+1} \right)}$$

$$a_0 = c_0 \cdot (a_0)^{\sum_{j=0}^{\infty}\left(k_j \cdot \frac{b^{j+1}}{j+1}\right)} \cdot \prod_{i=1}^{\infty} (a_i)^{\sum_{j=0}^{\infty}\left(k_j \cdot \frac{b^{i+j+1}}{i+j+1}\right)}$$

$$a_0 = \left[c_0 \cdot \prod_{i=1}^{\infty} (a_i)^{\sum_{j=0}^{\infty}\left(k_j \cdot \frac{b^{i+j+1}}{i+j+1}\right)}\right]^{\frac{1}{1-\sum_{j=0}^{\infty}\left(k_j \cdot \frac{b^{j+1}}{j+1}\right)}} \blacksquare$$

Example 3.1. Suppose the MFIE

$$\varphi(x) = e^{x-\frac{b^2}{2}} \cdot_* \int_0^b \varphi(t)^{dt}$$

is given for $b \in \mathbb{R}^+ \backslash \{1\}$. Let us solve this equation via the multiplicative power series method.

Solution. Let $\varphi(x) = \prod_{n=0}^{\infty} (a_n)^{(x-x_0)^n}$ be the multiplicative power series of the unknown function $\varphi(x)$.

The given function $f(x) = e^{x-\frac{b^2}{2}}$ has the multiplicative power series expansion as

$$e^{x-\frac{b^2}{2}} = e^{-\frac{b^2}{2}} \cdot e^x = e^{-\frac{b^2}{2}} \cdot \left(1^1 \cdot e^x \cdot 1^{x^2} \cdot \ldots \cdot 1^{x^n} \cdot \ldots\right)$$

$$e^{x-\frac{b^2}{2}} = \left(e^{-\frac{b^2}{2}}\right)^1 \cdot e^x \cdot \prod_{n=2}^{\infty} 1^{x^n}$$

Now let us look at the solution steps

$$\varphi(x) = e^{x-\frac{b^2}{2}} \cdot_* \int_0^b \varphi(t)^{dt}$$

$$\prod_{n=0}^{\infty} (a_n)^{(x-x_0)^n} = \left(e^{-\frac{b^2}{2}}\right)^1 \cdot e^x \cdot \prod_{n=2}^{\infty} 1^{x^n} \cdot {}_* \int_0^b \left[\prod_{k=0}^{\infty} (a_k)^{(t-x_0)^k}\right]^{dt}$$

If we take $x_0 = 0$ then

$$\prod_{n=0}^{\infty} (a_n)^{x^n} = \left(e^{-\frac{b^2}{2}}\right)^1 \cdot e^x \cdot \prod_{k=0}^{\infty} {}_* \int_0^b \left[(a_k)^{t^k}\right]^{dt}$$

$$\prod_{n=0}^{\infty} (a_n)^{x^n} = \left(e^{-\frac{b^2}{2}}\right)^1 \cdot e^x \cdot \prod_{k=0}^{\infty} (a_k)^{\left[\frac{t^{k+1}}{k+1}\right]_{t=0}^{b}}$$

$$\prod_{n=0}^{\infty} (a_n)^{x^n} = \left(e^{-\frac{b^2}{2}}\right)^1 \cdot e^x \cdot \prod_{k=0}^{\infty} (a_k)^{\left[\frac{b^{k+1}}{k+1}\right]}$$

Consequently, we write

$$a_0 \cdot a_1^x \cdot \prod_{n=0}^{\infty} (a_n)^{x^n} = \left\{\left(e^{-\frac{b^2}{2}}\right) \cdot \prod_{k=0}^{\infty} (a_k)^{\left[\frac{b^{k+1}}{k+1}\right]}\right\}^1 \cdot e^x$$

$a_1 = e$, $a_n = 1$, for $n \geq 2$ and

$$a_0 = \left(e^{-\frac{b^2}{2}}\right) \cdot \prod_{k=0}^{\infty} (a_k)^{\left[\frac{b^{k+1}}{k+1}\right]}$$

$$a_0 = e^{-\frac{b^2}{2}} \cdot (a_0)^b \cdot (a_1)^{\frac{b^2}{2}} \cdot 1$$

$$(a_0)^{1-b} = 1 \Rightarrow a_0 = 1.$$

So we have

$a_0 = 1, a_1 = e$

$a_n = 1, n \geq 2$

Hence the solution is

$$\varphi(x) = 1^1 \cdot e^x \cdot \prod_{n=2}^{\infty} 1^{x^n}$$

$$\varphi(x) = e^x.$$

4. MULTIPLICATIVE VOLTERRA INTEGRAL EQUATION (MVIE)

Theorem 4.1. Suppose the below MVIE is given

$$\varphi(x) = f(x) \cdot_* \int_0^x \varphi(t)^{K(x,t)dt}$$

And also let's assume that $\varphi(x) = \prod_{n=0}^{\infty} (a_n)^{x^n}$ and $f(x) = \prod_{n=0}^{\infty} (c_n)^{x^n}$ are multiplicative power series of the unknown function $\varphi(x)$ and the given function $f(x)$, respectively and let $K(x,t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} k_{i,j} \cdot x^i t^j$ be the additive power series of $K(x,t)$. Then the solution of the above equation is given as:

$$\varphi(x) = \prod_{n=0}^{\infty} (a_n)^{x^n}$$

Where

$$\begin{cases} a_0 = c_0, \\ a_n = c_n \cdot \prod_{i=0}^{n-1} (a_i)^{\sum_{j=0}^{n-1-i} \frac{k_{n-1-i-j,j}}{i+j+1}}, \quad n \geq 1. \end{cases}$$

Proof.

$$\varphi(x) = f(x) \cdot_* \int_0^x \varphi(t)^{K(x,t)dt}$$

$$\prod_{n=0}^{\infty} (a_n)^{x^n} = \prod_{n=0}^{\infty} (c_n)^{x^n} \cdot {}_* \int_0^x \left[\prod_{n=0}^{\infty} (a_n)^{t^n}\right]^{\sum_{i,j=0}^{\infty} k_{i,j} \cdot x^i t^j dt}$$

$$\prod_{n=0}^{\infty} (a_n)^{x^n} = \prod_{n=0}^{\infty} (c_n)^{x^n} \cdot \prod_{n=0}^{\infty} {}_* \int_0^x (a_n)^{t^n \cdot \sum_{i,j=0}^{\infty} k_{i,j} \cdot x^i t^j dt}$$

$$\prod_{n=0}^{\infty} (a_n)^{x^n} = \prod_{n=0}^{\infty} (c_n)^{x^n} \cdot \prod_{n=0}^{\infty} (a_n)^{\int_0^x \sum_{i=0}^{\infty}\left(x^i \cdot \sum_{j=0}^{\infty} k_{i,j} \cdot t^{n+j}\right)dt}$$

$$= \prod_{n=0}^{\infty} (c_n)^{x^n} \cdot \prod_{n=0}^{\infty} (a_n)^{\sum_{i=0}^{\infty}\left[x^i \cdot \sum_{j=0}^{\infty}\left(k_{i,j} \cdot \int_0^x t^{n+j} dt\right)\right]}$$

$$= \prod_{n=0}^{\infty} (c_n)^{x^n} \cdot \prod_{n=0}^{\infty} (a_n)^{\sum_{i=0}^{\infty}\left[x^i \cdot \sum_{j=0}^{\infty}\left(k_{i,j} \cdot \left(\frac{t^{n+j+1}}{n+j+1}\right)_{t=0}^{x}\right)\right]}$$

$$= \prod_{n=0}^{\infty} (c_n)^{x^n} \cdot \prod_{n=0}^{\infty} (a_n)^{\sum_{i=0}^{\infty}\left[x^i \cdot \sum_{j=0}^{\infty}\left(k_{i,j} \cdot \frac{x^{n+j+1}}{n+j+1}\right)\right]}$$

Consequently, we write

$$\prod_{n=0}^{\infty}(a_n)^{x^n}=\prod_{n=0}^{\infty}(c_n)^{x^n} \cdot \prod_{n=0}^{\infty}(a_n)^{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left(\frac{k_{i,j}}{n+j+1}\cdot x^{n+i+j+1}\right)}$$

$$=\prod_{n=0}^{\infty}(c_n)^{x^n}\cdot\prod_{n=0}^{\infty}(a_n)^{\sum_{i=0}^{\infty}\sum_{j=0}^{i}\left(\frac{k_{i-j,j}}{n+j+1}\cdot x^{n+i+1}\right)}$$

$$=\prod_{n=0}^{\infty}(c_n)^{x^n}\cdot\prod_{n=0}^{\infty}(a_n)^{\sum_{i=0}^{\infty}\sum_{j=0}^{i}\left(\frac{k_{i-j,j}}{n+j+1}\right)\cdot x^{n+i+1}}$$

$$=\prod_{n=0}^{\infty}(c_n)^{x^n}\cdot\prod_{n=0}^{\infty}\prod_{i=0}^{\infty}\left[(a_n)^{\sum_{j=0}^{i}\left(\frac{k_{i-j,j}}{n+j+1}\right)\cdot x^{n+i+1}}\right]$$

Here we define

$$B_{u,v}=(a_u)^{\sum_{j=0}^{v}\left(\frac{k_{v-j,j}}{u+j+1}\right)}$$

and we have

$$\prod_{n=0}^{\infty}(a_n)^{x^n}=\prod_{n=0}^{\infty}(c_n)^{x^n}\cdot\prod_{n=0}^{\infty}\prod_{i=0}^{\infty}\left[(a_n)^{\left(\sum_{j=0}^{i}\frac{k_{i-j,j}}{n+j+1}\right)x^{n+i+1}}\right]$$

$$\prod_{n=0}^{\infty}(a_n)^{x^n}=\prod_{n=0}^{\infty}(c_n)^{x^n}\cdot\prod_{n=0}^{\infty}\prod_{i=0}^{\infty}\left[\left(B_{n,i}\right)^{x^{n+i+1}}\right]$$

$$\prod_{n=0}^{\infty}(a_n)^{x^n}=\prod_{n=0}^{\infty}(c_n)^{x^n}\cdot\prod_{n=0}^{\infty}\left[\prod_{i=0}^{n}\left(B_{n-i,i}\right)^{x^{n+1}}\right]$$

$$\prod_{n=0}^{\infty}(a_n)^{x^n}=\prod_{n=0}^{\infty}(c_n)^{x^n}\cdot\prod_{n=0}^{\infty}\left[\prod_{i=0}^{n}\left(B_{n-i,i}\right)\right]^{x^{n+1}}$$

$$\prod_{n=0}^{\infty}(a_n)^{x^n}=\prod_{n=0}^{\infty}(c_n)^{x^n}\cdot\prod_{n=1}^{\infty}\left[\prod_{i=0}^{n-1}\left(B_{n-i-1,i}\right)\right]^{x^{n}}$$

$$\prod_{n=0}^{\infty}(a_n)^{x^n}=c_0\cdot\prod_{n=1}^{\infty}\left[c_n\cdot\prod_{i=0}^{n-1}\left(B_{n-i-1,i}\right)\right]^{x^{n}}$$

$$a_0\cdot\prod_{n=1}^{\infty}(a_n)^{x^n}=c_0\cdot\prod_{n=1}^{\infty}\left[c_n\cdot\prod_{i=0}^{n-1}\left(B_{i,n-i-1}\right)\right]^{x^{n}}$$

and thus

$$a_0=c_0,$$

$$a_n=c_n\cdot\prod_{i=0}^{n-1}\left(B_{i,n-i-1}\right),\qquad n\geq 1$$

At this point calculating $B_{i,n-i-1}$

$$B_{u,v}=(a_u)^{\left(\sum_{j=0}^{v}\frac{k_{v-j,j}}{u+j+1}\right)}$$

$$B_{i,n-i-1}=(a_i)^{\left(\sum_{j=0}^{n-i-1}\frac{k_{n-i-1-j,j}}{i+j+1}\right)}$$

and using this we have

$$a_n=c_n\cdot\prod_{i=0}^{n-1}\left(B_{i,n-i-1}\right)$$

$$a_n=c_n\cdot\prod_{i=0}^{n-1}(a_i)^{\left(\sum_{j=0}^{n-i-1}\frac{k_{n-i-1-j,j}}{i+j+1}\right)},\qquad n\geq 1.$$

Hence the solution $\varphi(x)=\prod_{n=0}^{\infty}(a_n)^{x^n}$ has bases

$$\begin{cases}a_0=c_0,\\ a_n=c_n\cdot\prod_{i=0}^{n-1}(a_i)^{\sum_{j=0}^{n-1-i}\frac{k_{n-1-i-j,j}}{i+j+1}},\ n\geq 1.\end{cases}\qquad\blacksquare$$

Example 4.1. Suppose the MFIE

$$\varphi(x)=e^{-x^2-x+3}\cdot_*\int_0^x\varphi(t)^{dt}$$

is given. Solve this equation using the multiplicative power series method (MPSM).

Solution. Let $\varphi(x)=\prod_{n=0}^{\infty}(a_n)^{x^n}$ be the MPS of the unknown function $\varphi(x)$. Now we write the power series expansion of two sides of the given equation.

$$\varphi(x)=(e^3)^1\cdot(e^{-1})^x\cdot(e^{-1})^{x^2}\cdot_*\int_0^x\varphi(t)^{dt}$$

$$\prod_{n=0}^{\infty}(a_n)^{x^n}=e^3\cdot(e^{-1})^x\cdot(e^{-1})^{x^2}\cdot_*\int_0^x\left[\prod_{n=0}^{\infty}(a_n)^{t^n}\right]^{dt}$$

$$\prod_{n=0}^{\infty}(a_n)^{x^n}=(e^3)^1\cdot(e^{-1})^x\cdot(e^{-1})^{x^2}\cdot\prod_{n=0}^{\infty}{}_*\int_0^x\left[(a_n)^{t^n}\right]^{dt}$$

$$\prod_{n=0}^{\infty}(a_n)^{x^n}=(e^3)^1\cdot(e^{-1})^x\cdot(e^{-1})^{x^2}\cdot\prod_{n=0}^{\infty}(a_n)^{\frac{x^{n+1}}{n+1}}$$

$$a_0\cdot\prod_{n=1}^{\infty}(a_n)^{x^n}=(e^3)^1\cdot(e^{-1})^x\cdot(e^{-1})^{x^2}\cdot\prod_{n=1}^{\infty}(a_{n-1})^{\frac{x^n}{n}}$$

So we can write

$$\begin{cases}a_0=e^3, & a_1=e^{-1}\cdot a_0,\\ a_2=e^{-1}\cdot {a_1}^{1/2}, & a_n=(a_{n-1})^{1/n},\ n\geq 3\end{cases}$$

Thus, we have

$a_0=e^3;\ a_1=e^2;\ (a_n)=1,\ n\geq 2$

So we get the solution

$$\varphi(x)=\prod_{n=0}^{\infty}(a_n)^{x^n}=a_0\cdot(a_1)^x\cdot\prod_{n=2}^{\infty}1^{x^n}$$

$$\varphi(x)=e^{2x+3}.$$

5. MULTIPLICATIVE VOLTERRA-FREDHOLM INTEGRAL EQUAION (MVFIE)

5.1 Solution of the MVFIE with Kernels $K_1(x,t) = K_2(x,t) = 1$

Theorem 5.1. Suppose the MVFIE

$$\varphi(x) = f(x) \cdot * \int_0^x \varphi(t)^{dt} \cdot * \int_0^b \varphi(t)^{dt}$$

is given. Here $b \in \mathbb{R}^+ \backslash \{\ln 2\}$. Also, let $f(x) = \prod_{n=0}^{\infty}(c_n)^{x^n}$ be the multiplicative power series of the given function $f(x)$. Then the multiplicative power series solution

$$\varphi(x) = \prod_{n=0}^{\infty}(a_n)^{x^n}$$

has bases a_n, $n \geq 0$ which can be calculated by the below formulas.

$$a_0 = \left\{c_0 \cdot (c_1)^{\frac{b^2}{2}} \cdot \prod_{n=2}^{\infty}\left[c_n\left(\prod_{k=1}^{n-1}(c_k)^{\frac{1}{(k+1)\cdot(k+2)\cdot\ldots\cdot n}}\right)\right]^{\frac{b^{n+1}}{n+1}}\right\}^{\frac{1}{(2-e^b)}},$$

$$a_1 = c_1 \cdot a_0,$$

$$a_n = c_n \cdot \left(\prod_{k=1}^{n-1}(c_k)^{\frac{1}{(k+1)\cdot(k+2)\cdot\ldots\cdot n}}\right) \cdot (a_0)^{\frac{1}{n!}}, \qquad n \geq 2$$

Proof.
Let

$$\varphi(x) = \prod_{n=0}^{\infty}(a_n)^{x^n}$$

$$f(x) = \prod_{n=0}^{\infty}(c_n)^{x^n}$$

be the multiplicative power series of $\varphi(x)$ and $f(x)$, respectively.

$$\varphi(x) = f(x) \cdot \left(*\int_0^x \varphi(t)^{dt}\right) \cdot \left(*\int_0^b \varphi(t)^{dt}\right)$$

$$\prod_{n=0}^{\infty}(a_n)^{x^n} = \prod_{n=0}^{\infty}(c_n)^{x^n} \cdot \left[*\int_0^x \left(\prod_{n=0}^{\infty}(a_n)^{t^n}\right)^{dt}\right] \cdot \left[*\int_0^b \left(\prod_{n=0}^{\infty}(a_n)^{t^n}\right)^{dt}\right]$$

$$\prod_{n=0}^{\infty}(a_n)^{x^n} = \left[\prod_{n=0}^{\infty}(c_n)^{x^n}\right] \cdot \prod_{n=0}^{\infty}\left[*\int_0^x \left([a_n]^{t^n}\right)^{dt}\right] \cdot \prod_{n=0}^{\infty}\left[*\int_0^b \left([a_n]^{t^n}\right)^{dt}\right]$$

$$= \left[\prod_{n=0}^{\infty}(c_n)^{x^n}\right] \cdot \left[\prod_{n=0}^{\infty}(a_n)^{\frac{x^{n+1}}{n+1}}\right] \cdot \left[\prod_{n=0}^{\infty}(a_n)^{\frac{b^{n+1}}{n+1}}\right]$$

$$= \left[\prod_{n=0}^{\infty}(c_n)^{x^n}\right] \cdot \left[\prod_{n=1}^{\infty}(a_{n-1})^{\frac{x^n}{n}}\right] \cdot \left[\prod_{n=0}^{\infty}(a_n)^{\frac{b^{n+1}}{n+1}}\right]$$

$$a_0 \cdot \prod_{n=1}^{\infty}(a_n)^{x^n} = \left\{c_0 \cdot \prod_{n=0}^{\infty}(a_n)^{\frac{b^{n+1}}{n+1}}\right\} \cdot \prod_{n=1}^{\infty}\left[c_n \cdot (a_{n-1})^{\frac{1}{n}}\right]^{x^n}$$

Thus we have

$$a_0 = c_0 \cdot \prod_{n=0}^{\infty}(a_n)^{\frac{b^{n+1}}{n+1}} \quad and \quad a_n = c_n \cdot (a_{n-1})^{\frac{1}{n}}, \quad n \geq 1$$

$$a_0 = c_0 \cdot (a_0)^b \cdot \prod_{n=1}^{\infty}(a_n)^{\frac{b^{n+1}}{n+1}}$$

$$(a_0)^{1-b} = c_0 \cdot \prod_{n=1}^{\infty}(a_n)^{\frac{b^{n+1}}{n+1}}$$

$$a_0 = \left\{c_0 \cdot \prod_{n=1}^{\infty}(a_n)^{\frac{b^{n+1}}{n+1}}\right\}^{\frac{1}{1-b}} \blacksquare$$

First, we'll write a_n with respect to a_0

$$a_n = c_n \cdot (a_{n-1})^{\frac{1}{n}}$$
$$a_1 = c_1 \cdot a_0$$
$$a_2 = c_2 \cdot (a_1)^{\frac{1}{2}} = c_2 \cdot (c_1 \cdot a_0)^{\frac{1}{2}}$$
$$a_2 = c_2 \cdot (c_1)^{\frac{1}{2}} \cdot (a_0)^{\frac{1}{2}}$$
$$a_3 = c_3 \cdot (c_2)^{\frac{1}{3}} \cdot (c_1)^{\frac{1}{3!}} \cdot (a_0)^{\frac{1}{3!}}$$

$\vdots$

So we have

$$a_1 = c_1 \cdot a_0$$
$$a_n = c_n \cdot \left(\prod_{k=1}^{n-1}(c_k)^{\frac{1}{(k+1)\cdot(k+2)\cdot\ldots\cdot n}}\right) \cdot (a_0)^{\frac{1}{n!}}, \qquad n \geq 2$$

Then let's find a_0

$$a_0 = \left\{\left(c_0 \cdot (a_1)^{\frac{b^2}{2}}\right) \cdot \prod_{n=2}^{\infty}(a_n)^{\frac{b^{n+1}}{n+1}}\right\}^{\frac{1}{1-b}}$$

$$a_0 = \left\{ \left(c_0 \cdot (c_1 \cdot a_0)^{\frac{b^2}{2}} \right) \cdot \prod_{n=2}^{\infty} \left[c_n \cdot \left(\prod_{k=1}^{n-1} (c_k)^{\frac{1}{(k+1)\cdot(k+2)\cdot\ldots\cdot n}} \right) \cdot (a_0)^{\frac{1}{n!}} \right]^{\frac{b^{n+1}}{n+1}} \right\}^{1/1-b}$$

$$a_0 = \left[c_0 \cdot (c_1)^{\frac{b^2}{2}} \cdot (a_0)^{\frac{b^2}{2}} \right]^{1/1-b} \cdot \prod_{n=2}^{\infty} \left[c_n \cdot \left(\prod_{k=1}^{n-1} (c_k)^{\frac{1}{(k+1)\cdot(k+2)\cdot\ldots\cdot n}} \right) \cdot (a_0)^{\frac{1}{n!}} \right]^{\frac{1}{(1-b)}\cdot\frac{b^{n+1}}{n+1}}$$

$$a_0 = \left[c_0 \cdot (c_1)^{\frac{b^2}{2}} \right]^{1/1-b} \cdot \prod_{n=2}^{\infty} \left[c_n \left(\prod_{k=1}^{n-1} (c_k)^{\frac{1}{(k+1)\cdot(k+2)\cdot\ldots\cdot n}} \right) \right]^{\frac{1}{(1-b)}\cdot\frac{b^{n+1}}{n+1}} \cdot \prod_{n=1}^{\infty} (a_0)^{\frac{1}{n!}\cdot\frac{1}{(1-b)}\cdot\frac{b^{n+1}}{n+1}}$$

$$a_0 = \left[c_0 \cdot (c_1)^{\frac{b^2}{2}} \right]^{1/1-b} \cdot \prod_{n=2}^{\infty} \left[c_n \left(\prod_{k=1}^{n-1} (c_k)^{\frac{1}{(k+1)\cdot(k+2)\cdot\ldots\cdot n}} \right) \right]^{\frac{1}{(1-b)}\cdot\frac{b^{n+1}}{n+1}} \cdot \prod_{n=1}^{\infty} (a_0)^{\frac{1}{(1-b)}\cdot\frac{b^{n+1}}{(n+1)!}}$$

$$a_0 = \left[c_0 \cdot (c_1)^{\frac{b^2}{2}} \right]^{1/1-b} \cdot \prod_{n=2}^{\infty} \left[c_n \left(\prod_{k=1}^{n-1} (c_k)^{\frac{1}{(k+1)\cdot(k+2)\cdot\ldots\cdot n}} \right) \right]^{\frac{1}{(1-b)}\cdot\frac{b^{n+1}}{n+1}} \cdot (a_0)^{\frac{1}{(1-b)}\sum_{n=1}^{\infty}\frac{b^{n+1}}{(n+1)!}}$$

$$a_0 = \left[c_0 \cdot (c_1)^{\frac{b^2}{2}} \right]^{1/1-b} \cdot \prod_{n=2}^{\infty} \left[c_n \left(\prod_{k=1}^{n-1} (c_k)^{\frac{1}{(k+1)\cdot(k+2)\cdot\ldots\cdot n}} \right) \right]^{\frac{1}{(1-b)}\cdot\frac{b^{n+1}}{n+1}} \cdot (a_0)^{\frac{\left(\sum_{n=0}^{\infty}\frac{b^n}{n!}\right)-1-b}{(1-b)}}$$

$$a_0 = \left[c_0 \cdot (c_1)^{\frac{b^2}{2}} \right]^{1/1-b} \cdot \prod_{n=2}^{\infty} \left[c_n \left(\prod_{k=1}^{n-1} (c_k)^{\frac{1}{(k+1)\cdot(k+2)\cdot\ldots\cdot n}} \right) \right]^{\frac{1}{(1-b)}\cdot\frac{b^{n+1}}{n+1}} \cdot (a_0)^{\frac{e^b-1-b}{(1-b)}}$$

and

$$a_0^{1-\frac{e^b-1-b}{(1-b)}} = \left[c_0 \cdot (c_1)^{\frac{b^2}{2}} \right]^{1/1-b} \cdot \prod_{n=2}^{\infty} \left[c_n \left(\prod_{k=1}^{n-1} (c_k)^{\frac{1}{(k+1)\cdot(k+2)\cdot\ldots\cdot n}} \right) \right]^{\frac{1}{(1-b)}\cdot\frac{b^{n+1}}{n+1}}$$

$$a_0^{\frac{2-e^b}{(1-b)}} = \left[c_0 \cdot (c_1)^{\frac{b^2}{2}} \right]^{1/1-b} \cdot \prod_{n=2}^{\infty} \left[c_n \left(\prod_{k=1}^{n-1} (c_k)^{\frac{1}{(k+1)\cdot(k+2)\cdot\ldots\cdot n}} \right) \right]^{\frac{1}{(1-b)}\cdot\frac{b^{n+1}}{n+1}}$$

$$a_0^{2-e^b} = \left[c_0 \cdot (c_1)^{\frac{b^2}{2}} \right] \cdot \prod_{n=2}^{\infty} \left[c_n \left(\prod_{k=1}^{n-1} (c_k)^{\frac{1}{(k+1)\cdot(k+2)\cdot\ldots\cdot n}} \right) \right]^{\frac{b^{n+1}}{n+1}}$$

$$a_0 = \left\{ c_0 (c_1)^{\frac{b^2}{2}} \cdot \prod_{n=2}^{\infty} \left[c_n \left(\prod_{k=1}^{n-1} (c_k)^{\frac{1}{(k+1)\cdot(k+2)\cdot\ldots\cdot n}} \right) \right]^{\frac{b^{n+1}}{n+1}} \right\}^{\frac{1}{(2-e^b)}}$$

5.2 Solution of MVFIE with Kernel $K_1(x,t) = K(x,t)$, $K_2(x,t) = 1$

Theorem 5.2. Suppose the MVFIE

$$\varphi(x) = f(x) \cdot_* \int_0^x \varphi(t)^{K(x,t)dt} \cdot_* \int_0^b \varphi(t)^{dt}$$

is given. Here $b > 0$. Also, let $f(x) = \prod_{n=0}^{\infty}(c_n)^{x^n}$ be the multiplicative power series of $f(x)$ and $K(x,t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} k_{i,j} \cdot x^i t^j$ be the power series of $K(x,t)$. Then the multiplicative power series solution

$$\varphi(x) = \prod_{n=0}^{\infty}(a_n)^{x^n}$$

has bases a_n, $n \geq 0$ which can be calculated by the below formulas.

$$a_n = c_n \cdot \prod_{\mu=0}^{n-1}(a_\mu)^{\sum_{\beta=0}^{n-1-\mu}\left(\frac{k_{n-1-\mu-\beta,\beta}}{\mu+\beta+1}\right)},$$

$$a_0 = \left[c_0 \cdot \prod_{n=1}^{\infty}(a_n)^{\frac{b^{n+1}}{n+1}} \right]^{\frac{1}{1-b}}.$$

Proof.

$$\varphi(x) = f(x) \cdot \left[* \int_0^x \varphi(t)^{K(x,t)dt} \right] \cdot \left[* \int_0^b \varphi(t)^{dt} \right]$$

$$\prod_{n=0}^{\infty} (a_n)^{x^n} = \prod_{n=0}^{\infty} (c_n)^{x^n} \cdot * \int_0^x \left[\prod_{n=0}^{\infty} (a_n)^{t^n} \right]^{\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} k_{i,j} \cdot x^i t^j dt} \cdot * \int_0^b \left[\prod_{n=0}^{\infty} (a_n)^{t^n} \right]^{dt}$$

$$\prod_{n=0}^{\infty} (a_n)^{x^n} = \prod_{n=0}^{\infty} (c_n)^{x^n} \cdot \prod_{n=0}^{\infty} (a_n)^{\int_0^x \sum_{i,j=0}^{\infty} k_{i,j} \cdot x^i t^j dt} \cdot \prod_{n=0}^{\infty} (a_n)^{\int_0^b t^n dt}$$

$$\prod_{n=0}^{\infty} (a_n)^{x^n} = \prod_{n=0}^{\infty} (c_n)^{x^n} \cdot \prod_{n=0}^{\infty} (a_n)^{\sum_{i,j=0}^{\infty} \left(k_{i,j} \cdot x^i \cdot \frac{t^{n+j+1}}{n+j+1} \right)_{t=0}^{x}} \cdot \prod_{n=0}^{\infty} (a_n)^{\frac{b^{n+1}}{n+1}}$$

$$\prod_{n=0}^{\infty} (a_n)^{x^n} = \prod_{n=0}^{\infty} (c_n)^{x^n} \cdot \prod_{n=0}^{\infty} (a_n)^{\sum_{i,j=0}^{\infty} \left(k_{i,j} \cdot \frac{x^{i+n+j+1}}{n+j+1} \right)} \cdot \prod_{n=0}^{\infty} (a_n)^{\frac{b^{n+1}}{n+1}}$$

$$\prod_{n=0}^{\infty} (a_n)^{x^n} = \prod_{n=0}^{\infty} (c_n)^{x^n} \cdot \prod_{n=1}^{\infty} \left(\prod_{\tau=0}^{n-1} B_{n-1-\tau,\tau} \right)^{x^n} \cdot \prod_{n=0}^{\infty} (a_n)^{\frac{b^{n+1}}{n+1}}$$

$$a_0 \cdot \prod_{n=1}^{\infty} (a_n)^{x^n} = \left[c_0 \cdot \prod_{n=0}^{\infty} (a_n)^{\frac{b^{n+1}}{n+1}} \right] \cdot \prod_{n=1}^{\infty} \left[c_n \cdot \prod_{\tau=0}^{n-1} B_{n-1-\tau,\tau} \right]^{x^n}$$

Thus we get

$$a_0 = c_0 \cdot \prod_{n=0}^{\infty} (a_n)^{\frac{b^{n+1}}{n+1}}$$

$$a_0 = \left[c_0 \cdot \prod_{n=1}^{\infty} (a_n)^{\frac{b^{n+1}}{n+1}} \right]^{\frac{1}{1-b}}.$$

And also we have

$$a_n = c_n \cdot \prod_{\tau=0}^{n-1} B_{n-1-\tau,\tau}, \qquad n \geq 1$$

$$a_n = c_n \cdot \prod_{\tau=0}^{n-1} (a_{n-1-\tau})^{\sum_{\beta=0}^{\tau} \left(\frac{k_{\tau-\beta,\beta}}{n-1-\tau+1+\beta} \right)}$$

$$a_n = c_n \cdot \prod_{\tau=0}^{n-1} (a_{n-1-\tau})^{\sum_{\beta=0}^{\tau} \left(\frac{k_{\tau-\beta,\beta}}{n-\tau+\beta} \right)}$$

Introducing a new indice $\mu = n - 1 - \tau$ we can write

$$a_n = c_n \cdot \prod_{\mu=0}^{n-1} (a_\mu)^{\sum_{\beta=0}^{n-1-\mu} \left(\frac{k_{n-1-\mu-\beta,\beta}}{\mu+\beta+1} \right)} \quad \blacksquare$$

Example 5.1. Suppose the MVFIE

$$\varphi(x) = e^{-x^2+2x-\frac{1}{4}} \cdot * \int_0^x \varphi(t)^{dt} \cdot * \int_0^{1/2} \varphi(t)^{dt}$$

is given. Solve this equation using the multiplicative power series method (MPSM)

Solution. Let $\varphi(x) = \prod_{n=0}^{\infty} (a_n)^{x^n}$ be the MPS of the function $\varphi(x)$. The function $f(x) = e^{-x^2+2x-\frac{1}{4}}$ has MPS given below

$$e^{-x^2+2x-\frac{1}{4}} = e^{\frac{-1}{4}} \cdot (e^2)^x \cdot (e^{-1})^{x^2} \cdot \prod_{k=3}^{\infty} 1^{x^k}$$

First, we write the MPS of the functions on either side of the given equation

$$\prod_{n=0}^{\infty} (a_n)^{x^n} = e^{\frac{-1}{4}} \cdot (e^2)^x \cdot (e^{-1})^{x^2} \cdot \prod_{k=3}^{\infty} 1^{x^k} \cdot * \int_0^x \left(\prod_{n=0}^{\infty} (a_n)^{t^n} \right)^{dt} \cdot * \int_0^{\frac{1}{2}} \left(\prod_{k=0}^{\infty} (a_k)^{t^k} \right)^{dt}$$

$$\prod_{n=0}^{\infty} (a_n)^{x^n} = e^{\frac{-1}{4}} \cdot (e^2)^x \cdot (e^{-1})^{x^2} \cdot \prod_{n=0}^{\infty} * \int_0^x \left[(a_n)^{t^n} \right]^{dt} \cdot \prod_{k=0}^{\infty} * \int_0^{\frac{1}{2}} \left[(a_k)^{t^k} \right]^{dt}$$

$$\prod_{n=0}^{\infty} (a_n)^{x^n} = e^{\frac{-1}{4}} \cdot (e^2)^x \cdot (e^{-1})^{x^2} \cdot \prod_{n=0}^{\infty} (a_n)^{\frac{x^{n+1}}{n+1} - 0} \cdot \prod_{k=0}^{\infty} (a_k)^{\left[\frac{t^{k+1}}{k+1} \right]_{t=0}^{1/2}}$$

$$\prod_{n=0}^{\infty} (a_n)^{x^n} = e^{\frac{-1}{4}} \cdot (e^2)^x \cdot (e^{-1})^{x^2} \cdot \prod_{n=0}^{\infty} (a_n)^{\frac{x^{n+1}}{n+1} - 0} \cdot \prod_{k=0}^{\infty} (a_k)^{\frac{\left(\frac{1}{2}\right)^{k+1}}{k+1}}$$

$$\prod_{n=0}^{\infty} (a_n)^{x^n} = \left[e^{\frac{-1}{4}} \cdot \prod_{k=0}^{\infty} (a_k)^{\frac{\left(\frac{1}{2}\right)^{k+1}}{k+1}} \right]^{x^0} \cdot (e^2)^x \cdot (e^{-1})^{x^2} \cdot \prod_{n=1}^{\infty} \left[(a_{n-1})^{\frac{1}{n}} \right]^{x^n}$$

$$\prod_{n=0}^{\infty}(a_n)^{x^n}=\left[e^{\frac{-1}{4}}\cdot\prod_{k=0}^{\infty}(a_k)^{\frac{\left(\frac{1}{2}\right)^{k+1}}{k+1}}\right]^{x^0}\cdot(e^2\cdot a_0)^x \cdot\left(e^{-1}\cdot {a_1}^{1/2}\right)^{x^2}\cdot\prod_{n=3}^{\infty}\left[(a_{n-1})^{\frac{1}{n}}\right]^{x^n}$$

So we have

$$a_0=e^{\frac{-1}{4}}\cdot\prod_{k=0}^{\infty}(a_k)^{\frac{\left(\frac{1}{2}\right)^{k+1}}{k+1}}$$
$$a_1=e^2\cdot a_0$$
$$a_2=e^{-1}\cdot {a_1}^{1/2}$$
$$a_2=(a_0)^{1/2}$$
$$a_n=(a_{n-1})^{1/n},\qquad n\geq 3$$

Second, we will give the value of $a_n,\ n\geq 2$ with respect to a_0.

$$a_3=(a_2)^{1/3}\Rightarrow a_3=\left({a_0}^{1/2}\right)^{\frac{1}{3}}\Rightarrow a_3=(a_0)^{\frac{1}{3!}}$$
$$a_4=(a_3)^{1/4}\Rightarrow a_4=\left({a_0}^{1/3!}\right)^{\frac{1}{4}}\Rightarrow a_4=(a_0)^{\frac{1}{4!}}$$
$$\vdots$$
$$a_n=(a_0)^{1/n!},\qquad n\geq 2.$$

Consequently, we will calculate the value of a_0

$$a_0=e^{\frac{-1}{4}}\cdot\prod_{k=0}^{\infty}(a_k)^{\frac{\left(\frac{1}{2}\right)^{k+1}}{k+1}}$$
$$a_0=e^{\frac{-1}{4}}\cdot(a_0)^{\frac{1}{2}}\cdot(a_1)^{\frac{\left(\frac{1}{2}\right)^{2}}{2}}\cdot\prod_{k=2}^{\infty}(a_k)^{\frac{\left(\frac{1}{2}\right)^{k+1}}{k+1}}$$
$$a_0=e^{\frac{-1}{4}}\cdot(a_0)^{\frac{1}{2}}\cdot(e^2\cdot a_0)^{\frac{1}{8}}\cdot\prod_{k=2}^{\infty}\left({a_0}^{\frac{1}{k!}}\right)^{\frac{\left(\frac{1}{2}\right)^{k+1}}{k+1}}$$

and

$$a_0=(a_0)^{\frac{5}{8}}\cdot\prod_{k=2}^{\infty}(a_0)^{\frac{\left(\frac{1}{2}\right)^{k+1}}{(k+1)!}}$$
$$a_0=(a_0)^{\frac{5}{8}}\cdot\prod_{k=3}^{\infty}(a_0)^{\frac{\left(\frac{1}{2}\right)^{k}}{(k)!}}$$
$$a_0=(a_0)^{\frac{5}{8}}\cdot(a_0)^{\sum_{k=3}^{\infty}\frac{\left(\frac{1}{2}\right)^{k}}{(k)!}}$$
$$a_0=(a_0)^{\frac{5}{8}}\cdot(a_0)^{\left(\frac{-13}{8}+e^{\frac{1}{2}}\right)}$$

$$a_0^{\left(1-\frac{5}{8}+\frac{13}{8}-e^{1/2}\right)}=1$$
$$(a_0)^{2-e^{1/2}}=1$$

Thus we find
$a_0=1.$
The multiplicative power series solution has bases

$$a_n=\begin{Bmatrix}e^2, & n=1\\ 1, & n\neq 1\end{Bmatrix}$$

Finally , we get the solution as

$$\varphi(x)=\prod_{n=0}^{\infty}(a_n)^{x^n}=1^1\cdot(e^2)^x\cdot 1^{x^2}\cdot 1^{x^3}\cdot\ldots\cdot 1^{x^n}\cdot\ldots$$
$$\varphi(x)=e^{2x}.$$

6. CONCLUSION

In this work, we give the definitions of MFIE of the second kind, MVIE of the second kind, and MVFIE of the second kind. Then, solutions of MIE via the MPSM are studied. We investigate solutions of MFIE with kernel equal to 1, MFIE with kernel depending only on the variable x, and MFIE with kernel depending only on the variable t. Consequently, we investigate the solution of MVIE. At last, we give the solutions of MVFIE with kernels equal to 1 and MVFIE with kernels $K_1(x,t)=K(x,t)$ and $K_2(x,t)=1$.

DECLARATION OF ETHICAL STANDARDS

The authors of this article declare that the materials and methods used in this study do not require ethical committee permission and/or legal-special permission.

AUTHORS' CONTRIBUTIONS

Arzu BAL: Did the research and wrote the manuscript.

Numan YALÇIN: Did the research and wrote the manuscript.

Mutlu DEDETÜRK: Did the research and wrote the manuscript.

CONFLICT OF INTEREST

There is no conflict of interest in this study.

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