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Research Article

Transformation formulae for terminating balanced ${}_4F_3$-series and implications

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Abstract

A new transformation from terminating balanced ${}_4F_3$-series to ${}_3F_2$-series is proved that contains a few known summation formulae as special cases. By means of Whipple's transformation, further closed form evaluations are given for terminating well–poised ${}_7F_6$-series as applications.

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1. Introduction and outline

Denote by $\mathbb{N}$ the set of natural numbers with $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. For an indeterminate x, define the shifted factorial by

$$(x)_0 = 1 \quad \text{and} \quad (x)_n = x(x+1)\cdots(x+n-1) \quad \text{with} \quad n \in \mathbb{N}.$$

For the sake of brevity, the quotient of shifted factorials will be abbreviated to

$$\begin{bmatrix} \alpha, & \beta, & \cdots, & \gamma \\ A, & B, & \cdots, & C \end{bmatrix}_n = \frac{(\alpha)_n(\beta)_n\cdots(\gamma)_n}{(A)_n(B)_n\cdots(C)_n}.$$

According to Bailey [2, §2.1], the classical hypergeometric series, for $p \in \mathbb{N}$ and an indeterminate z, is defined by

$${}_{1+p}F_p\left[\begin{matrix} a_0, & a_1, & \cdots, & a_p \\ & b_1, & \cdots, & b_p \end{matrix}\middle| z\right] = \sum_{k=0}^{\infty} \frac{(a_0)_k(a_1)_k\cdots(a_p)_k}{k!(b_1)_k\cdots(b_p)_k}\, z^k.$$

The hypergeoemtric series plays an important role in mathematics and physics (cf. [14, 15]). There exist numerous summation and transformation formulae of classical hypergeometric series (see [3, Chapter 8] and [6–10,12,13,16,17,20]). In this paper, we shall prove an unusual transformation formula (Theorem 2.1) that expresses a terminating balanced ${}_4F_3$-series in terms of a ${}_3F_2$-series. The two identities of balanced series due to Bailey [1] and Carlitz [4] as well as the terminating form of Whipple's formula for ${}_3F_2$-series are

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contained as special cases. As applications, further summation formulae for terminating well–poised ${}_7F_6$-series will be presented by making use of the Whipple transformation.

In order to assure the accuracy, all the displayed equations are tested by appropriately devised *Mathematica* commands.

2. Main results and proofs

Although there are many hypergeometric series transformations in the literature, the following one does not seem to have previously appeared.

Theorem 2.1 (Balanced series transformation: $n \in \mathbb{N}_0$)**.**

$$
{}_4F_3\left[\begin{matrix}-n, a-c+n, \frac{c}{2}, \frac{c+1}{2}\\ 1+a-e, \frac{e}{2}, \frac{e+1}{2}\end{matrix}\Big|1\right] = \left[\begin{matrix}1+a-c-e, e-c\\ 1+a-e,\ e\end{matrix}\right]_n \times {}_3F_2\left[\begin{matrix}-n, a-c+n, c\\ c+e-a-n, e+n\end{matrix}\Big|1\right].
$$

This theorem contains the following two important known special cases.

- $a = c+e$: Bailey [1] (cf. Chu [5, Eq. 2.8])

$$
{}_4F_3\left[\begin{matrix}-n, e+n, \frac{c}{2}, \frac{c+1}{2}\\ 1+c, \frac{e}{2}, \frac{e+1}{2}\end{matrix}\Big|1\right] = \frac{(e-c)_n}{(e)_n}. \tag{2.1}
$$

In fact, the transformation in Theorem 2.1 reduces to

$$
{}_4F_3\left[\begin{matrix}-n, e+n, \frac{c}{2}, \frac{c+1}{2}\\ 1+c, \frac{e}{2}, \frac{e+1}{2}\end{matrix}\Big|1\right] = \left[\begin{matrix}1, e-c\\ 1+c,\ e\end{matrix}\right]_n \times {}_2F_1\left[\begin{matrix}c, -n\\ -n\end{matrix}\Big|1\right].
$$

Then (2.1) follows by making use of the Chu–Vandermonde formula:

$$
{}_2F_1\left[\begin{matrix}c, -n\\ -n\end{matrix}\Big|1\right] = \frac{(-c-n)_n}{(-n)_n} = \frac{(1+c)_n}{n!}.
$$

- $a = c+e-1$: Carlitz [4] (cf. Chu [7, Eq. 5.2b]):

$$
{}_4F_3\left[\begin{matrix}-n, e-1+n, \frac{c}{2}, \frac{c+1}{2}\\ c, \quad \frac{e}{2}, \frac{e+1}{2}\end{matrix}\Big|1\right] = \frac{e-1+n}{e-1+2n}\,\frac{(e-c)_n}{(e)_n}. \tag{2.2}
$$

In this case for the right member displayed in Theorem 2.1, the factorial $(1+a-c-e)_n$ becomes a zero factor, while in the ${}_3F_2$-series only the last term contains a zero denominator. Therefore, we can determine the following limit:

$$
\begin{aligned}
&{}_4F_3\left[\begin{matrix}-n, e-1+n, \frac{c}{2}, \frac{c+1}{2}\\ c, \quad \frac{e}{2}, \frac{e+1}{2}\end{matrix}\Big|1\right]\\
&= \lim_{a\to c+e-1}\left[\begin{matrix}1+a-c-e, e-c\\ 1+a-e,\ e\end{matrix}\right]_n {}_3F_2\left[\begin{matrix}-n, a-c+n, c\\ c+e-a-n, e+n\end{matrix}\Big|1\right]\\
&= \lim_{a\to c+e-1}\left[\begin{matrix}1+a-c-e, e-c\\ 1+a-e,\ e\end{matrix}\right]_n \left[\begin{matrix}-n, a-c+n, c\\ 1, c+e-a-n, e+n\end{matrix}\right]_n\\
&= \lim_{a\to c+e-1}\left[\begin{matrix}1+a-c-e, e-c\\ 1+a-e,\ e\end{matrix}\right]_n \left[\begin{matrix}a-c+n, c\\ 1+a-c-e, e+n\end{matrix}\right]_n\\
&= \left[\begin{matrix}e-c\\ e\end{matrix}\right]_n \left[\begin{matrix}e-1+n\\ e+n\end{matrix}\right]_n = \frac{e-1+n}{e-1+2n}\,\frac{(e-c)_n}{(e)_n}. \qquad \square
\end{aligned}
$$

Proof of Theorem 2.1. According to the Chu–Vandermonde formula (cf. Bailey [2, §1.3]), we have the equality

$$
{}_2F_1\left[\begin{matrix}-k, e-c\\ e+k\end{matrix}\Big|1\right] = \frac{(e)_k(c)_{2k}}{(c)_k(e)_{2k}}.
$$

By substitution, we can manipulate the double sum

$$
\begin{aligned}
{}_4F_3\left[\begin{matrix}-n, a-c+n, \frac{c}{2}, \frac{c+1}{2}\\ 1+a-e, \frac{e}{2}, \frac{e+1}{2}\end{matrix}\Big|1\right] &= \sum_{k=0}^{n}\frac{(c)_{2k}}{(e)_{2k}}\left[\begin{matrix}-n, a-c+n\\ 1,\ 1+a-e\end{matrix}\right]_k\\
&= \sum_{k=0}^{n}\frac{(c)_{k}}{(e)_{k}}\left[\begin{matrix}-n, a-c+n\\ 1,\ 1+a-e\end{matrix}\right]_k {}_2F_1\left[\begin{matrix}-k, e-c\\ e+k\end{matrix}\Big|1\right]\\
&= \sum_{k=0}^{n}\frac{(c)_{k}}{(e)_{k}}\left[\begin{matrix}-n, a-c+n\\ 1,\ 1+a-e\end{matrix}\right]_k \sum_{i=0}^{k}\left[\begin{matrix}-k, e-c\\ 1,\ e+k\end{matrix}\right]_i\\
&= \sum_{i=0}^{n}\frac{(-1)^i}{(e)_{2i}}\left[\begin{matrix}-n, a-c+n, c, e-c\\ 1,\ 1+a-e\end{matrix}\right]_i\\
&\times {}_3F_2\left[\begin{matrix}i-n, a-c+n+i, c+i\\ 1+a-e+i, e+2i\end{matrix}\Big|1\right].
\end{aligned}
$$

Evaluating the last ${}_3F_2$-series by the Pfaff–Saalschutz theorem (cf. Bailey [2, §2.2])

$$
{}_3F_2\left[\begin{matrix}i-n, a-c+n+i, c+i\\ 1+a-e+i, e+2i\end{matrix}\Big|1\right] = \left[\begin{matrix}1+a-c-e, e-c+i\\ 1+a-e+i, e+2i\end{matrix}\right]_{n-i}
$$

and then simplifying the resulting expression, we confirm the transformation formula stated in Theorem 2.1. □

For the terminating balanced ${}_4F_3$-series with its parameters subject to the condition $1+a+c+e = b+d+\lambda+n$, there is a useful transformation (cf. Bailey [2, §7.2]):

$$
{}_4F_3\left[\begin{matrix}-n, a, c, e\\ b, d, \lambda\end{matrix}\Big|1\right] = \left[\begin{matrix}b-a, d-a\\ b,\ d\end{matrix}\right]_n {}_4F_3\left[\begin{matrix}-n,\ a,\ \lambda-c,\ \lambda-e\\ \lambda, 1+a-b-n, 1+a-d-n\end{matrix}\Big|1\right].
$$

Making the replacement $b \to 1+a+c+e-d-\lambda-n$ and then letting $e \to \infty$, we recover the following transformation:

$$
{}_3F_2\left[\begin{matrix}-n, a, c\\ b, d\end{matrix}\Big|1\right] = \left[\begin{matrix}d-a\\ d\end{matrix}\right]_n {}_3F_2\left[\begin{matrix}-n,\ a,\ b-c\\ b, 1+a-d-n\end{matrix}\Big|1\right]. \tag{2.3}
$$

Under transformation (2.3), the ${}_3F_2$-series in Theorem 2.1 can further be expressed in two different ${}_3F_2$-series, that are recored as follows.

Corollary 2.2 (Balanced series transformation: $n \in \mathbb{N}_0$).

$$
{}_4F_3\left[\begin{matrix}-n, a-c+n, \frac{c}{2}, \frac{c+1}{2}\\ 1+a-e, \frac{e}{2}, \frac{e+1}{2}\end{matrix}\Big|1\right] = \frac{(e-c)_n}{(e)_n} {}_3F_2\left[\begin{matrix}-n, c, c+e-a\\ e+n, 1+a-e\end{matrix}\Big|1\right].
$$

Corollary 2.3 (Balanced series transformation: $n \in \mathbb{N}_0$).

$$
\begin{aligned}
{}_4F_3\left[\begin{matrix}-n, a-c+n, \frac{c}{2}, \frac{c+1}{2}\\ 1+a-e, \frac{e}{2}, \frac{e+1}{2}\end{matrix}\Big|1\right] &= \left[\begin{matrix}1+2a-2c-e+n, e-c\\ 1+a-e,\ e\end{matrix}\right]_n\\
&\times {}_3F_2\left[\begin{matrix}-n, a-c+n, e-c+n\\ e+n, 1+2a-2c-e+n\end{matrix}\Big|1\right].
\end{aligned}
$$

When $a \to a+c-n$ and $e = 1+c$, the last corollary implies the following terminating series identity due to Whipple [18]:

$$
{}_3F_2\left[\begin{matrix}-n, 1+n, a\\ 1+c+n, 2a-c-n\end{matrix}\Big|1\right] = \left[\begin{matrix}1+c,\ a-\frac{3c}{2}\\ 2a-3c+n, 1+\frac{c}{2}\end{matrix}\right]_n. \tag{2.4}
$$

This is done by expressing the ${}_3F_2$-series on the right of Corollary 2.3 as

$$
{}_3F_2\left[\begin{matrix}-n, 1+n, a\\ 1+c+n, 2a-c-n\end{matrix}\Big|1\right] = \left[\begin{matrix}1+c, a-c\\ 1, 2a-3c+n\end{matrix}\right]_n {}_3F_2\left[\begin{matrix}-n, a-c+n,\ \frac{c}{2}\\ a-c,\ \frac{2+c}{2}\end{matrix}\Big|1\right]
$$

and then making use of the Pfaff–Saalschütz formula (cf. Bailey [2, §2.2])

$$ {}_3F_2\left[\begin{matrix}-n, a-c+n, \frac{c}{2}\\ a-c, \frac{2+c}{2}\end{matrix}\Big|1\right] = \left[\begin{matrix}a-\frac{3c}{2}, -n\\ a-c, -\frac{c}{2}-n\end{matrix}\right]_n = \left[\begin{matrix}a-\frac{3c}{2}, 1\\ a-c, 1+\frac{c}{2}\end{matrix}\right]_n. \qquad \square $$

3. Evaluations for terminating well–poised series

Between balanced series and well–poised series, there is an important transformation formula discovered by Whipple [19] (cf. Bailey [2, §4.3])

$$ \begin{aligned} W_n(a;b,c,d,e) &:= {}_7F_6\left[\begin{matrix}a, 1+\frac{a}{2}, b, c, d, e, -n\\ \frac{a}{2}, 1+a-b, 1+a-c, 1+a-d, 1+a-e, 1+a+n\end{matrix}\Big|1\right]\\ &= \frac{(1+a)_n(1+a-c-e)_n}{(1+a-c)_n(1+a-e)_n}{}_4F_3\left[\begin{matrix}-n, c, e, 1+a-b-d\\ 1+a-b, 1+a-d, c+e-a-n\end{matrix}\Big|1\right]. \end{aligned} \tag{3.1} $$

When $1+2a+n=b+c+d+e$, the above ${}_7F_6$-series is not only well–poised, but also 2-balanced. In this case, the following well–known formula due to Dougall [11] (cf. Bailey [2, §4.3]) holds:

$$ W_n(a;b,c,d,e) = \left[\begin{matrix}1+a, 1+a-b-c, 1+a-b-d, 1+a-c-d\\ 1+a-b, 1+a-c, 1+a-d, 1+a-b-c-d\end{matrix}\right]_n. $$

Observe that $W_n(a;b,c,d,e)$ is symmetric with respect to $\{b,c,d,e\}$. By combining Whipple's transformation (3.1) with (2.1), (2.2) and the Pfaff–Saalschütz formula, we can deduce further summation formulae for ${}_7F_6$-series in the sequel. Some of them can be found in [6,7,12]. The informed reader will notice that these evaluations for ${}_7F_6$-series are not particular cases of Dougall's one since they are not 2-balanced.

3.1. Well–poised ${}_7F_6$-series (I)

By choosing properly five parameters $\{a,b,c,d,e\}$ in Whipple's transformation and then applying (2.1), we find the following four well–poised series identities:

(3.1A)
$$ \begin{aligned} &W_n\Big(-\frac{1}{2}-n; \frac{c}{2}, \frac{c+1}{2}, \frac{1-e}{2}-n, -\frac{e}{2}-n\Big)\\ &= \left[\begin{matrix}\frac{1}{2}-n, -c-n\\ \frac{1-c}{2}-n, \frac{-c}{2}-n\end{matrix}\right]_n \times {}_4F_3\left[\begin{matrix}-n, e+n, \frac{c}{2}, \frac{c+1}{2}\\ 1+c, \frac{e}{2}, \frac{e+1}{2}\end{matrix}\Big|1\right]\\ &= \left[\begin{matrix}\frac{1}{2}, 1+c\\ \frac{1+c}{2}, \frac{2+c}{2}\end{matrix}\right]_n \frac{(e-c)_n}{(e)_n} = \left[\begin{matrix}\frac{1}{2}, 1+c, e-c\\ e, \frac{1+c}{2}, \frac{2+c}{2}\end{matrix}\right]_n. \end{aligned} $$

(3.1B)
$$ \begin{aligned} &W_n\Big(c-n-\frac{e}{2}; \frac{c}{2}, \frac{1+c}{2}, -\frac{e}{2}-n, 1+c-e-n\Big)\\ &= \left[\begin{matrix}\frac{1+e}{2}, \frac{e}{2}-c\\ \frac{e-c}{2}, \frac{1+e-c}{2}\end{matrix}\right]_n \times {}_4F_3\left[\begin{matrix}-n, e+n, \frac{c}{2}, \frac{c+1}{2}\\ 1+c, \frac{e}{2}, \frac{e+1}{2}\end{matrix}\Big|1\right]\\ &= \left[\begin{matrix}\frac{1+e}{2}, \frac{e}{2}-c\\ \frac{e-c}{2}, \frac{1+e-c}{2}\end{matrix}\right]_n \frac{(e-c)_n}{(e)_n} = \left[\begin{matrix}e-c, \frac{1+e}{2}, \frac{e}{2}-c\\ e, \frac{e-c}{2}, \frac{1+e-c}{2}\end{matrix}\right]_n. \end{aligned} $$

(3.1C)
$$ \begin{aligned} &W_n\Big(c-n+\frac{1-e}{2}; \frac{c}{2}, \frac{1+c}{2}, \frac{1-e}{2}-n, 1+c-e-n\Big)\\ &= \left[\begin{matrix}\frac{e}{2}, \frac{e-1}{2}-c\\ \frac{e-c}{2}, \frac{e-c-1}{2}\end{matrix}\right]_n \times {}_4F_3\left[\begin{matrix}-n, e+n, \frac{c}{2}, \frac{c+1}{2}\\ 1+c, \frac{e}{2}, \frac{e+1}{2}\end{matrix}\Big|1\right]\\ &= \left[\begin{matrix}\frac{e}{2}, \frac{e-1}{2}-c\\ \frac{e-c}{2}, \frac{e-c-1}{2}\end{matrix}\right]_n \frac{(e-c)_n}{(e)_n} = \left[\begin{matrix}e-c, \frac{e}{2}, \frac{e-1}{2}-c\\ e, \frac{e-c}{2}, \frac{e-c-1}{2}\end{matrix}\right]_n. \end{aligned} $$

(3.1D) $$W_n\Big(\frac{c+e}{2};\frac{1+c}{2},\frac{2+c}{2},\frac{e-c}{2},e+n\Big)$$
$$=\begin{bmatrix}\frac{2+c+e}{2},\frac{1-e}{2}-n\\ \frac{1+e}{2},\frac{2+c-e}{2}-n\end{bmatrix}_n\times{}_4F_3\begin{bmatrix}-n,e+n,\frac{c}{2},\frac{c+1}{2}\\ 1+c,\frac{e}{2},\frac{e+1}{2}\end{bmatrix}1\Big]$$
$$=\begin{bmatrix}\frac{1+e}{2},\frac{2+c+e}{2}\\ \frac{1+e}{2},\frac{e-c}{2}\end{bmatrix}_n\frac{(e-c)_n}{(e)_n}=\begin{bmatrix}e-c,1+\frac{c+e}{2}\\ e,\ \frac{e-c}{2}\end{bmatrix}_n.$$

3.2. Well–poised $_7F_6$-series (II)

By choosing properly five parameters $\{a,b,c,d,e\}$ in Whipple's transformation and then applying (2.2), we derive the following four identities:

(3.2A) $$W_n\Big(\frac12-n;\frac{c}{2},\frac{c+1}{2},\frac{2-e}{2}-n,\frac{3-e}{2}-n\Big)$$
$$=\begin{bmatrix}\frac32-n,\ 1-c-n\\ 1-n-\frac{c}{2},\frac{3-c}{2}-n\end{bmatrix}_n\times{}_4F_3\left[\begin{matrix}-n,e-1+n,\frac{c}{2},\frac{c+1}{2}\\ c,\qquad \frac{e}{2},\frac{e+1}{2}\end{matrix}\middle|1\right]$$
$$=\begin{bmatrix}c,-\frac12\\ \frac{c}{2},\frac{c-1}{2}\end{bmatrix}_n\frac{e-1+n}{e-1+2n}\frac{(e-c)_n}{(e)_n}=\begin{bmatrix}e-c,c,-\frac12,\frac{e-1}{2}\\ e-1,\frac{c}{2},\frac{c-1}{2},\frac{e+1}{2}\end{bmatrix}_n.$$

(3.2B) $$W_n\Big(c-n-\frac{e}{2};\frac{c}{2},\frac{1+c}{2},1-n-\frac{e}{2},1+c-e-n\Big)$$
$$=\begin{bmatrix}\frac{1+e}{2},\frac{e}{2}-c\\ \frac{e-c}{2},\frac{1+e-c}{2}\end{bmatrix}_n\times{}_4F_3\left[\begin{matrix}-n,e-1+n,\frac{c}{2},\frac{c+1}{2}\\ c,\qquad \frac{e}{2},\frac{e+1}{2}\end{matrix}\middle|1\right]$$
$$=\begin{bmatrix}\frac{1+e}{2},\frac{e}{2}-c\\ \frac{e-c}{2},\frac{1+e-c}{2}\end{bmatrix}_n\frac{e-1+n}{e-1+2n}\frac{(e-c)_n}{(e)_n}=\begin{bmatrix}e-c,\frac{e}{2}-c,\frac{e-1}{2}\\ e-1,\frac{1+e-c}{2},\frac{e-c}{2}\end{bmatrix}_n.$$

(3.2C) $$W_n\Big(c-n+\frac{1-e}{2};\frac{c}{2},\frac{1+c}{2},\frac{3-e}{2}-n,1+c-e-n\Big)$$
$$=\begin{bmatrix}\frac{e}{2},\frac{e-1}{2}-c\\ \frac{e-c}{2},\frac{e-c-1}{2}\end{bmatrix}_n\times{}_4F_3\left[\begin{matrix}-n,e-1+n,\frac{c}{2},\frac{c+1}{2}\\ c,\qquad \frac{e}{2},\frac{e+1}{2}\end{matrix}\middle|1\right]$$
$$=\begin{bmatrix}\frac{e}{2},\frac{e-1}{2}-c\\ \frac{e-c}{2},\frac{e-c-1}{2}\end{bmatrix}_n\frac{e-1+n}{e-1+2n}\frac{(e-c)_n}{(e)_n}=\begin{bmatrix}e-c,\frac{e}{2},\frac{e-1}{2},\frac{e-1}{2}-c\\ e-1,\frac{e+1}{2},\frac{e-c}{2},\frac{e-c-1}{2}\end{bmatrix}_n.$$

(3.2D) $$W_n\Big(\frac{c+e-1}{2};\frac{c}{2},\frac{1+c}{2},\frac{1-c+e}{2},e+n-1\Big)$$
$$=\begin{bmatrix}\frac{1+c+e}{2},\ 1-n-\frac{e}{2}\\ \frac{e}{2},\frac{3+c-e}{2}-n\end{bmatrix}_n\times{}_4F_3\left[\begin{matrix}-n,e-1+n,\frac{c}{2},\frac{c+1}{2}\\ c,\qquad \frac{e}{2},\frac{e+1}{2}\end{matrix}\middle|1\right]$$
$$=\begin{bmatrix}\frac{e}{2},\ \frac{1+c+e}{2}\\ \frac{e}{2},\frac{e-c-1}{2}\end{bmatrix}_n\frac{e-1+n}{e-1+2n}\frac{(e-c)_n}{(e)_n}=\begin{bmatrix}e-c,\frac{e-1}{2},\frac{1+c+e}{2}\\ e-1,\frac{e+1}{2},\frac{e-c-1}{2}\end{bmatrix}_n.$$

3.3. Well–poised $_7F_6$-series (III)

Finally by choosing properly five parameters $\{a,b,c,d,e\}$ in Whipple's transformation and then applying the Pfaff–Saalschütz formula (cf. Bailey [2, §2.2])

$${}_3F_2\left[\begin{matrix}-n,a-c,\ \frac{c}{2}\\ a-c-n,\ \frac{2+c}{2}\end{matrix}\middle|1\right]={}_4F_3\left[\begin{matrix}-n,a-c,\ \frac{c}{2},\frac{c+1}{2}\\ a-c-n,\frac{c+1}{2},\frac{c+2}{2}\end{matrix}\middle|1\right]=\begin{bmatrix}1,1-a+\frac{3c}{2}\\ 1-a+c,1+\frac{c}{2}\end{bmatrix}_n,$$

we establish the following four well–poised series identities:

$$(3.3\text{A})\quad W_n\Big(\frac{1}{2}-a+2c;\frac{c}{2},\frac{c+1}{2},\frac{1}{2}-a+\frac{3c}{2},1-a+\frac{3c}{2}\Big)$$
$$=\begin{bmatrix}1-a+c,\frac{3}{2}-a+2c\\ 1-a+\frac{3c}{2},\frac{3}{2}-a+\frac{3c}{2}\end{bmatrix}_n\times{}_3F_2\left[\begin{matrix}-n,a-c,\ \frac{c}{2}\\ a-c-n,\ \frac{2+c}{2}\end{matrix}\Big|1\right]$$
$$=\begin{bmatrix}1-a+c,\frac{3}{2}-a+2c\\ 1-a+\frac{3c}{2},\frac{3}{2}-a+\frac{3c}{2}\end{bmatrix}_n\begin{bmatrix}1,1-a+\frac{3c}{2}\\ 1-a+c,1+\frac{c}{2}\end{bmatrix}_n=\begin{bmatrix}1,\ \frac{3}{2}-a+2c\\ 1+\frac{c}{2},\frac{3}{2}-a+\frac{3c}{2}\end{bmatrix}_n.$$

$$(3.3\text{B})\quad W_n\Big(\frac{1+c}{2};\frac{1+c}{2},1,\frac{1}{2},a-c\Big)$$
$$=\begin{bmatrix}1-a+c,\frac{3+c}{2}\\ 1,\ \frac{3+3c}{2}-a\end{bmatrix}_n\times{}_3F_2\left[\begin{matrix}-n,a-c,\ \frac{c}{2}\\ a-c-n,\ \frac{2+c}{2}\end{matrix}\Big|1\right]$$
$$=\begin{bmatrix}1-a+c,\frac{3+c}{2}\\ 1,\ \frac{3+3c}{2}-a\end{bmatrix}_n\begin{bmatrix}1,1-a+\frac{3c}{2}\\ 1-a+c,1+\frac{c}{2}\end{bmatrix}_n=\begin{bmatrix}1-a+\frac{3c}{2},\frac{3+c}{2}\\ 1+\frac{c}{2},\frac{3+3c}{2}-a\end{bmatrix}_n.$$

$$(3.3\text{C})\quad W_n\Big(\frac{c-1}{2}-n;\frac{c}{2},\frac{1+c}{2},\frac{1+3c}{2}-a,-n\Big)$$
$$=\begin{bmatrix}1+\frac{c}{2},\frac{1-c}{2}\\ 1,\ \frac{1}{2}\end{bmatrix}_n\times{}_3F_2\left[\begin{matrix}-n,a-c,\ \frac{c}{2}\\ a-c-n,\ \frac{2+c}{2}\end{matrix}\Big|1\right]$$
$$=\begin{bmatrix}1+\frac{c}{2},\frac{1-c}{2}\\ 1,\ \frac{1}{2}\end{bmatrix}_n\begin{bmatrix}1,1-a+\frac{3c}{2}\\ 1-a+c,1+\frac{c}{2}\end{bmatrix}_n=\begin{bmatrix}\frac{1-c}{2},1-a+\frac{3c}{2}\\ \frac{1}{2},\ 1-a+c\end{bmatrix}_n.$$

$$(3.3\text{D})\quad W_n\Big(a-c-n-\frac{1}{2};\frac{1}{2},\frac{1+c}{2},a-n-\frac{3c}{2},a-c\Big)$$
$$=\begin{bmatrix}\frac{1}{2}-a+c,1+\frac{c}{2}\\ \frac{1}{2},1-a+\frac{3c}{2}\end{bmatrix}_n\times{}_3F_2\left[\begin{matrix}-n,a-c,\ \frac{c}{2}\\ a-c-n,\ \frac{2+c}{2}\end{matrix}\Big|1\right]$$
$$=\begin{bmatrix}\frac{1}{2}-a+c,1+\frac{c}{2}\\ \frac{1}{2},1-a+\frac{3c}{2}\end{bmatrix}_n\begin{bmatrix}1,1-a+\frac{3c}{2}\\ 1-a+c,1+\frac{c}{2}\end{bmatrix}_n=\begin{bmatrix}1,\ \frac{1}{2}-a+c\\ \frac{1}{2},\ 1-a+c\end{bmatrix}_n.$$

It should be pointed out that the last three ${}_7F_6$-series in "(3.3B), (3.3C), (3.3D)" are degenerated ones that can also be deduced directly from a formula for well–poised ${}_5F_4$-series (cf. Bailey [2, §4.3: Equation 3]). The remaining eleven evaluations of ${}_7F_6$-series in this section do not seem to have previously appeared in the literature.

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