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Approximately Near Rings in Proximal Relator Spaces

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Abstract

The motivation of this article is to define approximately near rings, some types of approximately near rings, approximately N-groups, approximately ideals, and approximately near rings of all descriptive approximately cosets. Moreover, some properties of these approximately algebraic structures are given. Furthermore, approximately near-ring homomorphisms are introduced and their some properties are investigated.

1. Introduction

Let X be a nonempty set and $\mathscr{R}_\delta$ be a set of proximity relations on X. Then $(X,\mathscr{R}_\delta)$ is called a proximal relator space. Efremovič proximity, descriptive proximity and Lodato proximity are different types of proximity relations [1]-[3]. Non-abstract points have locations and features. In proximal relator space, the sets consist of these points.
The aim of this work is to obtain algebraic structures in proximal relator spaces using descriptively upper approximations of the subsets of X. In 2017 and 2018, approximately semigroups and approximately ideals, approximately groups, approximately subgroups and approximately rings were introduced by İnan [4]-[7]. Approximately Γ-semigroups were also defined [8]. In these articles some examples of these approximately algebraic structures in digital images endowed with proximity relations were given as in this article. Approximately algebraic structures satisfy a framework for further applied areas such as image analysis or classification problems.
In 1983, Pilz introduced the near-rings as a generalization of rings. In near rings, the addition operation does not need to be commutative as only one distributive law is sufficient [9].
Essentially, the focus of this article is to define approximately near rings, some types of approximately near rings, approximately N-groups, approximately ideals and approximately near rings of all descriptive approximately cosets. Moreover, some properties of these approximately algebraic structures are given. Furthermore, approximately near ring homomorphisms are introduced and their some properties are investigated.

2. Preliminaries

Let X be a nonempty set and $\mathscr{R}$ be a family of relations on X. If $\mathscr{R}$ is a family of proximity relations on X, then $(X,\mathscr{R}_\delta)$ is called proximal relator space, where $\mathscr{R}_\delta$ contains proximity relations, for example Efremovič proximity δ_E [1], Lodato proximity $\delta_{\mathscr{L}}$ [2], Wallman proximity δ_ω or descriptive proximity δ_Φ [3, 10, 11].
Throughout this article, the Efremovič proximity [1] and the descriptive proximity relations are considered.
An Efremovič proximity δ is a relation on $P(X)$ that satisfies the conditions: For $I,J,K \subseteq X$

1^o $I \,\delta\, J \Rightarrow J \,\delta\, I$.

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2^o $I\ \delta\ J \Rightarrow I \neq \emptyset$ and $J \neq \emptyset$.
3^o $I \cap J \neq \emptyset \Rightarrow I\ \delta\ J$.
4^o $I\ \delta\ (J \cup K) \Leftrightarrow I\ \delta\ J$ or $I\ \delta\ K$.
5^o $\{x\}\ \delta\ \{y\} \Leftrightarrow x = y$.
6^o $I\ \underline{\delta}\ J \Rightarrow \exists E \subseteq X$ such that $I\ \underline{\delta}\ E$ and $E^c\ \underline{\delta}\ J$ (*Efremovič Axiom*).

Lodato proximity [2] swaps the *Efremovič Axiom* with:

$I\ \delta\ J$ and $\forall b \in J, \{b\}\ \delta\ K \Rightarrow I\ \delta\ K$ (*Lodato Axiom*).

Here, $I\ \delta\ J$ means that I is proximal to J. Also, $I\ \underline{\delta}\ J$ means that I is not proximal to J.
Let X be a set of non-abstract points which has a location and features [12, §3] in $(X, \mathscr{R}_{\delta_\Phi})$. Let $\Phi = \{\phi_1, \ldots, \phi_n\}$ be a set of probe functions that represents features of any $x \in X$.
A *probe function* $\phi_i : X \to \mathbb{R}$ represents features of a sample non-abstract point. Let $\Phi(x) = (\phi_1(x), \ldots, \phi_n(x)), (n \in \mathbb{N})$ be an object description denoting a feature vector of x, which is a description of each $x \in X$. After choosing a set of probe functions, one can obtain a descriptive proximity relation δ_Φ as follows:
[13] Let $I, J \subseteq X$.

$$Q(I) = \{\Phi(a) \mid a \in I\}$$

is a *set description* of $I \subseteq X$. And

$$I \underset{\Phi}{\cap} J = \{x \in I \cup J \mid \Phi(x) \in Q(I) \ \textit{and}\ \Phi(x) \in Q(J)\}\ .$$

is a descriptive intersection of I and J.
[10] If $Q(I) \cap Q(J) \neq \emptyset$, then I is called *descriptively proximal (near)* to J, denoted by $I \delta_\Phi J$.
Throughout the article, $\left(X, \mathscr{R}_{\delta_\Phi}\right)$ or shortly X is considered as descriptive proximal relator space, unless otherwise stated.
[14] Let X be a descriptive proximal relator space and $A \subseteq X$. Let $(A, \circ)$ and $(Q(A), \cdot)$ be groupoids. Consider the object description Φ by means of a function

$$\Phi : A \subseteq X \longrightarrow Q(A) \subset \mathbb{R}^n, x \mapsto \Phi(x), x \in A.$$

The object description Φ of A into $Q(A)$ is an *object descriptive homomorphism* if $\Phi(x \circ y) = \Phi(x) \cdot \Phi(y)$ for all $x, y \in A$.

Definition 2.1. *[5] Let $A \subseteq X$. A descriptively upper approximation of A is defined with*

$$\Phi^* A = \{x \in X \mid x \delta_\Phi A\}.$$

It is clear that $A \subseteq \Phi^* A$ for all $A \subseteq X$.

Lemma 2.2. *[5] Let I, J be subsets of X. Then*
(i) $Q(I \cap J) = Q(I) \cap Q(J)$,
(ii) $Q(I \cup J) = Q(I) \cup Q(J)$.

Definition 2.3. *[5] Let "$\cdot$" be a binary operation on X. $G \subseteq X$ is called an approximately groupoid if $x \cdot y \in \Phi^* G$ for all $x, y \in G$.*

Definition 2.4. *[4] Let "$\cdot$" be a binary operation on X. Then $G \subseteq X$ is called an approximately group if the following conditions are true:*

$(\mathscr{A}G_1)$ *$x \cdot y \in \Phi^* G$ for all $x, y \in G$,*
$(\mathscr{A}G_2)$ *$(x \cdot y) \cdot z = x \cdot (y \cdot z)$ property holds in $\Phi^* G$ for all $x, y, z \in G$,*
$(\mathscr{A}G_3)$ *There exists $e \in \Phi^* G$ such that $x \cdot e = e \cdot x = x$ for all $x \in G$ (e is called the approximately identity element of G),*
$(\mathscr{A}G_4)$ *There exists $y \in G$ such that $x \cdot y = y \cdot x = e$ for all $x \in G$ (y is called the inverse of x in G and denoted as x^{-1}).*

A subset S of X is called an *approximately semigroup* if

$(\mathscr{A}S_1)$ $x \cdot y \in \Phi^* S$ for all $x, y \in S$,
$(\mathscr{A}S_2)$ $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ property holds in $\Phi^* S$ for all $x, y, z \in S$
properties are satisfied.
If an approximately semigroup S has an approximately identity element $e \in \Phi^* S$ such that $x \cdot e = e \cdot x = x$ for all $x \in S$, then S is called an *approximately monoid*.
If $x \cdot y = y \cdot x$ for all $x, y \in S$ holds in $\Phi^* S$, then S is called *commutative approximately groupoid (semigroup, monoid or group)*.

Theorem 2.5. *[4] Let $G \subseteq X$ be an approximately group. Then the followings are true:*
(i) There is one and only one approximately identity element in G.
(ii) There is one and only one inverse of elements in G.
(iii) If either $x \cdot z = y \cdot z$ or $z \cdot x = z \cdot y$, then $x = y$ for all $x, y, z \in G$.

Theorem 2.6. *[4] Let G be an approximately group, H be a nonempty subset of G and $\Phi^* H$ be a groupoid. Then H is an approximately subgroup of G if and only if $x^{-1} \in H$ for all $x \in H$.*

Let G be an approximately groupoid in $\left(X, \mathscr{R}_{\delta_\Phi}\right)$, $x \in G$ and $A, B \subseteq G$. Then the subsets $x \cdot A$, $A \cdot x$, $A \cdot B \subseteq \Phi^* G \subseteq X$ are defined as:

$$\begin{aligned} x \cdot A &= xA = \{xa \mid a \in A\}, \\ A \cdot x &= Ax = \{ax \mid a \in A\}, \\ A \cdot B &= AB = \{ab \mid a \in A, b \in B\}. \end{aligned}$$

Lemma 2.7. *[4] Let $A, B \subseteq X$ and $A, B, Q(A), Q(B)$ be groupoids. If $\Phi : X \longrightarrow \mathbb{R}$ is an object descriptive homomorphism, then*

$$Q(A)Q(B) = Q(AB).$$

Theorem 2.8. *[6] Let G be an approximately group, H be an approximately subgroup of G and $G/_{\rho_l}$ be a set of all descriptive approximately left cosets of G by H. If $(\Phi^* G)/_{\rho_l} \subseteq \Phi^* \left(G/_{\rho_l}\right)$, then $G/_{\rho_l}$ is an approximately group with the binary operation $xH \odot yH = (x \cdot y)H$ for all $x, y \in G$.*

Definition 2.9. *[9] Let N be a nonempty set and "$+$" and "$\cdot$" be binary operations defined on N. Then N is called a (right) near-ring if the following properties are satisfied:*
(N_1) N is a group with "$+$" (need not be commutative),
(N_2) N is a semigroup with "$\cdot$",
(N_3) For all $x, y, z \in N$, $(x + y) \cdot z = (x \cdot z) + (y \cdot z)$.

3. Approximately near rings

Definition 3.1. *Let "$+$" and "$\cdot$" be binary operations on $\left(X, \mathscr{R}_{\delta_\Phi}\right)$. For a subset N of X is called an approximately near ring if the following conditions are satisfied:*

$(\mathscr{A}N_1)$ *N is an approximately group with "$+$" (need not be abelian),*
$(\mathscr{A}N_2)$ *N is an approximately semigroup with "$\cdot$",*
$(\mathscr{A}N_3)$ *For all $x, y, z \in N$,*
$(x + y) \cdot z = (x \cdot z) + (y \cdot z)$ property holds in $\Phi^* N$.

In addition,

$(\mathscr{A}N_4)$ *If $x \cdot y = y \cdot x$ for all $x, y \in N$,*

then N is a commutative approximately near ring.

$(\mathscr{A}N_5)$ *If $\Phi^* N$ contains an element 1_N such that $1_N \cdot x = x \cdot 1_N = x$ for all $x \in N$,*

then N is called an approximately near ring with identity.

Since $(\mathscr{A}N_3)$, instead of approximately near-ring it can be used approximately right near ring. Furthermore, if consider the condition $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$ for all $x, y, z \in N$ instead of $(\mathscr{A}N_3)$, then it can be named an approximately left near ring. Throughout this study approximately near ring will be used.
In general, the identity element of the approximately group $(N, +)$ is defined as zero of the approximately near ring N. Also, the set of all approximately near rings is shown with the notation $\mathscr{A}_{\mathscr{N}}$.
It should be noted here that, these conditions $(\mathscr{A}N_1) - (\mathscr{A}N_3)$ have to be hold in $\Phi^* N$. Addition or multiplying of finite number of elements in N may not always belong to $\Phi^* N$. Therefore we cannot always say that $kx \in \Phi^* N$ or $x^k \in \Phi^* N$ for all $x \in N$ and some $k \in \mathbb{Z}^+$. If $(\Phi^* N, +)$ and $(\Phi^* N, \cdot)$ are groupoids, then $kx \in \Phi^* N$ for all integer k or $x^k \in \Phi^* N$ for all positive integer k, for all $x \in N$.
An element x in approximately near ring N with identity is called a *left (resp. right) approximately invertible* if there exists $y \in N$ (resp. $z \in N$) such that $y \cdot x = 1_N$ (resp. $x \cdot z = 1_N$). The element y (resp. z) is called *a left (resp. right) approximately inverse* of x. If $x \in R$ is both left and right approximately invertible, then x is called an *approximately invertible* or an *approximately unit*. The set of approximately units in an approximately near ring N with identity forms is an approximately group with multiplication.

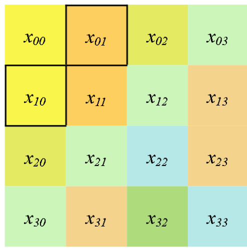

Figure 3.1: Digital Image I

Example 3.2. *Let I be a digital image endowed with δ_Φ. It is composed of 16 pixels (image elements) as shown in the Fig. 3.1. An image element x_{ij} is a pixel in the location (i,j). Let ϕ be a probe function that represents RGB (Red, Green, Blue) codes of pixels that are shown in Table 1.*

	x_{00}	x_{01}	x_{02}	x_{03}	x_{10}	x_{11}	x_{12}	x_{13}	x_{20}	x_{21}	x_{22}	x_{23}	x_{30}	x_{31}	x_{32}	x_{33}
Red	249	252	228	204	249	252	204	244	228	204	181	244	204	244	174	181
Green	245	207	234	245	245	207	245	212	234	245	232	212	245	212	220	232
Blue	75	94	98	185	75	94	185	140	98	185	231	140	185	140	124	231

Table 3.1: RGB codes of pixels

Let

$$+\ :\ \begin{array}{ll} I\times I & \longrightarrow I \\ (x_{ij},x_{kl}) & \longmapsto x_{ij}+x_{kl} \end{array},$$

$$x_{ij}+x_{kl}=x_{mn}, \quad i+k\equiv m \pmod 2 \text{ and } j+l\equiv n \pmod 2$$

be a binary operation on I such that $0\le i,j,k,l\le 3$. Let $N=\{x_{01},x_{10}\}\subseteq I$.
From Definition 2.1, descriptively upper approximation of N is $\Phi^*N=\{x_{ij}\in X \mid x_{ij}\delta_\phi N\}$. Hence $\phi(x_{ij})\cap Q(N)\neq\emptyset$ such that $x_{ij}\in I$, $Q(N)=\{\phi(x_{ij})\mid x_{ij}\in N\}$. From Table 1,

$$\begin{aligned} Q(N) &= \{\phi(x_{01}),\phi(x_{10})\} \\ &= \{(252,207,94),(249,245,75)\}. \end{aligned}$$

Hence we get $\Phi^*N=\{x_{00},x_{01},x_{10},x_{11}\}$ as in Fig. 3.2.

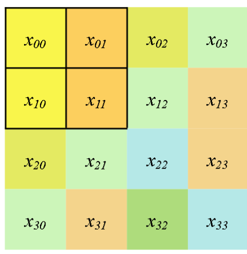

Figure 3.2: Upper Approximation of N

Hence N is an approximately group with "$+$" in $\left(I,\mathscr{R}_{\delta_\Phi}\right)$ from Definition 2.4. Furthermore, let

$$\begin{array}{ll} \cdot : I\times I & \longrightarrow I \\ (x_{ij},x_{kl}) & \longmapsto x_{ij}\cdot x_{kl}=x_{ij} \end{array}$$

be a binary operation on I. Then it is obvious that N is an approximately semigroup with "$\cdot$" in $\left(I,\mathscr{R}_{\delta_\Phi}\right)$. Also for all $x_{ij},x_{kl},x_{mn}\in N$,
$(x_{ij}+x_{kl})\cdot x_{mn}=x_{ij}\cdot x_{mn}+x_{kl}\cdot x_{mn}$ property holds in Φ^*N. But since $x_{01}\cdot(x_{01}+x_{01})\neq x_{01}\cdot x_{01}+x_{01}\cdot x_{01}$, so $x_{ij}\cdot(x_{kl}+x_{mn})=x_{ij}\cdot x_{kl}+x_{ij}\cdot x_{mn}$ property does not hold in Φ^*N. Consequently, N is an approximately right near ring.

Example 3.3. *Let I be a digital image endowed with δ_Φ. It is composed of 25 pixels (image elements) as shown in the Fig. 3.3. An image element x_{ij} is a pixel in the location (i,j). Let ϕ be a probe function that represents RGB (Red, Green, Blue) codes of pixels that are shown in Table 2.*
Let

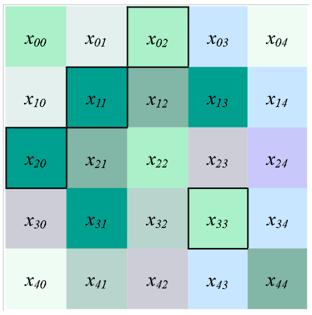

Figure 3.3: Digital Image I

	x_{00}	x_{01}	x_{02}	x_{03}	x_{04}	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}	x_{20}	x_{21}	x_{22}	x_{23}	x_{24}
Red	170	228	170	200	238	228	0	130	0	200	0	130	170	205	200
Green	240	240	240	230	252	240	160	182	160	230	160	182	240	205	200
Blue	200	237	200	255	244	237	145	167	145	255	145	167	200	216	250

	x_{30}	x_{31}	x_{32}	x_{33}	x_{34}	x_{40}	x_{41}	x_{42}	x_{43}	x_{44}
Red	205	0	183	170	200	238	183	205	200	130
Green	205	160	213	240	230	252	213	205	230	182
Blue	216	145	204	200	255	244	204	216	255	167

Table 3.2: RGB codes of pixels

$$+ \;:\; \begin{array}{ll} I\times I & \longrightarrow I \\ (x_{ij},x_{kl}) & \longmapsto x_{ij}+x_{kl} \end{array},$$

$$x_{ij}+x_{kl}=x_{mn}, \quad i+k\equiv m \pmod 4 \text{ and } j+l\equiv n \pmod 4$$

be a binary operation on I such that $0\leq i,j,k,l\leq 4$. Let $N=\{x_{02},x_{11},x_{20},x_{33}\}\subseteq I$.
From Definition 2.1, $\Phi^{*}N=\{x_{ij}\in I \mid x_{ij}\delta_{\phi}N\}$. Hence $\phi(x_{ij})\cap Q(N)\neq\emptyset$ such that $x_{ij}\in I$, $Q(N)=\{\phi(x_{ij}) \mid x_{ij}\in N\}$. From Table 2,

$$\begin{aligned} Q(N) &= \{\phi(x_{02}),\phi(x_{11}),\phi(x_{20}),\phi(x_{33})\} \\ &= \{(170,240,200),(0,160,145)\}. \end{aligned}$$

Hence we get $\Phi^{*}N=\{x_{00},x_{02},x_{11},x_{13},x_{20},x_{22},x_{31},x_{33}\}$.
And so N is an approximately group with "$+$" in $\left(I,\mathscr{R}_{\delta_\Phi}\right)$ from Definition 2.4. Furthermore, let

$$\begin{array}{ll} \cdot: I\times I & \longrightarrow I \\ (x_{ij},x_{kl}) & \longmapsto x_{ij}\cdot x_{kl}=x_{ij} \end{array}$$

be a binary operation on I. Then it is obivious that N is an approximately semigroup with "$\cdot$" in $\left(I,\mathscr{R}_{\delta_\Phi}\right)$. Also for all $x_{ij},x_{kl},x_{mn}\in N$,
$(x_{ij}+x_{kl})\cdot x_{mn}=x_{ij}\cdot x_{mn}+x_{kl}\cdot x_{mn}$ property holds in $\Phi^{*}N$. But since $x_{02}\cdot(x_{02}+x_{02})\neq x_{02}\cdot x_{02}+x_{02}\cdot x_{02}$, so $x_{ij}\cdot(x_{kl}+x_{mn})=x_{ij}\cdot x_{kl}+x_{ij}\cdot x_{mn}$ property does not hold in $\Phi^{*}N$. Consequently, N is an approximately right near ring.

Theorem 3.4. *All ordinary near rings in proximal relator spaces are approximately near rings.*

Proof. Let $N\subseteq X$ be a near ring. Since $N\subseteq\Phi^{*}N$, then the properties $(\mathscr{A}N_1)-(\mathscr{A}N_3)$ hold in $\Phi^{*}N$. Therefore N is an approximately near ring. □

Theorem 3.5. *All approximately rings in descriptive proximal relator space are approximately near rings.*

Proof. Let $N\subseteq X$ be an approximately ring. From definition of approximately ring, it is easily shown that N is an approximately near ring. □

Lemma 3.6. *Let $N\subseteq X$ be an approximately near ring and $0_N\in N$. If $0_N\cdot x\in N$ for all $x\in N$, then*
(i) $0_N\cdot x=0_N$,
(ii) $(-x)\cdot y=-(x\cdot y)$
for all $x,y\in N$.

Proof. (i) For all $x \in N$, $0_N \cdot x = (0_N + 0_N) \cdot x = 0_N \cdot x + 0_N \cdot x$.
From Theorem 2.5 (i), since the identity element is unique, $0_N \cdot x = 0_N$.
(ii) From (i), $0_N \cdot y = 0_N$ for all $y \in N$. Then $0_N = 0_N \cdot y = ((-x)+x) \cdot y = (-x) \cdot y + x \cdot y$.
From Theorem 2.5 (ii), since the approximately inverse element is unique, $(-x) \cdot y = -(x \cdot y)$. □

Definition 3.7. *Let N be an approximately near ring. The set*

$$N_0 = \{x \in N \mid x \cdot 0_N = 0_N\}$$

is called zero symmetric part of N and the set

$$N_c = \{x \in N \mid x \cdot 0_N = x\}$$

is called constant part of N.

If $N = N_0$, then N is called a zero symmetric approximately near ring and if $N = N_c$, then N is called constant approximately near ring. The set of all zero symmetric approximately near rings is represented as $\mathscr{N}_0$ and the set of all constant approximately near rings is represented as $\mathscr{N}_c$.
If the condition $d \cdot (x+y) = d \cdot x + d \cdot y$ holds in $\Phi^* N$ for all $x, y \in N$, then d is called distributive element. Also, the set of all approximately near ring with the identity is represented as $\mathscr{N}_1$ and the set of all distributive elements in N is represented as N_d.
If $N = N_d$, then N is called distributive approximately near ring.

Definition 3.8. *Let $(G,+)$ be an approximately group, N be an approximately near ring and*

$$\omega : \Phi^* N \times G \to \Phi^* G, \ \omega((x,g)) = xg.$$

The pair (G, ω) is called an approximately N-group if $(x+y)g = xg + yg$ and $(x \cdot y)g = x(yg)$ properties satisfy in $\Phi^* G$ for all $g \in G$ and all $x, y \in N$. It is denoted by ${}_N G$ and the set of all approximately N-groups is denoted by ${}_N\mathscr{G}$.

Theorem 3.9. *All approximately near-ring $(N,+,\cdot)$ are approximately N-groups.*

Definition 3.10. *Let $N \in \mathscr{N}_1$ and ${}_N G \in {}_N\mathscr{G}$. If $1_N g = g$ property holds in $\Phi^* G$ for all $g \in G$, then ${}_N G$ is called an unitary approximately N-group.*

Lemma 3.11. *Let N be an approximately near ring and G be an approximately N-group. Then*
(i) $0_N g = 0_G$ for all $g \in G$.
(ii) $(-x)g = -xg$ for all $g \in G$ and all $x \in N$.
(iii) $x 0_G = 0_G$ for all $x \in N_0$.
(iv) $xg = x 0_G$ for all $g \in G$ and all $x \in N_c$.

Proof. (i) For all $g \in G$, $0_N g = (0_N + 0_N) g = 0_N g + 0_N g$. From Theorem 2.5 (i), $0_N g = 0_G$.
(ii) From (i), $0_N g = 0_G$ for all $g \in G$. Then $0_G = 0_N g = ((-x)+x) g = (-x) g + xg$.From Theorem 2.5 (ii), $(-x) g = -xg$.
(iii) Since $x \cdot 0_N = 0_N$ for all $x \in N_0$, $x 0_G = x(0_N g) = (x \cdot 0_N) g = 0_N g = 0_G$ by (i).
(iv) Since $x \cdot 0_N = x$ for all $x \in N_c$, $xg = (x \cdot 0_N) g = x(0_N g) = x 0_G$ by (i). □

Definition 3.12. *Let N be an approximately near ring and M be an approximately subgroup of $(N,+)$. M is called an approximately subnear ring of N if $M \cdot M \subseteq \Phi^* M$.*

Theorem 3.13. *Let $N \subseteq X$ be an approximately near ring, $M \subseteq N$ and $(\Phi^* M,+)$, $(\Phi^* M,\cdot)$ be groupoids. Then M is an approximately subnear ring of N iff $-x \in M$ for all $x \in M$.*

Proof. ($\Rightarrow$) Let M is an approximately subnear ring of M. Then $(M,+)$ is an approximately group and hence $-x \in M$ for all $x \in M$.
($\Leftarrow$) Let $-x \in M$ for all $x \in M$. Since $(\Phi^* M,+)$ a groupoid, $(M,+)$ is an approximately group from Theorem 2.6. Therefore, since $(\Phi^* M,\cdot)$ is a groupoid and $M \subseteq N$, $x \cdot y \in \Phi^* M$ and $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ property holds in $\Phi^* M$ for all $x,y,z \in M$. Hence $(M,\cdot)$ is an approximately semigroup. Furthermore, since $(\Phi^* M,+)$ and $(\Phi^* M,\cdot)$ are groupoids and M is an approximately near ring, $(x+y) \cdot z = (x \cdot z) + (y \cdot z)$ property holds in $\Phi^* M$ for all $x,y,z \in M$. Consequently, M is an approximately subnear ring of N. □

Definition 3.14. *Let N be an approximately near ring, G be an approximately N-group and H be an approximately subgroup of $(G,+)$. Then H is called an approximately N-subgroup of G if $N \cdot H \subseteq \Phi^* H$.*

Definition 3.15. *Let N be an approximately near ring and I be an approximately subgroup of $(N,+)$. Then I is called an approximately ideal of N if the following properties are satisfied:*
(1) $I \cdot N \subseteq \Phi^* I$,
(2) $x \cdot (y+a) - x \cdot y \in \Phi^* I$ for all $x, y \in N$ and all $a \in I$.
Furthermore, I is called right approximately ideal of M if only the condition (1) satisfies. Also, I is called left approximately ideal of M if only the condition (2) satisfies.

Definition 3.16. *Let N be an approximately near ring, G be an approximately N-group and H be an approximately N-subgroup of G. Then H is called an approximately ideal of G if $x(g+h)-xg \in N_r(B)^* H$ for all $g \in G$, all $h \in H$ and all $x \in N$.*

Theorem 3.17. *Let $N \subseteq X$ be an approximately near ring, M_1 and M_2 two approximately subnear rings of N and Φ^*M_1, Φ^*M_2 be groupoids with the binary operations "$+$" and "$\cdot$". If*

$$(\Phi^*M_1) \cap (\Phi^*M_2) = \Phi^*(M_1 \cap M_2),$$

then $M_1 \cap M_2$ is an approximately subnear ring of N.

Corollary 3.18. *Let $N \subseteq X$ be an approximately near ring, $\{M_i : i \in \Delta\}$ be a nonempty family of approximately subnear rings of N and Φ^*M_i be groupoids for all $i \in \Delta$. If*

$$\bigcap_{i \in \Delta} (\Phi^*M_i) = \Phi^*\left(\bigcap_{i \in \Delta} M_i\right),$$

then $\bigcap_{i \in \Delta} M_i$ is an approximately subnear ring of N.

3.1. Approximately near rings of weak cosets

Let N be an approximately near ring and M be an approximately subnear ring of N. The relation "$\sim_r$" defined as

$$a \sim_r b \Leftrightarrow a+(-b) \in M \cup \{0_N\}$$

where $a,b \in N$.

Theorem 3.19. *Let N be an approximately near ring. Then "$\sim_r$" is a right weak equivalence relation on N.*

Proof. Since $(N,+)$ is an approximately group, $-a \in N$ for all $a \in N$. Due to $a+(-a) = 0_N \in M \cup \{0_N\}$, $a \sim_r a$. Let $a \sim_r b$ for all $a,b \in N$. Then $a+(-b) \in M \cup \{0_N\}$, that is $a+(-b) \in M$ or $a+(-b) \in \{0_N\}$. If $a+(-b) \in M$, since $(M,+)$ is an approximately group, then $-(a+(-b)) = b+(-a) \in M$. Hence $b \sim_r a$. Also if $a+(-b) \in \{0_N\}$, then $a+(-b) = 0_N$. Therefore $b+(-a) = -(a+(-b)) = -0_N = 0_N$ and so $b \sim_r a$. Consequently, "$\sim_r$" is a right weak equivalence relation on N. □

A weak class containing the element $a \in N$ according to the relation "$\sim_r$" is defined by

$$\tilde{a}_r = \{m+a | m \in M, a \in N, m+a \in N\} \cup \{a\}.$$

Definition 3.20. *Let N be an approximately near ring. A weak class determined by right weak equivalence relation "$\sim_r$" is called near right weak coset.*

Similarly, the relation "$\sim_\ell$" defined as

$$a \sim_\ell b \Leftrightarrow (-a)+b \in M \cup \{0_N\}$$

where $a,b \in N$.

Theorem 3.21. *Let N be an approximately near ring. Then "$\sim_\ell$" is a left weak equivalence relation on N.*

Proof. Since $(N,+)$ is an approximately group, $-a \in N$ for all $a \in N$. Due to $(-a)+a = 0_N \in M \cup \{0_N\}$, $a \sim_\ell a$. Let $a \sim_\ell b$ for all $a,b \in N$. Then $(-a)+b \in M \cup \{0_N\}$, that is, $(-a)+b \in M$ or $(-a)+b \in \{0_N\}$. If $(-a)+b \in M$, since $(M,+)$ is a an approximately group, then $-((-a)+b) = (-b)+a \in M$. Hence $b \sim_\ell a$. Also if $(-a)+b \in \{0_N\}$, then $(-a)+b = 0_N$. Therefore $(-b)+a = -((-a)+b) = -0_N = 0_N$ and so $b \sim_\ell a$. Consequently, "$\sim_\ell$" is a left weak equivalence relation on N. □

A class that contains the element $a \in N$, determined by relation "$\sim_\ell$" is

$$\tilde{a}_\ell = \{a+m | m \in M, a \in N, a+m \in N\} \cup \{a\}.$$

Definition 3.22. *Let N be an approximately near ring. A class determined by left weak equivalence relation "$\sim_\ell$" is called near left weak coset.*

We can easily show that $\tilde{a}_r = M+a$ and $\tilde{a}_\ell = a+M$. Approximately group $(M,+)$ may not always abelian. If $(M,+)$ is an abelian approximately group, $\tilde{a}_r = \tilde{a}_\ell$. Otherwise $\tilde{a}_r \neq \tilde{a}_\ell$.
Let N be an approximately near ring and M be an approximately subnear ring of N. Then

$$N/_{\sim_\ell} = \{a+M | a \in N\}$$

is a set of all near left weak cosets of N determined by M. If we consider Φ^*N instead of approximately near ring N

$$(\Phi^*N)/_{\sim_\ell} = \{a+M | a \in \Phi^*N\}.$$

Hence

$$a+M = \{a+m | m \in M, a \in \Phi^*N, a+m \in N\} \cup \{a\}.$$

Definition 3.23. *Let N be an approximately near ring and M be an approximately subnear ring of N. For $a,b \in N$, let $a+M$ and $b+M$ be two near left weak cosets that determined the elements a and b, respectively. Then sum of two near left weak cosets that determined by $a+b \in \Phi^*N$ can be defined as*

$$\{(a+b)+m | m \in M, a+b \in \Phi^*N, (a+b)+m \in N\} \cup \{a+b\}$$

and denoted by

$$(a+M) \oplus (b+M) = (a+b)+M.$$

Definition 3.24. *Let N be an approximately near ring and M be an approximately subnear ring of N. For $a,b \in N$, let $a+M$ and $b+M$ be two near left weak cosets that determined the elements a and b, respectively. Then product of two near left weak cosets that determined by $a \cdot b \in \Phi^*N$ can be defined as*

$$\{(a \cdot b)+m | m \in M, a \cdot b \in \Phi^*N, (a \cdot b)+m \in N\} \cup \{a \cdot b\}$$

and denoted by

$$(a+M) \odot (b+M) = (a \cdot b)+M.$$

Definition 3.25. *Let $N/_{\sim_\ell}$ be a set of all near left weak cosets of N determined by M and $\xi_\Phi(S)$ be a descriptive approximately collection of $S \in P(X)$. Then*

$$\Phi^*\left(N/_{\sim_\ell}\right) = \bigcup_{\xi_\Phi(S) \underset{\Phi}{\cap} N/_{\sim_\ell} \neq \emptyset} \xi_\Phi(S)$$

is called upper approximation of $N/_{\sim_\ell}$.

Theorem 3.26. *Let N be an approximately near ring, M be an approximately subnear ring of N and $N/_{\sim_\ell}$ be a set of all near left weak cosets of N determined by M. If*

$$(\Phi^*N)/_{\sim_\ell} \subseteq \Phi^*\left(N/_{\sim_\ell}\right),$$

then $N/_{\sim_\ell}$ is an approximately near ring with the operations given by

$$(a+M) \oplus (b+M) = (a+b)+M$$

and

$$(a+M) \odot (b+M) = (a \cdot b)+M$$

for all $a,b \in N$.

Proof. $(\mathscr{A}N_1)$ Let $(\Phi^*N)/_{\sim_\ell} \subseteq \Phi^*\left(N/_{\sim_\ell}\right)$. Since N is an approximately near ring, $\left(N/_{\sim_\ell}, \oplus\right)$ is an approximately group of all near left weak cosets of N determined by M from Theorem 2.8.
$(\mathscr{A}N_2)$

$(\mathscr{A}S_1)$ Since $(N,\cdot)$ is an approximately semigroup, $a \cdot b \in \Phi^*N$ for all $a,b \in N$ and $(a+M) \odot (b+M) = (a \cdot b)+M \in (\Phi^*N)/_{\sim_\ell}$ for all $(a+M),(b+M) \in N/_{\sim_\ell}$. From the hypothesis, $(a+M) \odot (b+M) = (a \cdot b)+M \in \Phi^*\left(N/_{\sim_\ell}\right)$ for all $(a+M),(b+M) \in N/_{\sim_\ell}$.

$(\mathscr{A}S_2)$ Since $(N,\cdot)$ is an approximately semigroup, associative property holds in Φ^*N. Hence

$$\begin{aligned}
& ((a+M) \odot (b+M)) \odot (c+M) \\
= \ & ((a \cdot b)+M) \odot (c+M) \\
= \ & ((a \cdot b) \cdot c)+M \\
= \ & (a \cdot (b \cdot c))+M \\
= \ & (a+M) \odot ((b \cdot c)+M) \\
= \ & (a+M) \odot ((b+M) \odot (c+M))
\end{aligned}$$

holds in $(\Phi^*N)/_{\sim_\ell}$ for all $(a+M),(b+M),(c+M) \in N/_{\sim_\ell}$. From the hypothesis, associative property holds in $\Phi^*\left(N/_{\sim_\ell}\right)$. So $\left(N/_{\sim_\ell}, \odot\right)$ is an approximately semigroup of all near left weak cosets of N determined by M.

($\mathscr{A}N_3$) Since N is an approximately near ring, right distributive property holds in Φ^*N for all $a,b,c \in N$. Then

$$\begin{aligned} & ((a+M)\oplus (b+M))\odot (c+M) \\ = & ((a+b)+M)\odot (c+M) \\ = & ((a+b)\cdot c)+M \\ = & ((a\cdot c)+(b\cdot c))+M \\ = & ((a\cdot c)+M)\oplus ((b\cdot c)+M) \\ = & ((a+M)\odot (c+M))\oplus ((b+M)\odot (c+M)) \end{aligned}$$

for all $(a+M),(b+M),(c+M) \in N/_{\sim_\ell}$.
Hence right distributive property holds in $\Phi^*\left(N/_{\sim_\ell}\right)$ from the hypothesis.
Consequently, $N/_{\sim_\ell}$ is an approximately near ring. □

Definition 3.27. *Let N be an approximately near ring and M be an approximately subnear ring of N. The approximately near ring $N/_{\sim_\ell}$ is called an approximately near ring of all near left weak cosets of N determined by M and denoted by $N/_wM$.*

3.2. Approximately near ring homomorphisms

Definition 3.28. *Let $N_1,N_2 \subseteq X$ be two approximately near rings and*

$$\psi : \Phi^*N_1 \to \Phi^*N_2$$

be a mapping. If

$$\psi(a+b) = \psi(a)+\psi(b)$$

and

$$\psi(a\cdot b) = \psi(a)\cdot\psi(b)$$

for all $a,b \in N_1$, then ψ is called an approximately near ring homomorphism. Furthermore, N_1 is called approximately homomorphic to N_2 and denoted by $N_1 \simeq_a N_2$.
An approximately near ring homomorphism $\psi : \Phi^*N_1 \to \Phi^*N_2$ is called
(1) an approximately near ring monomorphism if ψ is one-one,
(2) an approximately near ring epimorphism if ψ is onto,
(3) an approximately near ring isomorphism if ψ is one-one and onto.
Set of all approximately near ring homomorphisms from Φ^*N_1 into Φ^*N_2 is denoted by $Hom(\Phi^*N_1,\Phi^*N_2)$.

Theorem 3.29. *Let N_1,N_2 be two approximately near rings and ψ be an approximately near ring homomorphism from Φ^*N_1 into Φ^*N_2. Then*
(i) $\psi(0_{N_1}) = 0_{N_2}$, where $0_{N_2} \in \Phi^*N_2$ is the near zero of N_2.
(ii) $\psi(-a) = -\psi(a)$ for all $a \in N_1$.

Proof. (i) Since $0_{N_1} = 0_{N_1}+0_{N_1}$ and ψ is an approximately near ring homomorphism, $\psi(0_{N_1}) = \psi(0_{N_1}+0_{N_1}) = \psi(0_{N_1})+\psi(0_{N_1})$. Hence $\psi(0_{N_1}) = 0_{N_2}$ as the approximately identity element is unique.
(ii) $a+(-a) = 0_{N_1}$ for all $a \in N_1$. Then $0_{N_2} = \psi(0_{N_1}) = \psi(a+(-a)) = \psi(a)+\psi(-a)$ by (i). Similarly, $0_{N_2} = \psi(-a)+\psi(a)$ for all $a \in N_1$. By Theorem 2.5 (ii), since $\psi(a)$ has a unique approximately inverse, $\psi(-a) = -\psi(a)$ for all $a \in N_1$. □

Definition 3.30. *Let $N_1,N_2 \subseteq X$ be two approximately near rings and $\psi \in Hom(\Phi^*N_1,\Phi^*N_2)$. The set*

$$Ker\psi = \{a \in N_1 \mid \psi(a) = 0_{N_2}\}$$

is called kernel of approximately near ring homomorphism ψ.

Theorem 3.31. *Let $N_1,N_2 \subseteq X$ be two approximately near rings, $\psi \in Hom(\Phi^*N_1,\Phi^*N_2)$ and $(\Phi^*Ker\psi,+)$, $(\Phi^*Ker\psi,\cdot)$ be groupoids. Then $Ker\psi$ is a approximately subnear ring of N_1.*

Proof. Let $a \in Ker\psi$. Then $\psi(a) = 0_{N_2}$. Since $N_1,N_2 \subseteq X$ are two approximately near rings, $0_{N_1} \in \Phi^*N_1$ and $0_{N_2} \in \Phi^*N_2$, $\psi(0_{N_1}) = 0_{N_2}$ by Theorem 3.29 (i). Hence $0_{N_2} = \psi(0_{N_1}) = \psi(a+(-a)) = \psi(a)+\psi(-a)$ and so $\psi(-a) = 0_{N_2}$ from $\psi(a) = 0_{N_2}$. Thus from Definition 3.30, $-a \in Ker\psi$. Therefore $Ker\psi$ is an approximately subnear ring of N_1 from Theorem 3.13. □

Theorem 3.32. *Let $N_1, N_2 \subseteq X$ be two approximately near rings, $\psi \in Hom(\Phi^* N_1, \Phi^* N_2)$ and $(\Phi^* N_1, +)$, $(\Phi^* N_1, \cdot)$ be groupoids. If S is an approximately subnear ring of N_1 and*

$$\psi(\Phi^* S) = \Phi^* \psi(S),$$

then $\psi(S) = \{\psi(a) | a \in S\}$ is an approximately subnear ring of N_2.

Proof. Since $N_1, N_2 \subseteq X$ are two approximately near rings, $0_{N_1} \in \Phi^* N_1$ and $0_{N_2} \in \Phi^* N_2$, $\psi(0_{N_1}) = 0_{N_2}$ by Theorem 3.29 (i). Thus $0_{N_2} = \psi(0_{N_1}) \in \psi(\Phi^* S) = \Phi^* \psi(S)$. This means that $\Phi^* \psi(S) \neq \emptyset$, i.e., $\psi(S) \neq \emptyset$. Since S is an approximately subnear ring of N_1, $-a \in S$ for all $a \in S$ from Theorem 3.13. Therefore $-\psi(a) = \psi(-a) \in \psi(S)$ for all $\psi(a) \in \psi(S)$ by Theorem 3.29 (ii). Consequently, $\psi(S)$ is an approximately subnear ring of N_2 from Theorem 3.13. □

Theorem 3.33. *Let $N_1, N_2 \subseteq X$ be two approximately near rings, $T \subseteq N_2$, $\psi \in Hom(\Phi^* N_1, \Phi^* N_2)$ and $(\Phi^* T, +)$, $(\Phi^* T, \cdot)$ be groupoids. If T is an approximately subnear ring of N_2, then $\psi^{-1}(T) = \{a \in N_1 | \psi(a) \in T\}$ is an approximately subnear ring of N_1.*

Proof. Let $a \in \psi^{-1}(T)$. Then $\psi(a) \in T$. Since T is an approximately subnear ring of N_2, $-\psi(a) \in T$ from Theorem 3.13. Hence $\psi(-a) \in T$ and so $-a \in \psi^{-1}(T)$ by Theorem 3.29 (ii). Consequently, $\psi^{-1}(T)$ is an approximately subnear ring of N_1 from Theorem 3.13. □

Theorem 3.34. *Let N be an approximately near ring and M be an approximately subnear ring of N. Then the mapping $\Pi : \Phi^* N \to \Phi^*(N/_w M)$ defined by $\Pi(a) = a + M$ for all $a \in \Phi^* N$ is an approximately near ring homomorphism.*

Proof. From the definition of Π, Definitions 3.23 and 3.24,

$$\Pi(a+b) = (a+b)+M = (a+M) \oplus (b+M) = \Pi(a) \oplus \Pi(b), \Pi(a \cdot b) = (a \cdot b)+M = (a+M) \odot (b+M) = \Pi(a) \odot \Pi(b)$$

for all $a, b \in N$. Thus Π is an approximately near ring homomorphism from Definition 3.28. □

Definition 3.35. *The approximately near ring homomorphism Π is called a natural approximately near ring homomorphism from $\Phi^* N$ into $\Phi^*(N/_w M)$.*

Definition 3.36. *Let $N_1, N_2 \subseteq X$ be two approximately near rings, $S \subseteq N_1$. Let*

$$\tau : \Phi^* N_1 \longrightarrow \Phi^* N_2$$

be a mapping and

$$\tau_S = {}^{\tau}|_S : S \longrightarrow \Phi^* N_2$$

a restricted mapping. If

$$\tau(a+b) = \tau_S(a+b) = \tau_S(a) + \tau_S(b) = \tau(a) + \tau(b)$$

and

$$\tau(a \cdot b) = \tau_S(a \cdot b) = \tau_S(a) \cdot \tau_S(b) = \tau(a) \cdot \tau(b)$$

for all $a, b \in S$, then τ is called a restricted approximately near ring homomorphism and also, N_1 is called restricted approximately homomorphic to N_2, denoted by $N_1 \simeq_{ra} N_2$.

Theorem 3.37. *Let $N_1, N_2 \subseteq X$ be two approximately near rings and $\tau \in Hom(\Phi^* N_1, \Phi^* N_2)$. Let $(\Phi^* Ker\tau, +)$, $(\Phi^* Ker\tau, \cdot)$ be groupoids and $(\Phi^* N_1)/_{\sim_\ell}$ be a set of all approximately left weak cosets of $\Phi^* N_1$ determined by $Ker\tau$. If*

$$(\Phi^* N_1)/_{\sim_\ell} \subseteq \Phi^*\left(N_1/_{\sim_\ell}\right)$$

and

$$\Phi^* \tau(N_1) = \tau(\Phi^* N_1),$$

then

$$N_1/_{\sim_\ell} \simeq_{ra} \tau(N_1).$$

Proof. Since $(\Phi^* Ker\tau, +)$ and $(\Phi^* Ker\tau, \cdot)$ are groupoids, $Ker\tau$ is an approximately subnear ring of N_1 from Theorem 3.31. Since $Ker\tau$ is an approximately subnear ring of N_1 and $(\Phi^* N_1)/_{\sim_\ell} \subseteq \Phi^*\left(N_1/_{\sim_\ell}\right)$, then $N_1/_{\sim_\ell}$ is an approximately near ring of all near left weak cosets of N_1 determined by $Ker\tau$, from Theorem 3.26. Since $\Phi^* \tau(N_1) = \tau(\Phi^* N_1)$, $\tau(N_1)$ is an approximately subnear ring of N_2 from Theorem 3.32. Let

$$\begin{array}{rcl} \sigma : \Phi^*\left(N_1/_{\sim_\ell}\right) & \longrightarrow & \Phi^*\tau(N_1) \\ A & \longmapsto & \sigma(A) = \begin{cases} \sigma_{N_1/_{\sim_\ell}}(A) & ,A \in (\Phi^* N_1)/_{\sim_\ell} \\ 0_{\tau(N_1)} & ,A \notin (\Phi^* N_1)/_{\sim_\ell} \end{cases} \end{array}$$

be a mapping where

$$\begin{array}{rcl} \sigma_{N_1/_{\sim_\ell}} = \sigma\big|_{N_1/_{\sim_\ell}} : N_1/_{\sim_\ell} & \longrightarrow & \Phi^*\tau(N_1) \\ a + Ker\tau & \longmapsto & \sigma_{N_1/_{\sim_\ell}}(a + Ker\tau) = \tau(a) \end{array}$$

for all $a + Ker\tau \in N_1/_{\sim_\ell}$.
Since

$$\begin{aligned} a + Ker\tau &= \{a + k \mid k \in Ker\tau, a + k \in N_1\} \cup \{a\}, \\ b + Ker\tau &= \left\{b + k' \mid k' \in Ker\tau, b + k' \in N_1\right\} \cup \{b\} \end{aligned}$$

and the mapping τ is an approximately near ring homomorphism,

$$\begin{array}{rl} & a + Ker\tau = b + Ker\tau \\ \Rightarrow & a \in b + Ker\tau \\ \Rightarrow & a \in \{b + k' \mid k' \in Ker\tau, b + k' \in N_1\} \text{ or } a \in \{b\} \\ \Rightarrow & a = b + k',\, k' \in Ker\tau,\, b + k' \in N_1 \text{ or } a = b \\ \Rightarrow & -b + a = (-b + b) + k',\, k' \in Ker\tau \text{ or } \tau(a) = \tau(b) \\ \Rightarrow & -b + a = k',\, k' \in Ker\tau \\ \Rightarrow & -b + a \in Ker\tau \\ \Rightarrow & \tau(-b + a) = 0_{\tau(N_1)} \\ \Rightarrow & \tau(-b) + \tau(a) = 0_{\tau(N_1)} \\ \Rightarrow & -\tau(b) + \tau(a) = 0_{\tau(N_1)} \\ \Rightarrow & \tau(a) = \tau(b) \\ \Rightarrow & \sigma_{N_1/_{\sim_\ell}}(a + Ker\tau) = \sigma_{N_1/_{\sim_\ell}}(b + Ker\tau) \end{array}$$

Therefore $\sigma_{N_1/_{\sim_\ell}}$ is well defined.
For $A, B \in \Phi^*\left(\sigma_{N_1/_{\sim_\ell}}\right)$, we suppose that $A = B$. Since the mapping $\sigma_{N_1/_{\sim_\ell}}$ is well defined,

$$\begin{aligned} \sigma(A) &= \begin{cases} \sigma_{N_1/_{\sim_\ell}}(A) & , A \in (\Phi^* N_1)/_{\sim} \\ 0_{\tau(N_1)} & , A \notin (\Phi^* N_1)/_{\sim} \end{cases} \\ &= \begin{cases} \sigma_{N_1/_{\sim_\ell}}(B) & , B \in (\Phi^* N_1)/_{\sim} \\ 0_{\tau(N_1)} & , B \notin (\Phi^* N_1)/_{\sim} \end{cases} \\ &= \sigma(B). \end{aligned}$$

Consequently σ is well defined.
For all $a + Ker\tau, b + Ker\tau \in N_1/_{\sim_\ell} \subset \Phi^*\left(N_1/_{\sim_\ell}\right)$,

$$\begin{array}{rl} & \sigma((a + Ker\tau) \oplus (b + Ker\tau)) \\ = & \sigma((a + b) + Ker\tau) \\ = & \sigma_{N_1/_{\sim_\ell}}((a + b) + Ker\tau) \\ = & \tau(a + b) \\ = & \tau(a) + \tau(b) \\ = & \sigma_{N_1/_{\sim_\ell}}(a + Ker\tau) + \sigma_{N_1/_{\sim_\ell}}(b + Ker\tau) \\ = & \sigma(a + Ker\tau) + \sigma(b + Ker\tau) \end{array}$$

and

$$\begin{array}{rl} & \sigma((a + Ker\tau) \odot (b + Ker\tau)) \\ = & \sigma((a \cdot b) + Ker\tau) \\ = & \sigma_{N_1/_{\sim_\ell}}((a \cdot b) + Ker\tau) \\ = & \tau(a \cdot b) \\ = & \tau(a) \cdot \tau(b) \\ = & \sigma_{N_1/_{\sim_\ell}}(a + Ker\tau) \cdot \sigma_{N_1/_{\sim_\ell}}(b + Ker\tau) \\ = & \sigma(a + Ker\tau) \cdot \sigma(b + Ker\tau). \end{array}$$

Therefore σ is a restricted approximately near ring homomorphism by Definition 3.36. Consequently, $N_1/_{\sim_\ell} \simeq_{ra} \tau(N_1)$. $\square$

4. Conclusion

Algebraic structures provide a consistent theoretical background for all mathematical research. The theoretical background is very important in all problems of processing digital images. In this study, approximately near-rings as an algebraic structure on digital images were investigated. We hope this research will inspire the investigations in both some theoretical and applied sciences.

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Author's contributions

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*Author [Ayşegül Kocamaz]: Collected the data, contributed to research method or evaluation of data, wrote the manuscript (%40)

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