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Wintgen Inequalities for Submanifolds of δ-Lorentzian Trans-Sasakian Space Form

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Abstract

The outline of this research article is that, δ-Lorentzian trans Sasakian manifolds with a semi-symmetric-metric connection (briefly say *SSM*) have been investigated. Indeed, we obtain the expressions for Riemannian curvature tensor $\bar{R}$, Ricci curvature tensors $\bar{Ric}$ and scalar curvature $\bar{r}$ of the δ-Lorentzian trans-Sasakian manifolds with a *SSM* connection. Mainly, we discuss the generalized Wintgen inequalities for submanifolds in δ-Lorentzian trans-Sasakian space form with a *SSM* connection. Furthermore, we examine the generalized Wintgen inequality for submanifolds of δ-Lorentzian trans-Sasakian space form.

Keywords: *δ-Lorentzian trans-Sasakian manifold; semi-symmetric metric connection; Wintgen inequalities.*
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1. Introduction

The conception of manifolds with indefinite metrics has fruitful applications in mathematical physics and general theory of relatively. The study of differentiable manifolds with Lorentzian metric is a natural and interesting topic in differential geometry and physics. In 1969, Takahashi [1] has introduced the notion of almost contact metric manifolds equipped with pseudo Riemannian metric. These indefinite almost contact metric manifolds and indefinite Sasakian manifolds are known as (ε)-almost contact metric manifolds. The concept of (ε)-Sasakian manifolds is initiated by Bejancu and Duggal [2].

Notion of Lorentzian para-contact manifolds were introduced by Matsumoto [3]. *CR*-Submanifolds of Trans Lorentzian para Sasakian manifolds were investigated by Gill and Dube [4]. In [5], Pujar and Khairnar discussed some axioms of the Lorentzian trans-Sasakian manifolds and studied the some basic results. Siddiqi et al.[6] also studied some properties of trans-Sasakian manifolds with indefinite metric which are closely related to this topic. Semi-Riemannian manifolds has the index 1 and the structure vector field ξ is always a time like. This idea inspired to the Tripathi and others [7] to reveal (ε)-almost para-contact structure, in this circumstances the vector filed ξ is space like or time like according as $(\varepsilon) = 1$ or $(\varepsilon) = -1$.

In fact, if M has a Lorentzian metric g, that is a symmetric non-degenerate $(0,2)$ tensor field of index 1, then M is known as a Lorentzian manifold. Since the Lorentzian metric is of index 1, Lorentzian manifold M has exhibits three kind of vector fields (1) spacelike vector fields (2) timelike vector fields and (3) lightlike vector fields. This is the major difference with the Riemannian case gives interesting properties on the Lorentzian manifold. A differentiable manifold M has a Lorentzian metric if and only if M has a 1- dimensional distribution. Hence odd dimensional manifold is able to have a Lorentzian metric. Motivated by the above researches and remarks Bhati [8] developed the conception of δ-Lorentzian trans-Sasakian manifolds. Follow by Siddiqi et al. [9] who studied the contact *CR*-submanifold of a δ-Lorentzian trans-Sasakian manifold.

In 1924, the central idea of semi-symmetric linear connection on a differentiable manifold was initiated by Friedmann and Schouten [10]. In 1930, Bartolotti [11] gave a geometrical meaning of such a connection. In 1932, Hayden [12] devolved the discourse of semi-symmetric metric connection. In 1970, Yano [13], further enhance a systematic study of the semi-symmetric metric connection in a Riemannian manifold and this was more extensively studied by several geometers such as for more details (see [14], [15], [16], [17], [18], [19]). Semi-symmetric connections (*SSM*) play an important role in the study of Riemannian manifolds. There are various physical problems involving the semi-symmetric metric connection. For example, During the mathematical congress in Moscow in 1934, one evening mathematicians invented the "Moscow displacement." The streets of Moscow are approximately straight lines through the Kremlin and concentric circles around it. If a person walks in the street always facing the Kremlin, then this displacement is semi-symmetric and metric [10].

Let ∇ be a linear connection in an n-dimensional differentiable manifold M. The torsion tensor $\mathscr{T}$ and the curvature tensor R of ∇ are given

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respectively by

$$\mathscr{T}(E,F) = \nabla_E F - \nabla_F E - [E,F],$$

$$R(E,F)G = \nabla_E \nabla_F G - \nabla_F \nabla_E Z - \nabla_{[E,F]} G.$$

The connection ∇ is said to be symmetric if its torsion tensor $\mathscr{T}$ vanishes, otherwise it is non-symmetric. The connection ∇ is said to be metric connection if there is a Riemannian metric g in M such that $\nabla g = 0$, otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if it is the Levi-Civita connection.

A linear connection ∇ is said to be semi-symmetric connection if its torsion tensor $\mathscr{T}$ is of the form

$$\mathscr{T}(E,F) = u(F)E - u(E)F,$$

where u is a 1-form.

In contrast, in the year 1979, for any surface M^2 in E^4, the following inequality involving the Gauss curvature $\mathscr{K}$, the normal curvature $\mathscr{K}^\perp$ and the squared mean curvature $||\mathscr{H}||^2$ was obtained by P. Wintgen [20]

$$||\mathscr{H}||^2 \geq \mathscr{K} + |\mathscr{K}^\perp|.$$

Moreover, equality holds in the above relation if and only if the ellipse of curvature of M^2 in E^4 is a circle. Later on, an extension for arbitrary codimension m in real space forms $\overline{M}^{m+2}(c)$ was given in [21]

$$||\mathscr{H}||^2 + c \geq \mathscr{K} + |\mathscr{K}^\perp|.$$

De Smet, Dillen, Verstraelen and Vrancken conjectured the generalized Wintgen inequality for submanifolds in real space form. This conjecture is also known as DDVV conjecture and it was also proved by Ge and Tang [22]. In the recent years, DDVV inequality has been obtained by distinct researchers for different classes of submanifolds in different ambient manifolds (see [23]-[28]). So, in this manuscript we study generalized Wintgen inequalities for submanifolds in δ-Lorentzian trans-Sasakian space form with a semi-symmetric metric connection.

2. Preliminaries

Let M be a δ-almost contact metric manifold equipped with δ-almost contact metric structure $(\phi, \xi, \eta, g, \delta)$ consisting of a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η and an indefinite metric g such that

$$\phi^2 = E + \eta(E)\xi, \quad \eta \circ \phi = 0, \quad \phi\xi = 0, \tag{2.1}$$

$$\eta(\xi) = -1, \tag{2.2}$$

$$g(\xi,\xi) = -\delta, \tag{2.3}$$

$$\eta(E) = \delta g(E,\xi), \tag{2.4}$$

$$g(\phi E, \phi F) = g(E,F) + \delta\eta(E)\eta(F), \tag{2.5}$$

for all $E, F \in M$, where δ is such that $\delta^2 = 1$ so that $\delta = \pm 1$. The above structure $(\phi, \xi, \eta, g, \delta)$ on M is called the δ Lorentzian structure on M. If $\delta = 1$ and this is usual Lorentzian structure [8] on M, the vector field ξ is the time like [3], that is M contains a time like vector field. In [29] Tanno provided the classification of connected almost contact metric manifold. For such a manifold the sectional curvature of the plane section containing ξ is constant, say c. Tanno proved that they can be divided into three classes. (1) is homogeneous normal contact Riemannian manifolds with $c > 0$. Other two classes can be observe in [29].

In the classification of almost Hermitian manifolds, there develop a class W_4 of Hermitian manifolds which are merely related to the conformal Kaehler manifolds [30]. The class $C_6 \oplus C_5$ consist the structure namely trans-Sasakian of type (α, β) [31] and this class completely explain the characteristics of trans-Sasakian structures.

An almost contact metric structure [32] on M is called a trans-Sasakian [33] if $(M \times \mathbb{R}, J, \mathbb{G})$ belongs to the class W_4, where J is the almost complex structure on $M \times \mathbb{R}$ defined by

$$J\left(E, \psi\frac{d}{dt}\right) = \left(\phi(E) - \psi\xi, \eta(E)\frac{d}{dt}\right),$$

for all vector fields E on M and smooth functions ψ on $M \times \mathbb{R}$ and $\mathbb{G}$ is the product metric on $M \times \mathbb{R}$. This may be expressed by the condition

$$(\nabla_E \phi)F = \alpha(g(E,F)\xi - \eta(F)E) + \beta(g(\phi E, F)\xi - \eta(F)\phi E), \tag{2.6}$$

for any vector fields E and F on M, ∇ denotes the Levi-Civita connection with respect to g, α and β are smooth functions on M. The existence of condition (2.3) is ensured by the above discussion.

With the above literature, we recall the δ-Lorentzian trans-Sasakian manifolds [8] as follows.

Definition 2.1. *A δ-Lorentzian manifold with structure (ϕ,ξ,η,g,δ) is said to be δ-Lorentzian trans-Sasakian manifold of type (α,β) if it satisfies the condition*

$$(\nabla_E\phi)F=\alpha(g(E,F)\xi-\delta\eta(F)E)+\beta(g(\phi E,F)\xi-\delta\eta(F)\phi E), \tag{2.7}$$

for any vector fields $E,F\in M$.

If $\delta=1$, then the δ-Lorentzian trans-Sasakian manifold is the usual Lorentzian trans-Sasakian manifold of type (α,β) [33]. δ-Lorentzian trans-Sasakian manifold of type $(0,0)$, $(0,\beta)$ $(\alpha,0)$ are the Lorentzian cosymplectic, Lorentzian β-Kenmotsu and Lorentzian α-Sasakian manifolds respectively. In particular if $\alpha=1$, $\beta=0$ and $\alpha=0$, $\beta=1$, the δ-Lorentzian trans-Sasakian manifolds reduces to δ-Lorentzian Sasakian and δ-Lorentzian Kenmotsu manifolds respectively.
Form (2.4), we have

$$\nabla_E\xi=\delta\{-\alpha\phi(E)-\beta(E+\eta(E)\xi\}, \tag{2.8}$$

and

$$(\nabla_E\eta)F=\alpha g(\phi E,F)+\beta[g(E,F)+\delta\eta(E)\eta(F)]. \tag{2.9}$$

In a δ-Lorentzian trans-Sasakian manifold M, we have the following relations:

$$\begin{aligned}
R(E,F)\xi &= (\alpha^2+\beta^2)[\eta(F)E-\eta(E)F]+2\alpha\beta[\eta(F)\phi E-\eta(E)\phi F]+\delta[(F\alpha)\phi E-(E\alpha)\phi F+(Y\beta)\phi^2E-(E\beta)\phi^2F], && (2.10)\\
R(\xi,F)E &= (\alpha^2+\beta^2)[\delta g(E,F)\xi-\eta(E)F]+\delta(E\alpha)\phi F+\delta g(\phi E,F)(grad\alpha)\\
&+ \delta(E\beta)(F+\eta(F)\xi)-\delta g(\phi F,\phi E))(grad\beta)+2\alpha\beta[\delta g(\phi E,F)\xi+\eta(E)\phi F], && (2.11)\\
\eta(R(E,F)G) &= \delta(\alpha^2+\beta^2)[\eta(E)g(F,G)-\eta(F)g(E,G)+2\delta\alpha\beta[-\eta(E)g(\phi F,G)+\eta(F)g(\phi E,G)]\\
&- [(F\alpha)g(\phi E,G)+(E\alpha)g(F,\phi G)]-(F\beta)g(\phi^2E,G)+(E\beta)g(\phi^2F,G)], && (2.12)\\
Ric(E,\xi) &= [((n-1)(\alpha^2+\beta^2)-(\xi\beta)]\eta(E)+\delta((\phi E)\alpha)+(n-2)\delta(E\beta), && (2.13)\\
Ric(\xi,\xi) &= (n-1)(\alpha^2+\beta^2)-\delta(n-1)(\xi\beta), && (2.14)
\end{aligned}$$

where R is Riemannian curvature tensor and Ric is the Ricci curvature tensor.
Furthermore in an δ-Lorentzian trans-Sasakian manifold , we have

$$\delta\phi(grad\alpha)=\delta(n-2)(grad\beta), \tag{2.15}$$

and

$$2\alpha\beta-\delta(\xi\alpha)=0. \tag{2.16}$$

The ξ-sectional curvature K_ξ of M is the sectional curvature of the plane spanned by ξ and a unit vector field E. From (2.11), we have

$$K_\xi=g(R(\xi,E),\xi,E)=(\alpha^2+\beta^2)-\delta(\xi\beta). \tag{2.17}$$

It follows from (2.16) that ξ-sectional curvature does not depend on E. From (2.11)

$$g(R(\xi,F)G,\xi)=[(\alpha^2+\beta^2)-\delta(\xi\beta)]g(F,G)+[(\xi\beta)-\delta(\alpha^2+\beta^2)]\eta(F)\eta(G)+[2\alpha\beta+\delta(\delta\alpha)]g(\phi F,G), \tag{2.18}$$

An affine connection $\bar\nabla$ in M is called semi-symmetric connection [10], it is torsion tensor satisfies the following relations

$$\bar{\mathscr{T}}(E,F)=\bar\nabla_EF-\bar\nabla_EF-[E,F], \tag{2.19}$$

and

$$\bar{\mathscr{T}}(E,F)=\eta(E)F-\eta(F)E. \tag{2.20}$$

Moreover, a semi-symmetric connection is called semi-symmetric metric connection (SSM) if

$$(\bar\nabla g)(E,F)=0. \tag{2.21}$$

If ∇ is metric connection and $\bar\nabla$ is the semi-symmetric metric connection with non-vanishing torsion tensor $\mathscr{T}$ in M, then we have

$$\mathscr{T}(E,F)=\eta(F)E-\eta(E)F, \tag{2.22}$$

$$\bar\nabla_EF-\nabla_EF=\frac{1}{2}[\mathscr{T}(E,F)+\mathscr{T}'(E,F)+\mathscr{T}'(E,F)], \tag{2.23}$$

where

$$g(\mathscr{T}(G,E),F)=g(\mathscr{T}'(E,F),G). \tag{2.24}$$

By using (2.4), (2.21) and (2.22), we get

$$g(\mathscr{T}'(E,F),G)=g(\eta(E)G-\eta(G)E,F),$$

$$g(\mathscr{T}^{'}(E,F),G)=\eta(E)g(G,F)-\delta g(E,F)g(\xi,G),$$

$$\mathscr{T}^{'}(E,F)=\eta(E)F-\delta g(E,F)\xi,$$

$$\mathscr{T}^{'}(F,E)=\eta(F)E-\delta g(E,F)\xi. \tag{2.25}$$

From (2.20), (2.21),(2.23) and (2.24), we get

$$\bar{\nabla}_E F=\nabla_E F+\eta(F)E-\delta g(E,F)\xi.$$

Let M be an n-dimensional δ-Lorentzian trans-Sasakian manifold and ∇ be the metric connection on M. The relation between the semi-symmetric metric connection $\bar{\nabla}$ and the metric connection ∇ on M is given by

$$\bar{\nabla}_E F=\nabla_E F+\eta(F)E-\delta g(E,F)\xi. \tag{2.26}$$

3. Characteristics of Curvature on δ-Lorentzian trans-Sasakian manifold with a *SSM* connection

Let M be an n-dimensional δ-Lorentzian trans-Sasakian manifold. The curvature tensor $\bar{R}$ of M with respect to the *SSM* connection $\bar{\nabla}$ is defined by

$$\bar{R}(E,F)G=\bar{\nabla}_E\bar{\nabla}_F G-\bar{\nabla}_F\bar{\nabla}_E G-\bar{\nabla}_{[E,F]}G. \tag{3.1}$$

By using (2.1), (2.4), (2.25) and (3.1), we get

$$\begin{aligned}\bar{R}(E,F)G &= R(E,F)G+(\delta)[g(E,G)F-g(F,G)E]+(\beta+\delta)[g(F,G)\eta(E)-g(E,G)\eta(F)]\xi\\ &- (\beta\delta-1)[\eta(F)E-\eta(E)F]\eta(G),+\alpha[g(\phi E,G)F-g(\phi F,G)\phi E-g(E,G)\phi F+g(F,G)\phi E],\end{aligned} \tag{3.2}$$

where

$$R(E,F)G=\nabla_E\nabla_F G-\nabla_F\nabla_E G-\nabla_{[E,F]}G$$

is the Riemannian curvature tensor of connection ∇.

Lemma 3.1. *Let M be n-dimensional δ-Lorentzian trans-Sasakian manifold with a SSM connection, then*

$$(\bar{\nabla}_E\phi)(F)=\alpha g(\phi E,F)\xi-\delta\eta(F)E+\beta(g(\phi E,F)\xi-(\delta\beta+\delta)\eta(F)\phi E, \tag{3.3}$$

$$\bar{\nabla}_E\xi=-(1+\delta\beta)E-(1+\delta\beta)\eta(E)\xi-\delta\alpha\phi E, \tag{3.4}$$

$$(\bar{\nabla}_E\eta)F=\alpha g(\phi E,F)\xi+(\beta+\delta)g(E,F)-(1+\beta\delta)\eta(E)\eta(F). \tag{3.5}$$

Proof. By the covariant differentiation of ϕF with respect to E, we have

$$\bar{\nabla}_E\phi F=(\bar{\nabla}_E\phi)+\phi(\bar{\nabla}_E F).$$

By using (2.1) and (2.25), we have

$$(\bar{\nabla}_E\phi)F=(\nabla_E\phi)F-\eta(F)\phi E.$$

In view of (2.8), the last equation gives

$$(\bar{\nabla}_E\phi)(F)=\alpha(g(\phi E,F)\xi-\delta\eta(F)E+\beta(g(\phi E,F)\xi-(\delta\beta+\delta)\eta(F)\phi E.$$

To prove (3.4), we replace $F=\xi$ in (2.25) and we have

$$\bar{\nabla}_E\xi=\nabla_E\xi+\eta(\xi)E-\delta g(E,\xi)\xi.$$

By using (2.2), (2.4) and (2.9), the above equation gives

$$\bar{\nabla}_E\xi=-(1+\delta\beta)E-(1+\delta\beta)\eta(E)\xi-\delta\alpha\phi E.$$

In order to prove (3.5), we differentiate $\eta(F)$ covariantly with respect to E and using (2.25), we have

$$\bar{\nabla}_E\eta(Y)=(\nabla_E\eta)F+g(E,F)-\eta(E)\eta(F).$$

Using (2.10) in above equation, we get

$$(\bar{\nabla}_E\eta)F=\alpha g(\phi E,F)\xi+(\beta+\delta)g(E,F)-(1+\beta\delta)\eta(E)\eta(F).$$

□

Lemma 3.2. *Let M be n-dimensional δ-Lorentzian trans-Sasakian manifold with a SSM connection, then*

$$\begin{aligned}\bar{R}(E,F)\xi &= (\alpha^2+\beta^2-\delta\beta)[\eta(E)F-\eta(F)E]+(2\alpha\beta+\delta\alpha)[\eta(F)\phi E-\eta(E)\phi F]\\ &+ \delta[(F\alpha)\phi E-(E\alpha)\phi F-(E\beta)\phi^2F+(F\beta)\phi^2E].\end{aligned} \tag{3.6}$$

Proof. By replacing $G=\xi$ in (3.2), we have

$$\begin{aligned}\bar{R}(E,F)\xi &= R(E,F)\xi+(\delta)[g(E,\xi)F-g(F,\xi)E]+(\beta+\delta)[g(F,\xi)\eta(E)-g(E,\xi)\eta(F)]\xi\\ &- (\beta\delta-1)[\eta(F)E-\eta(E)F]\eta(\xi)+\alpha[g(\phi E,\xi)F-g(\phi F,\xi)\phi E-g(E,\xi)\phi F+g(F,\xi)\phi E]\end{aligned} \tag{3.7}$$

In view of (2.2), (2.4) and (2.10), the above equation reduces to

$$\bar{R}(E,F)\xi=(\alpha^2+\beta^2-\delta\beta)[\eta(E)F-\eta(F)E]+(2\alpha\beta+\delta\alpha)[\eta(F)\phi E-\eta(E)\phi F]+\delta[(F\alpha)\phi E-(E\alpha)\phi F-(E\beta)\phi^2F+(F\beta)\phi^2E].$$

□

Remark 1. Replace $F=\xi$ and using (3.2), (2.11), (2.2) and (2.4), we find

$$\bar{R}(E,\xi)\xi=(\alpha^2+\beta^2-\delta\beta)[-E-\eta(E)F]+(2\alpha\beta+\delta\alpha+\delta(\xi\alpha))[\phi E+\delta(\xi\beta)\phi^2F]. \tag{3.8}$$

Remark 2. Now, again replace $E=\xi$ in (3.6), using (2.1), (2.2) and (2.4), we find

$$\bar{R}(\xi,F)\xi=(\alpha^2+\beta^2-\delta\beta)[-\eta(F)\xi-F]-(2\alpha\beta+\delta\alpha+\delta(\xi\alpha))[\phi F-\delta(\xi\beta)\phi^2F]. \tag{3.9}$$

Remark 3. Replace $F=E$ in (3.9), we get

$$\bar{R}(\xi,E)\xi=-(\alpha^2+\beta^2-\delta\beta)[-E-\eta(E)\xi].-(2\alpha\beta+\delta\alpha+\delta(\xi\alpha))[\phi E-\delta(\xi\beta)\phi^2E]. \tag{3.10}$$

From (3.9) and (3.10), we obtain

$$\bar{R}(E,\xi)\xi=-\bar{R}(\xi,E)\xi. \tag{3.11}$$

Now, using contraction on E in (3.2), we get

$$\bar{Ric}(F,G)=Ric(E,G)-[(\delta)(n-2)+\beta]g(F,G)-(\beta\delta-1)(n-2)\eta(Z)\eta(Y)-\alpha(n-2)g(\phi F,G), \tag{3.12}$$

where $\bar{Ric}$ and Ric are the Ricci tensors of the connections $\bar{\nabla}$ and ∇, respectively on M.
Putting $F=G=e_i$ and taking summation over i, $1\le i\le n-1$ in (3.12), using (2.14) and also the relations $r=Ric(e_i,e_i)=\sum_{i=1}^{n}\delta_i R(e_i,e_i,e_i,e_i)$, we get

$$\bar{r}=r-(n-1)[(\delta)(n-2)+2\beta], \tag{3.13}$$

where $\bar{r}$ and r are the scalar curvatures of the connections $\bar{\nabla}$ and ∇, respectively on M.
Now, we have the following lemmas.

Lemma 3.3. *Let M be n-dimensional δ-Lorentzian trans-Sasakian manifold with the SSM connection, then the scalar curvature is constant.*

Lemma 3.4. *Let M be n-dimensional δ-Lorentzian trans-Sasakian manifold with the SSM connection, then*

$$\bar{Ric}(\phi F,G)=-\delta(\phi^2F)\alpha-\delta(n-2)(\phi F)\beta-\alpha(n-2)g(\phi F,\phi G), \tag{3.14}$$

$$\bar{Ric}(F,\xi)=[(n-1)(\alpha^2+\beta^2-\delta(\xi\beta)-\delta\beta(n-1)]\eta(F)+\delta(n-2)(F\beta)+\delta(\phi F)\beta, \tag{3.15}$$

$$\bar{Ric}(\xi,\xi)=[(n-1)(\alpha^2+\beta^2-\delta(\xi\beta)-\delta\beta(n-1)]\eta(F). \tag{3.16}$$

Proof. Substituting $F=\phi Y$ in equation (3.12) and using (2.13) and (2.6), we have (3.14). Taking $F=\xi$ in (3.12) and using (2.13) we get (3.15). (3.16) follows from assuming $F=\xi$ in (3.15) we get (2.18). □

Now, we provide a non trivial example of δ-Lorentzian trans-Sasakian manifold with a *SSM* connection.

Example: We consider the three dimensional manifold $M=[(x,y,z)\in R^3 \mid z\neq 0]$, where (x,y,z) are the Cartesian coordinates in R^3. Choosing the vector fields

$$v_1=z\frac{\partial}{\partial x},\quad v_2=z\frac{\partial}{\partial y},\quad v_3=-z\frac{\partial}{\partial z},$$

which are linearly independent at each point of M. Let g be the Riemannian metric define by

$$g(v_1,v_3)=g(v_2,v_3)=g(v_2,v_2)=0,\qquad g(v_1,v_1)=g(v_2,v_2)=g(v_3,v_3)=\delta,$$

where $\delta = \pm 1$. Let η be the 1-form defined by $\eta(Z) = \varepsilon g(Z, v_3)$ for any vector field Z on TM. Let ϕ be the $(1,1)$ tensor field defined by $\phi(v_1) = -v_2, \quad \phi(v_2) = e_1, \quad \phi(v_3) = 0$. Then by the linearity property of ϕ and g, we have

$$\phi^2 G = Z + \eta(G)v_3, \quad \eta(v_3) = 1 \quad \text{and} \quad g(\phi G, \phi H) = g(G,H) - \delta\eta(G)\eta(H)$$

for any vector fields G,H on M.

Let ∇ be the Levi-Civita connection with respect to the metric g. Then we have

$$[v_1, v_2] = 0, \qquad [v_1, v_3] = \delta v_1, \qquad [v_2, v_3] = \delta v_2.$$

The Riemannian connection ∇ with respect to the metric g is given by

$$2g(\nabla_E F, G) = Eg(F,G) + Fg(G,E) - Gg(E,F) + g([E,F],G) - g([F,G],E) + g([G,E],F).$$

From above equation which is known as Koszul's formula, we have

$$\nabla_{v_1} v_3 = \delta v_1, \quad \nabla_{v_2} v_3 = \delta v_2, \quad \nabla_{v_3} v_3 = 0, \tag{3.17}$$

$$\nabla_{v_1} v_2 = 0, \quad \nabla_{v_2} v_2 = -\delta v_3, \quad \nabla_{v_3} v_2 = 0,$$

$$\nabla_{v_1} v_1 = -\delta v_3, \quad \nabla_{v_2} v_1 = 0, \quad \nabla_{v_3} v_1 = 0.$$

Using the above relations, for any vector field E on M, we have

$$\nabla_E \xi = \delta(E + \eta(X)\xi)$$

for $\xi \in v_3$, $\alpha = 0$ and $\beta = 1$. Hence the manifold M under consideration is an δ-Lorentzian trans-Sasakian of type $(0,1)$ manifold of dimension three.
Now, we consider this structure for semi-symmetric metric connection, from (2.29), we obtain:

$$\bar{\nabla}_{v_1} v_3 = (1+\delta)v_1, \quad \bar{\nabla}_{v_2} v_3 = (1+\delta)v_2, \quad \bar{\nabla}_{v_3} v_3 = 0, \tag{3.18}$$

$$\bar{\nabla}_{v_1} v_2 = 0, \quad \bar{\nabla}_{v_2} v_2 = -(1+\delta)v_3, \quad \bar{\nabla}_{v_3} v_2 = 0,$$

$$\bar{\nabla}_{v_1} v_1 = -(1+\delta)v_3, \quad \bar{\nabla}_{v_2} v_1 = 0, \quad \bar{\nabla}_{v_3} v_1 = 0.$$

Then the Riemannian, Ricci curvature tensor and scalar curvature tensor with respect to *SSM* connection are given by:

$$\bar{R}(v_1,v_2)v_2 = -(1+\delta)^2 v_1, \quad \bar{R}(v_1,v_3)v_3 = -\delta(1+\delta)v_2, \quad \bar{R}(v_2,v_1)v_1 = -(1+\delta)^2 v_2,$$

$$\bar{R}(v_2,v_3)v_3 = -\delta(1+\delta)v_2, \quad \bar{R}(v_3,v_1)v_1 = \delta(1+\delta)v_3, \quad \bar{R}(v_3,v_2)v_2 = -\delta(1+\delta)v_3,$$

$$\bar{Ric}(v_1,v_1) = \bar{Ric}(v_2,v_2) = -(1+\delta)(1+2\delta), \qquad \bar{Ric}(v_3,v_3,) = 2\delta(1+\delta).$$

$$\bar{r} = -2(1+\delta)^2.$$

4. Wintgen inequalities for submanifolds in δ-Lorentzian trans-Sasakian space form with a *SSM* connection

The present section is devoted to obtain generalized Wintgen inequalities for submanifolds in δ-Lorentzian trans-Sasakian space form with a *SSM* connection.
From equation (3.2), recall that for an n-dimensional δ-Lorentzian trans-Sasakian manifold M, the curvature tensor $\bar{R}$ with respect to the semi-symmetric metric connection $\bar{\nabla}$ is defined by

$$\begin{aligned} \bar{R}(E,F)G &= R(E,F)G + (\delta)[g(E,G)F - g(F,G)E] + (\beta+\delta)[g(F,G)\eta(E) - g(E,G)\eta(F)]\xi - (\beta\delta - 1)[\eta(F)E - \eta(E)F]\eta(G) \\ &+ \alpha[g(\phi E,G)F - g(\phi F,G)\phi E - g(E,G)\phi F + g(F,G)\phi E]. \end{aligned} \tag{4.1}$$

Let M' be m-dimensional submanifold of n-dimensional δ-Lorentzian trans-Sasakian manifold M with a *SSM* connection and induced metric g. Let ∇ and $\nabla^{\perp}$ represent the induced connections on the tangent bundle TM' and $TM'^{\perp}$ of M', respectively and h be the second fundamental form of M'. For all $X,Y \in \Gamma(TM')$ and $N \in \Gamma(T^{\perp}M')$, we recall the Gauss and Weingarten formulas by

$$\overline{\nabla}_E F = \nabla_E F + h(E,F),$$

and

$$\overline{\nabla}_E N = -A_N E + \nabla^{\perp}_E N,$$

where A_N denotes the shape operator of M' with respect to N. We also have the following relation

$$g(A_N E, F) = g(h(E,F),N)$$

for all $E,F \in \Gamma(TM')$ and $N \in \Gamma(T^{\perp}M')$.
The equation of Gauss is written as

$$\overline{R}(E,F,G,H) = R(E,F,G,H) - g(h(E,H),h(F,G)) + g(h(E,G),h(F,H)), \tag{4.2}$$

for all vector fields $E,F,G,H \in TM'$.
Assume that $\{e_1,\dots,e_m\}$ and $\{e_{m+1},\dots,e_n\}$ represent local orthonormal tangent frame of the tangent bundle TM' of M' and a local orthonormal normal frame of the normal bundle $T^{\perp}M'$ of M' in M. Recall the mean curvature vector $\mathscr{H}$ of M' by

$$\mathscr{H} = \sum_{i=1}^{m} \frac{1}{m} h(e_i,e_i) \tag{4.3}$$

and squared norm of second fundamental form by

$$||h||^2 = \sum_{i,j=1}^{m} g\big(h(e_i,e_j),h(e_i,e_j)\big)^2. \tag{4.4}$$

We write the scalar curvature τ at $p \in M'$ as

$$\tau = \sum_{1\le i<j\le m} R(e_i,e_j,e_j,e_i) \tag{4.5}$$

and define the normalized scalar curvature ρ of M' by

$$\rho = \frac{2\tau}{m(m-1)} = \frac{2}{m(m-1)} \sum_{1\le i<j\le m} \mathscr{K}(e_i \wedge e_j) \tag{4.6}$$

where $\mathscr{K}$ is the sectional curvature function on M'. We define the scalar normal curvature $\mathscr{K}_N$ in terms of the components of the second fundamental form by the following expression [23]

$$\mathscr{K}_N = \sum_{1\le i<j\le m} \sum_{1\le r<s\le n} \Big(\sum_{k=1}^{m} h^r_{jk}h^s_{ik} - h^r_{ik}h^s_{jk}\Big)^2. \tag{4.7}$$

Also recall the scalar normal curvature as [23]

$$\rho_N = \frac{2}{m(m-1)}\sqrt{\mathscr{K}_N}. \tag{4.8}$$

Now, we prove the generalized Wintgen inequality for submanifolds of δ-Lorentzian trans-Sasakian space form M with a *SSM* connection.

Theorem 4.1. *Let M' be an m-dimensional submanifold of a δ-Lorentzian trans-Sasakian manifold M with a SSM connection. Then*

$$\rho_N \quad \le \quad ||\mathscr{H}||^2 + 2(\alpha^2+\beta^2-\delta\xi\beta) - \frac{2}{m}[(\delta)(m-2)+2\beta] - 2\rho. \tag{4.9}$$

Proof. Assume that $\{e_1,\dots,e_m\}$ and $\{e_{m+1},\dots,e_n\}$ denotes the local orthonormal tangent frame and local orthonormal normal frame on M' respectively. Then, in view of Gauss equation, we have

$$\begin{aligned}\sum_{1\le i<j\le m} R(e_i,e_j,e_j,e_i) \quad &= \quad m(m-1)(\alpha^2+\beta^2-\delta\xi\beta) - (m-1)[(\delta)(m-2)+2\beta] \\ &+ \quad \sum_{r=m+1}^{n}\sum_{1\le i<j\le m}\Big[h^r_{ii}h^r_{jj} - (h^r_{ij})^2\Big].\end{aligned} \tag{4.10}$$

Also

$$2\tau = \sum_{1\le i<j\le m} R(e_i,e_j,e_j,e_i). \tag{4.11}$$

Using (4.10) and (4.11), we obtain

$$\begin{aligned}2\tau \quad &= \quad m(m-1)(\alpha^2+\beta^2-\delta\xi\beta) - (m-1)[(\delta)(m-2)+2\beta] \\ &+ \quad \sum_{r=m+1}^{n}\sum_{1\le i<j\le m}\Big[h^r_{ii}h^r_{jj} - (h^r_{ij})^2\Big].\end{aligned} \tag{4.12}$$

We also note that

$$\begin{aligned}m^2||\mathscr{H}||^2 \quad &= \quad \sum_{r=m+1}^{n}\Big(\sum_{i=1}^{m} h^r_{ii}\Big)^2 = \frac{1}{m-1}\sum_{r=m+1}^{n}\sum_{1\le i<j\le m}(h^r_{ii}-h^r_{jj})^2 \\ &\quad + \frac{2m}{m-1}\sum_{r=m+1}^{n}\sum_{1\le i<j\le m} h^r_{ii}h^r_{jj}.\end{aligned} \tag{4.13}$$

But, from [34] it is known

$$\begin{aligned}&\sum_{r=m+1}^{n}\sum_{1\le i<j\le m}(h^r_{ii}-h^r_{jj})^2 + 2m\sum_{r=m+1}^{n}\sum_{1\le i<j\le m}(h^r_{ij})^2 \ge \\ &\qquad 2m\Big[\sum_{m+1\le r<s\le n}\sum_{1\le i<j\le m}\Big(\sum_{k=1}^{m}(h^r_{jk}h^s_{ik} - h^r_{ik}h^s_{jk})\Big)^2\Big]^{\frac{1}{2}}\end{aligned} \tag{4.14}$$

Thanks to (4.13), (4.14) and (4.7), we have

$$m^2||\mathscr{H}||^2 - m^2\rho_N \geq \frac{2m}{m-1}\sum_{r=m+1}^{n}\sum_{1\leq i<j\leq m}[h_{ii}^r h_{jj}^r - (h_{ij}^r)^2]. \tag{4.15}$$

Hence, taking view of (4.8), (4.12) and (4.15), we find

$$\rho_N - ||\mathscr{H}||^2 \quad \leq \quad 2(\alpha^2+\beta^2-\delta\xi\beta) - \frac{2}{m}[(\delta)(m-2)+2\beta] - 2\rho$$

whereby proving the inequality (4.9). □

As a consequence of theorem 4.1, we give the following results:

Corollary 4.2. *Let M' be an m-dimensional submanifold of a usual Lorentzian trans Sasakian manifold M of type (α,β) with a SSM connection. Then*

$$\rho_N \quad \leq \quad ||\mathscr{H}||^2 + 2(\alpha^2+\beta^2-\xi\beta) - \frac{2}{m}[(m-2)+2\beta] - 2\rho. \tag{4.16}$$

Corollary 4.3. *Let M be a Lorentzian cosymplectic manifold (δ-Lorentzian trans Sasakian manifold of type $(0,0)$) with a SSM connection and M' be an m-dimensional submanifold of M. Then*

$$\rho_N \quad \leq \quad ||\mathscr{H}||^2 - \frac{2}{m}[(\delta)(m-2)] - 2\rho. \tag{4.17}$$

Corollary 4.4. *Let M be a Lorentzian β-Kenmotsu manifold (δ-Lorentzian trans-Sasakian manifold of type $(0,\beta)$) with a SSM connection and M' be an m-dimensional submanifold of M. Then*

$$\rho_N \quad \leq \quad ||\mathscr{H}||^2 + 2(\beta^2-\delta\xi\beta) - \frac{2}{m}[(\delta)(m-2)+2\beta] - 2\rho. \tag{4.18}$$

Corollary 4.5. *Let M be a Lorentzian α-Sasakian manifold (δ-Lorentzian trans-Sasakian manifold of type $(\alpha,0)$) with a SSM connection and M' be an m-dimensional submanifold of M. Then*

$$\rho_N \quad \leq \quad ||\mathscr{H}||^2 + 2\alpha^2 - \frac{2}{m}[(\delta)(m-2)] - 2\rho. \tag{4.19}$$

Corollary 4.6. *Let M' be an m-dimensional submanifold of a δ-Lorentzian Sasakian manifold M with a SSM connection. Then*

$$\rho_N \quad \leq \quad ||\mathscr{H}||^2 + 2 - \frac{2}{m}[(\delta)(m-2)] - 2\rho. \tag{4.20}$$

Corollary 4.7. *Let M' be an m-dimensional submanifold of a δ-Lorentzian Kenmotsu manifolds M with a SSM connection. Then*

$$\rho_N \quad \leq \quad ||\mathscr{H}||^2 + 2(1-\delta\xi) - \frac{2}{m}[(\delta)(m-2)+2] - 2\rho. \tag{4.21}$$

Next, we derive the generalized Wintgen inequality for submanifolds of δ-Lorentzian trans-Sasakian space form.

Theorem 4.8. *For a m-dimensional submanifold M' of a δ-Lorentzian trans-Sasakian manifold M, we have*

$$\rho_N \quad \leq \quad ||\mathscr{H}||^2 + 2(\alpha^2+\beta^2-\delta\xi\beta) - 2\rho. \tag{4.22}$$

As an application of above theorem, we have the following results.

Corollary 4.9. *For a m-dimensional submanifold M' of a usual Lorentzian trans-Sasakian manifold M of type (α,β), we have*

$$\rho_N \quad \leq \quad ||\mathscr{H}||^2 + 2(\alpha^2+\beta^2-\xi\beta) - 2\rho. \tag{4.23}$$

Corollary 4.10. *Let M be a Lorentzian cosymplectic manifold (δ-Lorentzian trans Sasakian manifold of type $(0,0)$) and M' be an m-dimensional submanifold of M. Then*

$$\rho_N \quad \leq \quad ||\mathscr{H}||^2 - 2\rho. \tag{4.24}$$

Corollary 4.11. *Let M be a Lorentzian β-Kenmotsu manifold (δ-Lorentzian trans-Sasakian manifold of type $(0,\beta)$) and M' be an m-dimensional submanifold of M. Then*

$$\rho_N \quad \leq \quad ||\mathscr{H}||^2 + 2(\beta^2-\delta\xi\beta) - 2\rho. \tag{4.25}$$

Corollary 4.12. *Let M be a Lorentzian α-Sasakian manifold (δ-Lorentzian trans-Sasakian manifold of type $(\alpha,0)$) and M' be an m-dimensional submanifold of M. Then*

$$\rho_N \quad \leq \quad ||\mathscr{H}||^2 + 2\alpha^2 - 2\rho. \tag{4.26}$$

Corollary 4.13. *Let M' be an m-dimensional submanifold of a δ-Lorentzian Sasakian manifold M. Then*

$$\rho_N \quad \leq \quad ||\mathscr{H}||^2 + 2 - 2\rho. \tag{4.27}$$

Corollary 4.14. *Let M' be an m-dimensional submanifold of a δ-Lorentzian Kenmotsu manifolds M. Then*

$$\rho_N \quad \leq \quad ||\mathscr{H}||^2 + 2(1-\delta\xi) - 2\rho. \tag{4.28}$$

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