Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 52 (4) (2023), 850 – 875
DOI : 10.15672/hujms.1117320

Research Article

Optimal order uniform convergence of weak Galerkin finite element method on Bakhvalov-type meshes for singularly perturbed convection dominated problems

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Abstract

In this paper, we propose a weak Galerkin finite element method (WG-FEM) for solving two-point boundary value problems of convection-dominated type on a Bakhvalov-type mesh. A special interpolation operator which has a simple representation and can be easily extended to higher dimensions is introduced for convection-dominated problems. A robust optimal order of uniform convergence has been proved in the energy norm with this special interpolation using piecewise polynomials of degree $k \geq 1$ on interior of the elements and piecewise constant on the boundary of each element. The proposed finite element scheme is parameter-free formulation and since the interior degrees of freedom can be eliminated efficiently from the resulting discrete system, the number of unknowns of the proposed method is comparable with the standard finite element methods. An optimal order of uniform convergence is derived on Bakhvalov-type mesh. Finally, numerical experiments are given to support the theoretical findings and to show the efficiency of the proposed method.

Mathematics Subject Classification (2020). 65L11, 65L20, 65L60, 65L70

Keywords. singularly perturbed differential equation, weak Galerkin finite element method, Bakhvalov-type meshes, optimal uniform convergence

1. Introduction

It is well known that singularly perturbed problems (SPPs) exhibit so-called boundary layers which are regions where the derivatives of the solution vary rapidly. The presence of these boundary layers leads to oscillations in numerical approximation and causes difficulties for finding an accurate numerical approximation in standard numerical methods such as the classical finite difference (FD) methods or the standard finite element methods (FEMs) and makes these methods unstable and unsatisfactory unless the mesh size is moderately smaller than the perturbation parameter [11]. Thus, uniform convergent numerical methods which produce stable and more accurate approximate solution independent of the perturbation parameter have been proposed and analyzed in the literature; see e.g., the books [11, 16, 17, 21] and references therein. One of the efficient way of handling singularly perturbed problems is to use layer-adapted meshes. Boundary layers can be resolved

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Received: 16.05.2022; Accepted: 18.11.2022

by designing layer-adapted meshes if we know a priori knowledge of the structure of the layers. The commonly used layer-adapted meshes for solving SPPs include Shishkin-type meshes and Bakhvalov-type meshes, see e.g., the standard FD methods [6, 11] and conforming FEMs [25, 34] on Shishkin-type meshes or Bakhvalov-type meshes. The authors in [5] show that the numerical solution of convection-dominated problems still has some oscillatory behaviour even if the layer-adapted meshes are used in the discretization. To overcome these oscillations, an additional stabilization is added in the numerical schemes. Examples of stabilized numerical schemes for SPP of convection-diffusion type include the FD method of up-winding flavor [1, 7, 12, 27], the streamline-diffusion finite element method [13, 15, 26] and the discontinuous Galerkin methods [8, 9, 22, 33, 37, 38]. Further descriptions and investigations on these methods for SPPs can be found in the recent books [11, 21] and references therein. Since Shishkin-type meshes have simpler structure and clear analysis, many articles have been devoted to uniform convergence of SPPs on Shishkin-type meshes; see e.g., [12, 14, 21, 24] and references therein. Unfortunately, a logarithmic factor will be present in the error bounds when one uses a Shishkin-type mesh and this factor deteriorates the optimal order of convergence. Consequently, in general, Bakhvalov-type meshes have better numerical results than Shishkin-type meshes. This superior feature is much more noticeable in higher-order schemes (see, e.g., [11, p. 10] and [35, p. 10]). On the other hand, unlike Shishkin-type meshes, the transition points of Bakhvalov-type meshes which are located between the fine and the rough parts of the meshes are independent of the number of mesh points. However, these transition points make convergence analysis of FEM more subtle on Bakhvalov-type meshes and require construction of different numerical approaches. More precisely, the difficulty arises from instability of the standard Lagrange interpolant in L^2-norm on the interval which is the last mesh interval of the fine part and is neighbor to the coarse part of Bakhvalov-type meshes for convection dominated problems. Therefore, there are limited papers dealing with SPPs of convection-dominated type on Bakhvalov-type meshes. The authors in [23] and [3] consider the conforming linear FEM using a quasi-interpolation technique and investigate the optimal order uniform convergence of the conforming FEM using linear elements on Bakhvalov-type meshes. However, the obtained results in these works are limited to only linear finite element in one dimensional cases. It is not easy to extend these obtained results to the uniform convergence of FEM using higher-order finite elements in one and higher dimensions for SPPs of convection-dominated type. Recently, the optimal uniform convergence of FEMs using high order elements for SPPs of convection-dominated type on Bakhvalov-type meshes has been studied in [35]. The standard FEM is applied to SPPs of convection-dominated problems on Bakhvalov-type meshes and the optimal order of uniform convergence is obtained with the introduction of a novel interpolation operator in [35]. The conforming FEM can still produce some little oscillations in the discrete solution when it is applied to SPPs of convection-dominated problems. Therefore, we propose a stabilized FEM to improve the convergence results of [35].

In this paper, we consider the WG-FEM initially developed in [31] for solving second order elliptic problems. The key feature of this method is that the classical derivative is replaced by *weak derivative* in the corresponding variational formulation in a way that completely discontinuous functions have been allowed to use in the numerical scheme which has a parameter independent stabilizer. The weak Galerkin method has been studied and applied to a variety of problems including Stokes equations [32], interface problem [18], Maxwell equation [19], fractional time convection-diffusion problems [28], and singularly perturbed elliptic equations in one and higher dimensions [10, 29, 30].

The uniform convergent weak Galerkin method has been presented in [39] for convection dominated problems. However, the obtained results are only available for linear convection-dominated problems on a piecewise uniform Shishkin mesh. To the best of the authors knowledge, optimal uniform convergence of the WG-FEM on Bakhvalov-type meshes has not been presented for SPPs of convection-diffusion type so far. The main goal of this paper is to present robust optimal order uniform convergence in the energy norm on Bakhvalov-type mesh for convection-dominated problems. Unlike reaction-diffusion problems, the standard Lagrange interpolation for convection-dominated problems on Bakhvalov-type meshes is not suitable for robust uniform convergence because of instability issues of the Lagrange interpolation. Therefore, we adapt a special interpolation operator introduced by Zhang and Liu in [35] on Bakhvalov-type meshes for convection-dominated problem.

The rest of the paper is organized as follows. In Section 2, the WG-FEM scheme for the singularly perturbed convection-diffusion problems is constructed and the stability of the proposed method is studied. We also discuss the error estimates of the proposed method for SPPs of convection dominated type in Section 2. Various numerical examples are given to confirm the theoretical findings in Section 3. Finally, we summarize the theoretical findings in Section 4.

Throughout this article, we use C for generic constants independent of ε, N, and the mesh size h which may be different in each location.

2. Convection-diffusion problems

In this section, we consider the following singularly perturbed convection-diffusion problem: Find $u \in C^2(0,1) \cap C[0,1]$ such that

$$\begin{aligned} \mathcal{L}u := -\varepsilon u''(x) - b(x)u'(x) + c(x)u(x) = g(x) \text{ in } \Omega = (0,1), \\ u(0) = 0, \quad u(1) = 0, \end{aligned} \tag{2.1}$$

where $0 < \varepsilon \ll 1$ is a small perturbation parameter and b, c and g are sufficiently smooth functions such that

$$b(x) \geq \beta > 0,\ c(x) \geq 0,\ c(x) + \frac{1}{2}b'(x) \geq \gamma^2 > 0, \quad \forall x \in \bar{\Omega}, \tag{2.2}$$

for some positive constants β and γ. Under the assumption (2.2), the problem (2.1) has a unique solution in $H^2(\Omega) \cap H_0^1(\Omega)$ for all $g \in L^2(\Omega)$ [25], [34]. The analytical solution of problem (2.1) exhibits an exponential boundary layer of width $\mathcal{O}(\varepsilon|\ln\varepsilon|)$ at $x = 0$ if the perturbation parameter $0 < \varepsilon \ll 1$ is arbitrarily small (see [21]).

2.1. A decomposition of the solution

In this section, we recall some important properties for the derivatives of the solution of (2.1). The following lemma provides the bounds for the solution of (2.1) and its derivatives and the solution decomposition. The proof of the lemma can be found in [Lemma 8.1,[21]].

Lemma 2.1 ([21])**.** *Let q be a positive integer. Assume that the condition (2.2) is satisfied and b, c and g are sufficiently smooth functions. The solution u of (2.1) has the following solution decomposition*

$$u = R + L, \tag{2.3}$$

where the regular part R and the layer part L satisfy $\mathcal{L}R = g$ and $\mathcal{L}L = 0$ and

$$|R^{(i)}(x)| \leq C, \quad |L^{(i)}(x)| \leq C\varepsilon^{-i} \exp\left(-\frac{\beta x}{\varepsilon}\right) \quad \text{for } 0 \leq i \leq q. \tag{2.4}$$

2.2. Bakhvalov-type mesh

Bakhvalov mesh is originally introduced and constructed for the layer functions in SPPs in [2]. The mesh points of the Bakhvalov mesh are given in terms of a piecewise C^1 continuous mesh generating function. The transition point of the Bakhvalov mesh is not explicitly determined since it contains a nonlinear equation, see e.g., [21]. For this reason, the mesh generating function of Bakhvalov mesh leads to various kind of meshes that are called Bakhvalov-type meshes.

For convection-diffusion problems, we will consider the following Bakhvalov-type mesh presented in [23]. The main feature of this mesh is that the mesh generating functions are not in C^1 and its transition point is known. The mesh generating function for the Bakhvalov-type mesh is given by [23]

$$x = \varphi(t) = \begin{cases} -\frac{\sigma\varepsilon}{\beta} \ln\left(1 - 2(1-\varepsilon)t\right), & \text{for } t \in [0, 1/2], \\ 1 - d(1-t), & \text{for } t \in [1/2, 1]. \end{cases} \tag{2.5}$$

Here, σ will be determined later and d is the constant such that $\varphi(t)$ is continuous at $t = 1/2$.

Assume that $N \geq 4$ is an integer. We define the mesh points as

$$x_n = \varphi(t_n), \quad t_n = \frac{n}{N} \text{ for } n = 0, 1, \dots, N.$$

Figure 1 depicts the Bakhvalov-type mesh generated by the mesh generating function (2.5) on $[0, 1]$ with $\sigma = 3, \beta = 1$ and $\varepsilon = 10^{-2}$.

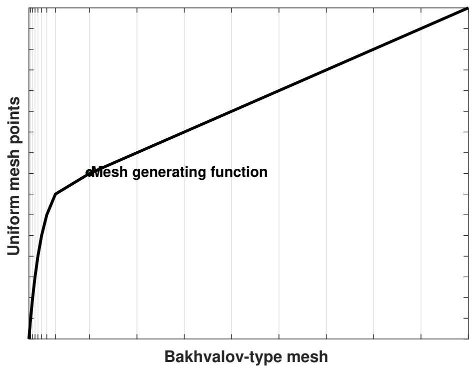


Figure 1. Bakhvalov-type mesh with transition point (denoted by green circle).

Denote the mesh by $I_n = [x_{n-1}, x_n]$ and the mesh size by $h_n = x_n - x_{n-1}$ for $n = 1, \dots, N$ and set $\mathcal{T}_N = \{I_n, n = 1, \dots, N\}$. For each interval $I_n \in \mathcal{T}_N$, the unit normal is defined as $\mathbf{n}_{I_n}(x_n) = 1$ and $\mathbf{n}_{I_n}(x_{n-1}) = -1$. For simplicity, we use the notation $\mathbf{n}$ rather than $\mathbf{n}_{I_n}$.

Note that the transition point is $x_{N/2} = \tau = -\frac{\sigma\varepsilon}{\beta} \ln\varepsilon$ and that $\exp(-x_{N/2}/\varepsilon) = \varepsilon^\sigma$. Thus, in order to resolve fully the boundary layer, we choose σ such that ε^σ is sufficiently small.

We have the following lemma on Bakhvalov-type mesh (2.5) which can be proved using the ideas in [35].

Lemma 2.2. *On Bakhvalov-type mesh (2.5), the mesh sizes* $h_n = x_n - x_{n-1}, n = 1, 2 \dots, N$ *have the following properties*

$$h_1 \leq h_2 \leq \cdots \leq h_{N/2-1}, \tag{2.6}$$

$$h_{N/2-1} = \mathcal{O}(\varepsilon), \tag{2.7}$$

$$C\varepsilon \leq h_{N/2} \leq CN^{-1}, \tag{2.8}$$

$$h_n = \mathcal{O}(N^{-1}), \quad n = N/2 + 1, \dots, N, \tag{2.9}$$

$$h_n^\nu \exp(-\beta x_{n-1}/\varepsilon) \leq C\varepsilon^\nu N^{-\nu} \quad \textit{for } 0 \leq \nu \leq k+1 \textit{ and } 1 \leq n \leq \frac{N}{2} - 1, \tag{2.10}$$

where C is a positive constant independent of N and ε.

Lemma 2.3. *On Bakhvalov-type mesh (2.5), the mesh sizes* $h_n = x_n - x_{n-1}, n = 1, 2 \dots, N$ *have the following lower bound*

$$x_{\frac{N}{2}-1} \geq -\frac{\sigma\varepsilon}{\beta}\ln(\varepsilon + 2N^{-1}), \tag{2.11}$$

$$x_n \geq -\frac{\sigma\varepsilon}{\beta}\ln(\varepsilon), \quad n = \frac{N}{2}, \dots, N, \tag{2.12}$$

$$h_n \geq C\varepsilon N^{-1}, \quad n = 1, 2, \dots, \frac{N}{2}. \tag{2.13}$$

where C *is a positive constant independent of* N *and* ε.

Proof. It follows from the definition of the mesh points (2.5) that

$$\begin{aligned} x_{\frac{N}{2}-1} &= -\frac{\sigma\varepsilon}{\beta}\ln(1 - 2(1-\varepsilon)(N/2-1)N^{-1}) \\ &= -\frac{\sigma\varepsilon}{\beta}\ln(1 + (1-\varepsilon)(2/N - 1)) \\ &= -\frac{\sigma\varepsilon}{\beta}\ln(\varepsilon + \frac{2}{N}(1-\varepsilon)) \\ &\geq -\frac{\sigma\varepsilon}{\beta}\ln(\varepsilon + \frac{2}{N}), \end{aligned}$$

which proves (2.11). Using (2.5), we get

$$x_n = 1 - d(1 - \frac{n}{N}), \text{ for } n = \frac{N}{2}, \dots, N. \tag{2.14}$$

Note that if $\frac{N}{2} \leq n \leq N$, then $\frac{1}{2} \leq \frac{n}{N} \leq 1$ holds true. This implies that $0 \leq 1 - \frac{n}{N} \leq \frac{1}{2}$. Since $d = 2(1 + \frac{\sigma\varepsilon}{\beta}\ln(\varepsilon))$ is a positive number for a given $\varepsilon > 0$, we are led to

$$-\frac{d}{2} \leq -d(1 - \frac{n}{N}) \leq 0. \tag{2.15}$$

Using (2.15) in (2.14), we obtain, for $n = \frac{N}{2}, \dots, N$

$$\begin{aligned} x_n &= 1 - d(1 - \frac{n}{N}) \\ &\geq 1 - \frac{d}{2} = 1 - (1 + \frac{\sigma\varepsilon}{\beta}\ln(\varepsilon)) \\ &= -\frac{\sigma\varepsilon}{\beta}\ln(\varepsilon), \end{aligned}$$

which proves (2.12). From Lemma 2.2 and (2.5), we have

$$h_n \geq h_1 = x_1 - x_0 = -\frac{\sigma\varepsilon}{\beta}\ln(1 - 2(1-\varepsilon)N^{-1}) \text{ for } n = 1, \dots, \frac{N}{2}. \tag{2.16}$$

Since $\varepsilon \ll 1$, we have $1 - \frac{2}{N}(1-\varepsilon) \leq 1 - \frac{1}{N}$ (e.g., $\varepsilon \leq \frac{1}{2}$ which is realistic in the SPPs). Using the basic fact that if $0 < 1 - x \leq 1 - y \leq 1$, then $-\ln(1-y) \leq -\ln(1-x)$, we obtain

$$-\ln(1 - \frac{1}{N}) \leq -\ln(1 - \frac{2}{N}(1-\varepsilon)). \tag{2.17}$$

Let $f(x) := x + \ln(1-x)$ for $x \in (0,1)$. Then $f'(x) = -\frac{x}{1-x} < 0$ for $x \in (0,1)$. This shows that f is a decreasing function on $(0,1)$. Thus we have $x + \ln(1-x) = f(x) \leq f(0) = 0$ when $x \in (0,1)$, or equivalently, we obtain $-\ln(1-x) \geq x$ if $0 < x < 1$. Hence, it follows that $-\ln(1 - \frac{1}{N}) \geq \frac{1}{N}$ for $N \geq 4$. This last inequality, (2.17) and (2.16) imply that $h_n \geq C\varepsilon N^{-1}$ which proves (2.13). Thus, the proof is completed. □

2.3. WG-FEM for convection-diffusion problems

In this subsection, we introduce the notions of weak functions and weak derivatives. Then, we will construct the WG-FEM scheme for the problem (2.1) based on the weak derivatives.

A function $u = \{u_0, u_b\}$ on the interval $I_n = (x_{n-1}, x_n)$ is called a weak function such that $u_0 \in L^2(I_n)$ and $u_b \in L^\infty(\partial I_n)$ with $\partial I_n = \{x_{n-1}, x_n\}$. Here, u_0 is the value of u inside of the interval (x_{n-1}, x_n) and u_b is the value of u on the boundary of the interval ∂I_n which can be different from the trace of u_0 on the boundary.

We denote the space of weak functions $\mathcal{W}(I_n)$ on the interval I_n by

$$\mathcal{W}(I_n) = \{u = \{u_0, u_b\} : u_0 \in L^2(I_n), v_b \in L^\infty(\partial I_n)\}.$$

The inclusion map

$$\mathcal{I}_{\mathcal{W}}(u) = \{u|_{I_n}, u|_{\partial I_n}\}, \quad \forall u \in H^1(I_n)$$

embeds the local Sobolev space $H^1(I_n)$ into the weak function space $\mathcal{W}(I_n)$.

For a given integer $k \geq 1$, we define a local WG finite element space $S_N(I_n)$ as follows:

$$S_N(I_n) = \{u = \{u_0, u_b\} : u_0|_{I_n} \in \mathbb{P}_k(I_n), u_b|_{\partial I_n} \in \mathbb{P}_0(\partial I_n) \quad \forall I_n \in \mathcal{T}_N\}, \tag{2.18}$$

where $\mathbb{P}_k(I_n)$ is the set of polynomials on I_n of degree at most k and $\mathbb{P}_0(\partial I_n)$ is the set of constant polynomials on ∂I_n. A global WG finite element space S_N consists of $u = \{u_0, u_b\}$ such that $u_0|_{I_n} \in \mathbb{P}_k(I_n)$ for $n = 1, \ldots, N$ and u_b has a single value at the nodes x_n of the partition $\mathcal{T}_N$. Let S_N^0 denote the subspace of S_N defined by

$$S_N^0 = \{u = \{u_0, u_b\} : u \in S_N, u_b(0) = u_b(1) = 0\}. \tag{2.19}$$

Now, we define the weak derivative of a weak function $u = \{u_0, u_b\} \in S_N$ as follows.

Definition 2.4. For any weak function $u \in S_N(I_n)$, the **weak derivative** $d_{w,I_n} u \in \mathbb{P}_{k-1}(I_n)$ of $u = \{u_0, u_b\}$ is defined on I_n as the unique polynomial satisfying the following equation

$$(d_{w,I_n} u, v)_{I_n} = -(u_0, v')_{I_n} + \langle u_b, v\mathbf{n}\rangle_{\partial I_n}, \qquad \forall v \in \mathbb{P}_{k-1}(I_n), \tag{2.20}$$

where

$$(w, z)_{I_n} = \int_{I_n} w(x)z(x)\,dx, \text{ and } \langle w, z\mathbf{n}\rangle_{\partial I_n} = w(x_n)z(x_n) - w(x_{n-1})z(x_{n-1}).$$

We also define a weak convection derivative for approximating the convection part $\beta u'$ as follows.

Definition 2.5. For any weak function $u \in S_N(I_n)$, the **weak convection derivative** $d^b_{w,I_n} u \in \mathbb{P}_k(I_n)$ of $u = \{u_0, u_b\}$ is defined on I_n as the unique polynomial satisfying the following equation

$$(d^b_{w,I_n} u, v)_{I_n} = -(u_0, (\beta v)')_{I_n} + \langle u_b, \beta v\mathbf{n}\rangle_{\partial I_n} \qquad \forall v \in \mathbb{P}_k(I_n). \tag{2.21}$$

Then the weak derivatives $d_w u$ and d^b_w of a weak function u on S_N is given by

$$(d_w u)|_{I_n} = d_{w,I_n}(u|_{I_n}), \quad (d^b_w u)|_{I_n} = d^b_{w,I_n}(u|_{I_n}), \forall u \in S_N.$$

We adapt the following notations for the sake of simplicity:

$$(u, v)_{I_n} = \int_{I_n} u(x)v(x)\,dx, \quad \langle u, v\rangle_{\partial I_n} = u(x_n)v(x_n) + u(x_{n-1})v(x_{n-1})$$

$$(u, v) = \sum_{n=1}^{N} (u, v)_{I_n}, \quad \langle u, v\rangle = \sum_{n=1}^{N} \langle u, v\rangle_{\partial I_n}, \quad \|u\|^2 = \sum_{n=1}^{N} (u, u)^2_{I_n}.$$

With the aid of the above definitions, the WG-FEM solution of the problem (2.1) is to find an approximate solution $u_N = \{u_0, u_b\} \in S_N^0$ satisfying the following equation [39]:

$$a(u_N, v_N) = (g, v_0), \quad \forall v_N = \{v_0, v_b\} \in S_N^0, \tag{2.22}$$

where the bilinear form $a(v_N, v_N)$ is defined as follows: for any $v_N \in S_N$

$$a(u_N, v_N) = b(u_N, v_N) + s_d(u_N, v_N) + s_c(u_N, v_N), \tag{2.23}$$

$$b(u_N, v_N) = \varepsilon(d_w u_N, d_w v_N) + (cu_0 - d_w^b u_N, v_0), \tag{2.24}$$

$$s_d(u_N, v_N) = \sum_{n=1}^{N} \langle \varrho_n(u_0 - u_b), (v_0 - v_b)\rangle_{\partial I_n}, \tag{2.25}$$

$$s_c(u_N, v_N) = \sum_{n=1}^{N} \langle b\mathbf{n}(u_0 - u_b), v_0 - v_b\rangle_{\partial_- I_n}, \tag{2.26}$$

where $\partial_- I_n = \{x \in \partial I_n : b(x)\mathbf{n} \le 0\}$, and $\varrho_n \ge 0, n = 1, \dots N$ is the penalization parameter given by

$$\varrho_n = \begin{cases} N, & \text{for } n = 1, 2, \dots, N/2, \\ 1, & \text{for } n = N/2 + 1, \dots, N. \end{cases} \tag{2.27}$$

The choice of the penalization parameter is an important issue in the uniform error analysis below. For the uniform convergence of the proposed method, the penalization parameter is taken as N in the fine part of the domain, see Lemma 2.13.

2.4. Stability of the WG-FEM

The following multiplicative trace inequality and the inverse inequality from [20] will be used frequently in the analysis.

$$\|v\|^2_{L^2(\partial I_n)} \le C(h_n^{-1}\|v\|^2_{L^2(I_n)} + \|v\|_{L^2(I_n)}\|v'\|_{L^2(I_n)}), \quad \forall v \in H^1(I_n), \tag{2.28}$$

$$\|v_N'\|_{L^2(I_n)} \le Ch_n^{-1}\|v_N\|_{L^2(I_n)}, \quad \forall v_N \in \mathbb{P}_k(I_n), \tag{2.29}$$

$$\|v_N\|_{L^p(\partial I_n)} \le Ch_n^{-1/p}\|v_N\|_{L^p(I_n)}, \quad \forall 1 \le p \le \infty, \forall v_N \in \mathbb{P}_k(I_n). \tag{2.30}$$

Following [39], we introduce an energy norm $|||\cdot|||_\varepsilon$ in S_N as follows: for $v_N = \{v_0, v_b\} \in S_N$,

$$|||v_N|||^2_\varepsilon = \varepsilon||d_w v_N\|^2 + \|\gamma v_0\|^2 + |v_N|^2_u + s_d(v_N, v_N), \tag{2.31}$$

where the seminorm $|\cdot|_u$ is given by

$$|v_N|^2_u := \sum_{n=0}^{N-1} c_n|\sqrt{b}(v_0 - v_b)|^2(x_n^+),$$

with $c_n = \begin{cases} \frac{1}{2}, & n = 0, \\ 1, & n = 1, \dots, N-1, \end{cases}$ and $x_n^+ = \lim_{t\to 0, t>0} u(x_n + t)$.

We also introduce the discrete H^1 energy norm $||\cdot||_S$ in $S_N + H_0^1(\Omega)$ defined as

$$\|v_N\|^2_S = \varepsilon\|Dv_0\|^2 + \|\gamma v_0\|^2 + |v_N|^2_u + s_d(v_N, v_N), \tag{2.32}$$

where $Dw := \dfrac{dw}{dx}$ which is occasionally denoted by w' is the ordinary derivatives of a functions $w(x)$.

We point out that a function $w \in H_0^1(\Omega)$ can be interpreted as a weak function $w = \{w_0, w_b\}$ with $w_0 = w|_{I_n}$ and $w_b = w|_{\partial I_n}$ for I_n.

The following lemmas show that the norms $|||\cdot|||_\varepsilon$ and $||\cdot||_S$ defined by (2.31) and (2.32), respectively, are equivalent in the WG finite element space S_N and the bilinear form $a(\cdot,\cdot)$ given in (2.22) is coercive in the $|||\cdot|||_\varepsilon$-norm.

Theoretically, the preferred norm does not affect the measured errors much. However, we take advantages of the norm defined by (2.32) numerically. The H^1-seminorm involves only the interior value u_0 of the weak function u_N, therefore computing the errors in this norm is less expensive.

Lemma 2.6 ([39]). *Let $v_N=\{v_0,v_b\}\in S_N^0$. Then there are two positive constants C_l and C_s such that*

$$C_l|||v_N|||_\varepsilon \le |||v_N||_S \le C_s|||v_N|||_\varepsilon. \tag{2.33}$$

Proof. The proof is similar to the ones given in Lemma 3.1 and Lemma 3.2 of [39], so we omit the proof here. □

Lemma 2.7. *Let $v_N=\{v_0,v_b\}\in S_N^0$. Then there is a positive constant C such that*

$$a(v_N,v_N)\ge |||v_N|||_\varepsilon^2. \tag{2.34}$$

Proof. For any $v_N=\{v_0,v_b\}, w_N=\{w_0,w_b\}\in S_N^0$, it follows from the definition of the weak convection derivative (2.21) and integration by parts that

$$\begin{aligned}-(d_w^b v_N,w_0) &= (v_0,(bw_0)')-\langle v_b,b\mathbf{n}w_0\rangle\\ &= -(bv_0',w_0)+\langle b\mathbf{n}(v_0-v_b),w_0\rangle,\end{aligned} \tag{2.35}$$

and

$$\begin{aligned}-(d_w^b w_N,v_0) &= (w_0,(bv_0)')-\langle w_b,b\mathbf{n}v_0\rangle\\ &= (w_0,(bv_0)')-\langle w_b,b\mathbf{n}(v_0-v_b)\rangle,\end{aligned} \tag{2.36}$$

where we have used the fact that

$$\begin{aligned}\langle b\mathbf{n}v_b,w_b\rangle &= \sum_{n=1}^{N}[(bv_bw_b)(x_n)-(bv_bw_b)(x_{n-1})]\\ &= (bv_bw_b)(1)-(bv_bw_b)(0)=0,\end{aligned}$$

because v_b and w_b are well-defined at the inter-boundaries and $v_N,w_N\in S_N^0$, that is, $v_b(1)=w_b(1)=v_b(0)=w_b(0)=0$.

Taking $v_N=w_N$ and summing together (2.35) and (2.36), we get

$$-(d_w^b v_N,v_0)=\frac{1}{2}(b'v_0,v_0)+\frac{1}{2}\langle b\mathbf{n}(v_0-v_b),v_0-v_b\rangle. \tag{2.37}$$

We can easily derive the following relation

$$s_c(v_N,v_N)+\frac{1}{2}\langle b\mathbf{n}(v_0-v_b),v_0-v_b\rangle=|v_N|_u^2.$$

The above equality, (2.37) and the assumption (2.2) reveal that

$$-(d_w^b v_N+cv_0,v_0)+s_c(v_N,v_N)=((c+\frac{1}{2}b')v_0,v_0)+|v_N|_u^2\ge\|\gamma v_0\|^2+|v_N|_u^2. \tag{2.38}$$

The definition of the bilinear form $a(\cdot,\cdot)$ and (2.38) lead to

$$a(v_N,v_N)\ge C\Big(\varepsilon(d_wv_N,d_wv_N)+\|\gamma v_0\|^2+|v_N|_u^2+s_d(v_N,v_N)\Big)=C|||v_N|||_\varepsilon^2.$$

We complete the proof.

□

In light of Lemma 2.7 and the bilinear form (2.22) , we deduce that

$$|||u_N|||_\varepsilon\le C||g||,$$

which in turn implies the discrete problem (2.22) has a unique solution. The existence follows from the uniqueness.

As a result of Lemma 2.6 and Lemma 2.7, we conclude that the bilinear form $a(\cdot,\cdot)$ is also coercive in the energy like norm $||\cdot||_S$ defined by (2.32).

Lemma 2.8. *Let $v_N=\{v_0,v_b\}\in S_N^0$. Then there is a positive constant C such that*

$$a(v_N,v_N)\ge C||v_N||_S^2. \tag{2.39}$$

2.5. Error analysis of the WG-FEM on Bakhvalov-type mesh

In general, the standard Lagrange interpolation is used in error estimates of FEM for SPPs. In [23], the author pointed out the Lagrange interpolation on Bakhvalov-type meshes leads to instability in some part of the mesh intervals in the error analysis of convection dominated problems. Recently, Brdar and Zarin studied FEM for SPPs using a Clément quasi-interpolant operator in [3]. Unfortunately, the analysis is limited to only linear FEM and can not be applied to higher order methods in [3]. We consider a special interpolation operator introduced by Zhang and Liu [35] for our uniform error analysis. This interpolation operator is defined as follows: Write $x_n^m := x_n + \frac{m}{k}h_{n+1}$ for $n = 0, \dots, N-1$ and $m = 0, \dots, k-1$ and set $x_N^0 = x_N$. We define the interpolation $\mathcal{P}u$ of the solution u based on the regularity (2.3) of the solution u as

$$\mathcal{P}u := R^I + \pi L, \tag{2.40}$$

where R^I is the standard interpolation of R given by

$$R^I(x) = \sum_{n=0}^{N} R(x_n^0)\theta_n^0(x) + \sum_{n=0}^{N-1}\sum_{m=1}^{k-1} R(x_n^m)\theta_n^m(x),$$

and

$$\pi L(x) = \sum_{n=0, n\neq \frac{N}{2}}^{N} L(x_n^0)\theta_n^0(x) + \sum_{n=0, n\neq \frac{N}{2}-1}^{N-1}\sum_{m=1}^{k-1} L(x_n^m)\theta_n^m(x), \tag{2.41}$$

where θ_n and θ_n^m are the piecewise nodal basis functions with respect to nodes x_n and x_n^m, respectively.

The well-known interpolation result [4, Theorem 3.1.4] states that for $k = 1, 2, \dots,$

$$|w - w^I|_{H^l(I_n)} \leq C h_n^{k+1-l} |w|_{H^{k+1}(I_n)}, \quad \forall w \in H^{k+1}(I_n), \quad 0 \leq l \leq k+1, \tag{2.42}$$

$$\|w - w^I\|_{L^\infty(I_n)} \leq C h_n^{k+1} |w|_{W^{k+1,\infty}(I_n)}, \quad \forall w \in W^{k+1,\infty}(I_n), \tag{2.43}$$

where $W^{k+1,\infty}(I_n)$ is the standard Sobolev spaces.

Clearly, $\mathcal{P}u$ is continuous on I_n and the weak function $\{\mathcal{P}u|_{I_n}, \mathcal{P}u|_{\partial I_n}\}$ which is again denoted by $\mathcal{P}u$ belongs to S_N.

Let us define the operator

$$\mathbf{P}L(x) = L(x_{\frac{N}{2}}^0)\theta_{\frac{N}{2}}^0(x) + \sum_{m=1}^{k-1} L(x_{\frac{N}{2}-1}^m)\theta_{\frac{N}{2}-1}^m(x).$$

Observe that the operators $\mathcal{P}, \pi$ and $\mathbf{P}$ are related as follows [35]:

$$\pi L(x) = L^I - \mathbf{P}L(x), \quad \mathcal{P}u = u^I - \mathbf{P}L(x), \tag{2.44}$$

$$\pi L|_{[x_0, x_{N/2-1}]\cup[x_{N/2}, x_N]} = L^I|_{[x_0, x_{N/2-1}]\cup[x_{N/2}, x_N]}, \tag{2.45}$$

where L^I and u^I are the standard Lagrange interpolation of L and the solution u, respectively. Observe that $\pi L(x)$ is the Lagrange interpolation of L except on the problematic region which is contained in the last interval of the fine part and is neighbor to the coarse part of the Bakhvalov-type mesh defined by (2.5) while $\mathbf{P}L(x)$ is the redefined standard Lagrange interpolation on the problematic region. This explains the equation (2.44). Since the standard Lagrange interpolation of L and the interpolation operator $\pi L(x)$ agree expect the problematic region, the equation (2.45) follows.

Since $\mathbf{P}L$ is a continuous operator we have $\|\mathbf{P}L\|_S^2 = \varepsilon\|\mathbf{P}L'\|^2 + \|\gamma\mathbf{P}L\|^2$. Hence we have the following result.

Lemma 2.9 ([36]). *Assume that $\sigma \geq k+1$. Then there holds*

$$\|\mathbf{P}L\|_S \leq CN^{-(k+1)}. \tag{2.46}$$

Lemma 2.10. *Assume that $\sigma \geq k+1$. On Bakhvalov-type mesh (2.5), we have*

$$\|(R-R^I)^{(l)}\|_{L^2(\Omega)} \leq CN^{l-(k+1)}, \quad l=0,1,2,\ldots, \tag{2.47}$$

$$\|L-L^I\|_{L^\infty(\Omega)} + \|R-R^I\|_{L^\infty(\Omega)} + \|u-u^I\|_{L^\infty(\Omega)} \leq CN^{-(k+1)}, \tag{2.48}$$

$$\|L-L^I\|_{L^2(\Omega)} + \|R-R^I\|_{L^2(\Omega)} + \|u-u^I\|_{L^2(\Omega)} \leq CN^{-(k+1)}, \tag{2.49}$$

$$\sum_{n=1}^{N/2} \|(L-L^I)^{(l)}\|_{L^2(I_n)} \leq C\varepsilon^{1/2-l}N^{l-(k+1)}, \quad l=1,2, \tag{2.50}$$

$$\sum_{n=N/2+1}^{N} \|(L-L^I)^{(l)}\|_{L^2(I_n)} \leq CN^{l-(k+1)}, \quad l=1,2. \tag{2.51}$$

Proof. The first estimate follows from the interpolation bounds (2.42) and the fact that $h_n \leq CN^{-1}$ for $n=1,\ldots,N$.

From (2.43) and (2.4), one can show that for $1 \leq n \leq N/2-1$

$$\|L-L^I\|_{L^\infty(I_n)} \leq Ch_n^{k+1}|L^{(k+1)}|_{L^\infty(I_n)} \leq C\varepsilon^{-(k+1)}h_n^{k+1}e^{-\beta x_{n-1}/\varepsilon} \leq CN^{-(k+1)}, \tag{2.52}$$

where (2.10) with $\nu = k+1$ is used in the last step. For $N/2 \leq n \leq N$, using the fact that $\|L^I\|_{L^\infty(I_n)} \leq C\|L\|_{L^\infty(I_n)}$, (2.4), (2.11) and (2.12) we obtain

$$\|L-L^I\|_{L^\infty(I_n)} \leq \|L\|_{L^\infty(I_n)} + \|L^I\|_{L^\infty(I_n)} \leq Ce^{-\beta x_{n-1}/\varepsilon} \leq CN^{-\sigma}, \tag{2.53}$$

where $\varepsilon \leq N^{-1}$ is used. Combining (2.52) and (2.53) and using the fact $\sigma \geq k+1$, we conclude that

$$\|L-L^I\|_{L^\infty(\Omega)} \leq CN^{-(k+1)}. \tag{2.54}$$

Similarly, using the fact that $h_n \leq CN^{-1}$ for $n=1,\ldots,N$ and $|R^{(k+1)}| \leq C$, we have $\|R-R^I\|_{L^\infty(\Omega)} \leq Ch_n^{k+1}|R^{(k+1)}|_{L^\infty(\Omega)} \leq CN^{-(k+1)}$. Collecting this estimate, (2.54) and (2.3) give (2.48). Holder inequalities and (2.48) imply the estimate (2.49).

Using again (2.42), (2.4) and (2.10) with $\nu = k+3/2-l$ for $l=1,2$, we obtain

$$\begin{aligned}
\sum_{n=1}^{N/2-1} \|(L-L^I)^{(l)}\|^2_{L^2(I_n)} &\leq C\sum_{n=1}^{N/2-1} h_n^{2(k+1-l)}|L|^2_{H^{k+1}(I_n)} \\
&\leq C\sum_{n=1}^{N/2-1} h_n^{2(k+3/2-l)}|L^{(k+1)}|^2_{L^\infty(I_n)} \\
&\leq C\sum_{n=1}^{N/2-1} h_n^{2(k+3/2-l)}\varepsilon^{-2(k+1)}\exp(-2\beta x_{n-1}/\varepsilon) \\
&= C\varepsilon^{-2(k+1)}\sum_{n=1}^{N/2-1}\left(h_n^{k+3/2-l}\exp(-\beta x_{n-1}/\varepsilon)\right)^2 \\
&\leq C\varepsilon^{-2(k+1)}\sum_{n=1}^{N/2-1}\varepsilon^{2(k+3/2-l)}N^{-2(k+3/2-l)} \\
&\leq C\varepsilon^{1-2l}N^{-2(k+1-l)},
\end{aligned}$$

where we use inclusion relationships among the $L^p(I_n)$ spaces in the second inequality since I_n has a finite measure. More precisely, $\|L^{(k+1)}\|^2_{L^2(I_n)} \leq h_n\|L^{(k+1)}\|^2_{L^\infty(I_n)}$.

For $n=N/2$, using the triangle inequality, the inverse estimate, (2.4) and Lemma 2.2 we obtain

$$\begin{aligned}\|(L-L^I)^{(l)}\|^2_{L^2(I_{N/2})} &\le C\|L^{(l)}\|^2_{L^2(I_{N/2})} + C\|(L^I)^{(l)}\|^2_{L^2(I_{N/2})}\\ &\le C\int_{N/2-1}^{N/2}(L^{(l)})^2\,dx + Ch_{N/2}^{-2l}\|L^I\|^2_{L^2(I_{N/2})}\\ &\le C\int_{N/2-1}^{N/2}\varepsilon^{-2l}\exp(-2\beta x/\varepsilon)\,dx + Ch_{N/2}^{-2l+1}\|L^I\|^2_{L^\infty(I_{N/2})}\\ &\le C\varepsilon^{1-2l}\exp(-2\beta x_{N/2-1}/\varepsilon) + Ch_{N/2}^{-2l+1}\exp(-2\beta x_{N/2-1}/\varepsilon)\\ &\le C\varepsilon^{1-2l}N^{-2\sigma} \le C\varepsilon^{1-2l}N^{-2(k+1)},\end{aligned}$$

where we use inclusion relationships among the $L^p(I_n)$ spaces in the third inequality and the fact that $\|L^I\|_{L^\infty(I_{N/2})} \le \|L\|_{L^\infty(I_{N/2})}$ in the fourth inequality.

For $\frac{N}{2}+1 \le n \le N$, using (2.42), (2.4) and Lemma 2.2, we arrive at

$$\begin{aligned}\sum_{n=\frac{N}{2}+1}^{N}\|(L-L^I)^{(l)}\|^2_{L^2(I_n)} &\le C\sum_{n=\frac{N}{2}+1}^{N} h_n^{2(k+1-l)}|L|^2_{H^{k+1}(I_n)}\\ &\le C\sum_{n=\frac{N}{2}+1}^{N} h_n^{2(k+3/2-l)}|L^{(k+1)}|^2_{L^\infty(I_n)}\\ &\le C\sum_{n=\frac{N}{2}+1}^{N} h_n^{2(k+3/2-l)}\varepsilon^{-2(k+1)}\exp(-2\beta x_{n-1}/\varepsilon)\\ &\le C\sum_{n=\frac{N}{2}+1}^{N} N^{-2(k+3/2-l)}\varepsilon^{-2(k+1)}\exp(-2\beta x_{N/2}/\varepsilon)\\ &\le CN^{-2(k+1-l)}\varepsilon^{-2(k+1)}\varepsilon^{2\sigma} \le CN^{-2(k+1-l)},\end{aligned}$$

where we use inclusion relationships among the $L^p(I_n)$ spaces in the second inequality. Thus we complete the proof. □

Lemma 2.11. *Assume that $\sigma \ge k+1$. Then there holds*

$$\|\pi L - L\|_S \le CN^{-k}. \tag{2.55}$$

Proof. Since πL is continuous we have $\|\pi L - L\|^2_S = \varepsilon\|(\pi L - L)'\|^2 + \|\gamma(\pi L - L)\|^2$. From (2.44) we get

$$\|\pi L - L\|^2_S \le \varepsilon\|(L-L^I)'\|^2 + \|\gamma(L-L^I)\|^2 + \|\mathbf{P}L\|^2_S.$$

Due to Lemma 2.9 and the inequalities (2.50) and (2.51), we get

$$\|\pi L - L\|_S \le CN^{-k}.$$

The proof is now completed. □

Now, we will derive the following error equations that will be play an essential role in the error analysis in the sequel.

Lemma 2.12. *Let u be the solution of the problem (2.1). Then for any $v_N = \{v_0, v_b\} \in S_N^0$, we have the following error equations*

$$-\varepsilon(u'', v_0) = \varepsilon(d_w(\mathcal{P}u), d_w v_N) - T_1(u, v_N), \tag{2.56}$$

$$(cu, v_0) = (c\mathcal{P}u, v_0) - T_2(u, v_N), \tag{2.57}$$

$$-(bu', v_0) = -(d_w^b(\mathcal{P}u), v_0) - T_3(u, v_N), \tag{2.58}$$

where

$$T_1^N(u, v_N) = -\varepsilon((u - \mathcal{P}u)', v_0') + \varepsilon\langle u' - (\mathcal{P}u)', (v_0 - v_b)\boldsymbol{n}\rangle, \tag{2.59}$$

$$T_2(u, v_N) = ((u - \mathcal{P}u), (bv_0)') + \langle u - \mathcal{P}u, bv_0\boldsymbol{n}\rangle, \tag{2.60}$$

$$T_3(u, v_N) = (c(\mathcal{P}u - u), v_0), \tag{2.61}$$

and $\mathcal{P}$ is defined by (2.40)

Proof. Notice that the interpolation operator $\mathcal{P}$ is continuous on I_n, that is, $\mathcal{P}u \in C(I_n)$ for any $u \in H^1(I_n)$, $n = 1, \ldots, N$. Thus, the weak derivative of the interpolation operator $d_w(\mathcal{P}u)$ is equivalent to the classical derivative, i.e., $d_w(\mathcal{P}u) = (\mathcal{P}u)'$ for any $u \in H^1(I_n)$. Therefore, the commutativity property in [39, Lemma 3.4] holds true for the interpolation operator $\mathcal{P}$ defined by (2.40). The rest of the proof is similar to the proofs of Lemma 3.5 and Lemma 3.6 in [39] where a special interpolation is used on the uniform Shishkin mesh. To avoid a repeat, we refer to the reader to [39]. □

Lemma 2.13. *Assume that $u \in H^{k+1}(\Omega)$ and ϱ_n is given by (2.27) and $\sigma \geq k+1$. Denote R^I by the standard Lagrange interpolation R and πL denotes the interpolation defined by (2.41). Then the interpolation $\mathcal{P}u = R^I + \pi L$ satisfies the following bound*

$$\left\{\sum_{n=1}^{N} \frac{\varepsilon^2}{\varrho_n} \|(u - \mathcal{P}u)'\|^2_{L^2(\partial I_n)}\right\}^{1/2} \leq CN^{-k}.$$

Proof. From (2.44), we have $u - \mathcal{P}u = u - u^I + \mathbf{P}L$. Owing to the triangle inequality and the inverse inequality (2.29)

$$\begin{aligned}\|(u - \mathcal{P}u)'\|_{L^2(\partial I_n)} &\leq \|(u - u^I)'\|_{L^2(\partial I_n)} + \|(\mathbf{P}L)'\|_{L^2(\partial I_n)} \\ &\leq \|(u - u^I)'\|_{L^2(\partial I_n)} + Ch_n^{-1}\|\mathbf{P}L\|_{L^2(I_n)}.\end{aligned}$$

Recalling (2.27), one has

$$\frac{\varepsilon h_n^{-1}}{\varrho_n} \leq C \quad \text{for } n = 1, \ldots, N.$$

Then, by Lemma 2.9 we have

$$\begin{aligned}\sum_{n=1}^{N} \frac{\varepsilon^2}{\varrho_n} \|(u - \mathcal{P}u)'\|^2_{L^2(\partial I_n)} &\leq C\Big(\sum_{n=1}^{N} \frac{\varepsilon^2}{\varrho_n} \|(u - u^I)'\|^2_{L^2(\partial I_n)} + \sum_{n=1}^{N} \frac{\varepsilon^2}{\varrho_n} h_n^{-2} \|\mathbf{P}L\|^2_{L^2(I_n)}\Big) \\ &\leq C\Big(\sum_{n=1}^{N} \frac{\varepsilon^2}{\varrho_n} \|(u - u^I)'\|^2_{L^2(\partial I_n)} + N^{-2(k+1)}\Big).\end{aligned} \tag{2.62}$$

It remains to bound the first term on the RHS of the above inequality. For the sake of simplicity, we write $u - u^I = \xi = \xi_R + \xi_L$ with $\xi_R = R - R^I$ and $\xi_L = L - L^I$.

The triangle inequality implies that

$$\sum_{I_n \in \mathcal{T}_N} \frac{\varepsilon^2}{\varrho_n} \|\xi'\|^2_{L^2(\partial I_n)} \leq \sum_{I_n \in \mathcal{T}_N} \frac{\varepsilon^2}{\varrho_n} \Big(\|\xi_R'\|^2_{L^2(\partial I_n)} + \|\xi_L'\|^2_{L^2(\partial I_n)}\Big). \tag{2.63}$$

The trace inequality (2.28) states that

$$\|\xi_R'\|^2_{L^2(\partial I_n)} \leq h_n^{-1} \|\xi_R'\|^2_{L^2(I_n)} + \|\xi_R'\|_{L^2(I_n)} \|\xi_R''\|_{L^2(I_n)}. \tag{2.64}$$

From (2.47), Lemma 2.3 and the penalty parameter (2.27), we get

$$\begin{aligned}\sum_{n=1}^{N}\frac{\varepsilon^2}{\varrho_n}\|\xi_R'\|^2_{L^2(\partial I_n)} &\le C\sum_{n=1}^{N}\frac{\varepsilon^2}{\varrho_n}(h_n^{-1}\|\xi_R'\|^2_{L^2(I_n)}+\|\xi_R'\|_{L^2(I_n)}\|\xi_R''\|_{L^2(I_n)})\\ &\le C\Big(\varepsilon\sum_{n=1}^{N/2}\|\xi_R'\|^2_{L^2(I_n)}+\varepsilon^2N\sum_{n=N/2+1}^{N}\|\xi_R'\|^2_{L^2(I_n)}\\ &\quad+\varepsilon^2N^{-1}\sum_{n=1}^{N/2}\|\xi_R'\|_{L^2(I_n)}\|\xi_R''\|_{L^2(I_n)}\\ &\quad+\varepsilon^2\sum_{n=N/2+1}^{N}\|\xi_R'\|_{L^2(I_n)}\|\xi_R''\|_{L^2(I_n)}\Big)\\ &\le C\varepsilon N^{-2k},\end{aligned} \tag{2.65}$$

where we have used that $\varepsilon N < 1$.

Next, using the same argument along with the inequalities (2.50) and (2.51) we arrive at

$$\begin{aligned}\sum_{n=1}^{N}\frac{\varepsilon^2}{\varrho_n}\|\xi_L'\|^2_{L^2(\partial I_n)} &\le C\sum_{n=1}^{N}\frac{\varepsilon^2}{\varrho_n}(h_n^{-1}\|\xi_L'\|^2_{L^2(I_n)}+\|\xi_L'\|_{L^2(I_n)}\|\xi_L''\|_{L^2(I_n)})\\ &\le C\Big(\varepsilon\sum_{n=1}^{N/2}\|\xi_L'\|^2_{L^2(I_n)}+\varepsilon^2N\sum_{n=N/2+1}^{N}\|\xi_L'\|^2_{L^2(I_n)}\\ &\quad+\varepsilon^2N^{-1}\sum_{n=1}^{N/2}\|\xi_L'\|_{L^2(I_n)}\|\xi_L''\|_{L^2(I_n)}\\ &\quad+\varepsilon^2\sum_{n=N/2+1}^{N}\|\xi_L'\|_{L^2(I_n)}\|\xi_L''\|_{L^2(I_n)}\Big)\\ &\le C\Big(N^{-2k}+\varepsilon N^{-2k}+N^{-2k}+\varepsilon N^{-2k}\Big)\\ &\le CN^{-2k},\end{aligned} \tag{2.66}$$

where again the fact that $\varepsilon N < 1$ is used.

Combining the inequalities (2.65) and (2.66) gives

$$\sum_{I_n\in\mathcal{T}_N}\frac{\varepsilon^2}{\varrho_n}\|\xi'\|^2_{L^2(\partial I_n)} \le CN^{-2k}.$$

This last inequality and (2.62) give the desired result. Thus, we complete the proof. □

We will derive an error equation for the discretization error $\eta = \mathcal{P}u - u_N$ which will be used in the error analysis below.

Lemma 2.14. *Let u and $u_N \in S_N^0$ be the exact solution and the WG-FEM solution of problem (2.1) and (2.22) on Bakhvalov-type mesh (2.5), respectively. Then we have the following error equation for $\eta = \mathcal{P}u - u_N$*

$$a(\mathcal{P}u - u_N, v_N) = T(u, v_N), \quad \forall v_N \in S_N^0, \tag{2.67}$$

where $T(u,v_N) = T_1(u,v_N)+T_2(u,v_N)+T_3(u,v_N)$. Here, $T_1(u,v_N), T_2(u,v_N)$ and $T_3(u,v_N)$ are defined by (2.59), (2.60) and (2.61), respectively.

Proof. Testing (2.1) by the test functions $v_N = \{v_0, v_b\} \in S_N^0$, we obtain

$$-\varepsilon(u'', v_0) - (bu', v_0) + (cu, v_0) = (g, v_0). \tag{2.68}$$

Using the fact that $\mathcal{P}u$ is continuous in Ω, we have $s_c(\mathcal{P}u, v_N) = s_d(\mathcal{P}u, v_N) = 0$. Plugging the equations in (2.59), (2.60) and (2.61) into the above equation (2.68), we arrive at

$$a(\mathcal{P}u, v_N) = b(\mathcal{P}u, v_N) = (g, v_0) + T(u, v_N). \tag{2.69}$$

Subtracting (2.22) from the equation (2.69) yields the desired equation (2.67). Thus, the proof is now completed. □

Lemma 2.15. *Assume that $u \in H^{k+1}(\Omega)$ and the penalization parameter ϱ_n is given by (2.27). If $\sigma \geq k+1$, then we have, for any $v_N = \{v_0, v_b\} \in S_N^0$*

$$T(u, v_N) \leq CN^{-k}||v_N||_S, \tag{2.70}$$

where C is independent of N and ε.

Proof. In order to prove (2.70), we estimate $T_1(u, v_N), T_2(u, v_N)$ and $T_3(u, v_N)$ individually. By the triangle inequality, we obtain

$$|T_1^N(u, v_N)| \leq \varepsilon|((u - \mathcal{P}u)', v_0')| + \varepsilon|\langle(u - \mathcal{P}u)', (v_0 - v_b)\mathbf{n}\rangle| =: \mathcal{R}_1 + \mathcal{R}_2.$$

We first estimate $\mathcal{R}_1$. Using Cauchy-Schwarz inequality, (2.47), (2.50) and (2.51) of Lemma 2.10 and Lemma 2.9 , we arrive at

$$\begin{aligned} |\mathcal{R}_1| &\leq \varepsilon^{1/2}\|(u - u^I)'\|\varepsilon^{1/2}\|v_0'\| + \varepsilon\|(\mathbf{P}L)'\|\|v_0\| \\ &\leq C\Big[\varepsilon^{1/2}\Big(N^{-k} + \varepsilon^{-1/2}N^{-k} + N^{-k}\Big) + N^{-(k+1)}\Big]\|v_N\|_S \\ &\leq CN^{-k}\|v_N\|_S. \end{aligned} \tag{2.71}$$

For $\mathcal{R}_2$, it follows from the Cauchy-Schwarz inequality and Lemma 2.13 that

$$\begin{aligned} |\mathcal{R}_2| &\leq \sum_{n=1}^{N} \varepsilon|\langle(u - \mathcal{P}u)', (v_0 - v_b)\mathbf{n}\rangle_{\partial I_n}| \\ &\leq \sum_{n=1}^{N} \varepsilon\|(u - \mathcal{P}u)'\|_{L^2(\partial I_n)}\|v_0 - v_b\|_{L^2(\partial I_n)} \\ &\leq \left\{\sum_{n=1}^{N} \frac{\varepsilon^2}{\varrho_n}\|(u - \mathcal{P}u)'\|^2_{L^2(\partial I_n)}\right\}^{1/2} \left\{\sum_{n=1}^{N} \varrho_n\|v_0 - v_b\|^2_{L^2(\partial I_n)}\right\}^{1/2} \\ &\leq CN^{-k}s_d^{1/2}(v_N, v_N). \end{aligned} \tag{2.72}$$

As a result of (2.71) and (2.72), we have

$$|T_1(u, v_N)| \leq CN^{-k}||v_N||_S. \tag{2.73}$$

Let $T_2(u, v_N) = (u - \mathcal{P}u, (bv_0)') + \langle u - \mathcal{P}u, bv_0\mathbf{n}\rangle =: \mathcal{S}_1 + \mathcal{S}_2$. Using the relation (2.40), we have

$$\begin{aligned} \mathcal{S}_1 &= (u - \mathcal{P}u, (bv_0)') = (\xi_R, (bv_0)') + (L - \pi L, (bv_0)') \\ &= (\xi_R, (bv_0)') + ((L - \pi L), b'v_0) + ((L - \pi L), bv_0') \end{aligned}$$

where $\xi_R = R - R^I$.

Using the fact that $R^I = R$ on ∂I_n, integration by parts on the first term on the RHS of the above equation and the Cauchy-Schwarz inequality yield

$$(\xi_R, (bv_0)') + ((L - \pi L), b'v_0) \leq C\Big(\|\xi_R'\| + \|L - \pi L\|\Big)\|v_N\|_S.$$

Then, using (2.47) and (2.55) we obtain

$$(\xi_R, (bv_0)') + ((L - \pi L), b'v_0) \leq CN^{-k}\|v_N\|_S \leq CN^{-k}\|v_N\|_S. \tag{2.74}$$

The last term $((L-\pi L), bv_0')$ can be estimated as follows:

$$
\begin{aligned}
((L-\pi L), bv_0') &= \int_0^{\frac{N}{2}-2} b(L-L^I)v_0'\,dx + \int_{\frac{N}{2}-2}^{\frac{N}{2}} b(L-\pi L)v_0'\,dx \\
&\quad + \int_{\frac{N}{2}}^{N} b(L-L^I)v_0'\,dx \\
&:= Z_1 + Z_2 + Z_3,
\end{aligned}
$$

where we have used (2.45). The Hölder inequalities, the Cauchy-Schwarz inequality, (2.4) and (2.10) with $\nu = k+1$ imply that

$$
\begin{aligned}
|Z_1| &\le C \sum_{n=1}^{\frac{N}{2}-2} \int_{x_{n-1}}^{x_n} |L-L^I||v_0'|\mathrm{d}x \\
&\le C \sum_{n=1}^{\frac{N}{2}-2} \|L-L^I\|_{L^\infty(I_n)} \|v_0'\|_{L^1(I_n)} \\
&\le C \sum_{n=1}^{\frac{N}{2}-2} h_n^{k+1}\varepsilon^{-(k+1)}e^{-\beta x_i/\varepsilon} \cdot h_n^{1/2}\|v_0'\|_{L^2(I_n)} \\
&\le C\varepsilon^{1/2} \sum_{n=1}^{\frac{N}{2}-2} N^{-(k+1)}\|v_0'\|_{L^2(I_n)} \\
&\le C \left(\sum_{n=1}^{\frac{N}{2}-2} N^{-2(k+1)}\right)^{1/2} \left(\varepsilon \sum_{n=1}^{\frac{N}{2}-2} \|v_0'\|^2_{L^2(I_n)}\right)^{1/2} \\
&\le CN^{-(k+1/2)}\|v_N\|_S,
\end{aligned}
\tag{2.75}
$$

where we have used (2.6), (2.7) and (2.43).

The Cauchy-Schwarz inequality, the inverse inequality and (2.48) reveal that

$$
\begin{aligned}
|Z_3| &\le C\|L-L^I\|_{L^2([x_{N/2},x_N])}\|v_0'\|_{L^2([x_{N/2},x_N])} \\
&\le CN^{-(k+1)} \cdot N\|v_0\|_{L^2([x_{N/2},x_N])} \le CN^{-k}\|v_0\|.
\end{aligned}
\tag{2.76}
$$

On $I_{\frac{N}{2}-1} = [\frac{N}{2}-2, \frac{N}{2}-1]$, it follows from (2.41) that $\pi L(x) = L^I - L(x^0_{\frac{N}{2}-1})\theta^0_{\frac{N}{2}-1}(x)$. Thus, if $\sigma \ge k+1$ we arrive at

$$
\begin{aligned}
\left|\int_{x_{\frac{N}{2}-2}}^{x_{\frac{N}{2}-1}} b(L-\pi L)v_0'\mathrm{d}x\right| &\le C\Big(\int_{x_{\frac{N}{2}-2}}^{x_{\frac{N}{2}-1}} |L-L^I|\,|v_0'|\mathrm{d}x + |L(x_{\frac{N}{2}-1})|\int_{x_{\frac{N}{2}-2}}^{x_{\frac{N}{2}-1}} |\theta^0_{x_{\frac{N}{2}-1}} v_0'|\mathrm{d}x\Big) \\
&\le C\left(\|L-L^I\|_{L^\infty(I_{\frac{N}{2}-1})} + |L(x_{\frac{N}{2}-1})|\right)\|v_0'\|_{L^1(I_{\frac{N}{2}-1})} \\
&\le C(h^{k+1}_{\frac{N}{2}-1}\varepsilon^{-(k+1)}e^{-\beta x_{\frac{N}{2}-2}/\varepsilon} + N^{-\sigma}) \cdot h^{1/2}_{\frac{N}{2}-1}\|v_0'\|_{L^2(I_{\frac{N}{2}-1})} \\
&\le C(N^{-(k+1)} + N^{-\sigma})\|v_0\|_S \le CN^{-(k+1)}\|v_0\|_S,
\end{aligned}
\tag{2.77}
$$

where we have used the Hölder inequalities, (2.42), (2.43), (2.10) with $\nu = k+1$ and $\sigma \ge k+1$ and (2.7).

On $I_{\frac{N}{2}} = [\frac{N}{2} - 1, \frac{N}{2}]$, it follows from (2.44) that $\pi L(x) = L(x^0_{\frac{N}{2}})\theta^0_{\frac{N}{2}}(x)$. From (2.48), when $\sigma \geq k+1$ we get

$$\begin{aligned}
\left|\int_{x_{\frac{N}{2}-1}}^{x_{\frac{N}{2}}} b(L-\pi L)v_0'\mathrm{d}x\right| &\leq C\Big(|L(x_{\frac{N}{2}})|\int_{x_{\frac{N}{2}-1}}^{x_{\frac{N}{2}}} |\theta^0_{\frac{N}{2}}(x)|\,|v_0'|\mathrm{d}x + \int_{x_{\frac{N}{2}-1}}^{x_{\frac{N}{2}}} |L|\,|v_0'|\mathrm{d}x\Big) \\
&\leq C\Big(\varepsilon^\sigma \|\theta^0_{\frac{N}{2}}\|_{L^2([x_{\frac{N}{2}-1},x_{\frac{N}{2}}])}\|v_0'\|_{L^2([x_{\frac{N}{2}-1},x_{\frac{N}{2}}])} \\
&\quad + \|L\|_{L^2([x_{\frac{N}{2}-1},x_{\frac{N}{2}}])}\|v_0'\|_{L^2([x_{\frac{N}{2}-1},x_{\frac{N}{2}}])}\Big) \\
&\leq C\left(\varepsilon^\sigma h^{1/2}_{\frac{N}{2}} + \varepsilon^{1/2}N^{-\sigma}\right)\|v_0'\|_{L^2([x_{\frac{N}{2}-1},x_{\frac{N}{2}}])} \\
&\leq C(\varepsilon^{\sigma-1/2}N^{-1/2} + N^{-\sigma})\|v_0\|_S \leq CN^{-(k+1)}\|v_0\|_S,
\end{aligned} \tag{2.78}$$

where we have used $\varepsilon \leq N^{-1}$ and the facts that

$$\|L\|_{L^2([x_{\frac{N}{2}-1},x_{\frac{N}{2}}])} \leq \varepsilon^{1/2}N^{-\sigma} \text{ and } \|\theta^0_{\frac{N}{2}}\|_{L^2([x_{\frac{N}{2}-1},x_{\frac{N}{2}}])} \leq Ch^{1/2}_{N/2}.$$

Thus, it follows from (2.77) and (2.78) that

$$|Z_2| \leq CN^{-(k+1)}\|v_N\|_S. \tag{2.79}$$

From (2.75), (2.76) and (2.79) we have

$$((L-\pi L), bv_0') \leq CN^{-k}\|v_N\|_S. \tag{2.80}$$

As a result of the estimates above, we obtain

$$|\mathcal{S}_1| \leq CN^{-k}\|v_N\|_S. \tag{2.81}$$

We next estimate the second term $\mathcal{S}_2$. With the help of (2.44) and using the assumption that $\sigma \geq k+1$, we arrive at

$$\begin{aligned}
&|\langle \mathbf{P}L, bv_0\mathbf{n}\rangle| = |\langle \mathbf{P}L, b\,(v_0 - v_b)\,\mathbf{n}\rangle| \\
&\leq \left|L\left(x_{N/2-1}\right)(b\,(v_0-v_b))\left(x^-_{N/2-1}\right)\right| + \left|L\left(x_{N/2-1}\right)(b\,(v_0-v_b))\left(x^+_{N/2-1}\right)\right| \\
&\leq CN^{-(k+1)}\left(\left|(v_0-v_b)\left(x^-_{N/2-1}\right)\right| + \left|(v_0-v_b)\left(x^+_{N/2-1}\right)\right|\right) \\
&\leq CK(\varrho)N^{-(k+1)}\left(\varrho_{\frac{N}{2}-2}\left|(v_0-v_b)\left(x^-_{\frac{N}{2}-1}\right)\right|^2 + \varrho_{\frac{N}{2}-1}\left|(v_0-v_b)\left(x^+_{\frac{N}{2}-1}\right)\right|^2\right)^{1/2} \\
&\leq CN^{-(k+1)}K(\varrho)s_d\,(v_N, v_N) \leq CN^{-(k+1)}\|v_N\|_S,
\end{aligned} \tag{2.82}$$

where $K(\varrho) := \left(\varrho^{-1}_{N/2-2} + \varrho^{-1}_{N/2-1}\right)^{1/2}$ and we have used (2.4) and (2.11) in the second inequality. From (2.81) and (2.82), we get

$$|T_2(u, v_N)| \leq CN^{-k}\|v_N\|_S. \tag{2.83}$$

We finally estimate $T_3(u,v_N)$. By making use of the Cauchy-Schwarz inequality and the estimates (2.47)-(2.49) of Lemma 2.10 and Lemma 2.9 we get

$$\begin{aligned}|T_3(u,v_N)| \le& C\Big(\|\xi_R\|\|v_0\| + \sum_{n=1}^{N/2}\|\xi_L\|_{L^2(I_n)}\|v_0\|_{L^2(I_n)} \\ &+ \sum_{n=\frac{N}{2}+1}^{N}\|\xi_L\|_{L^2(I_n)}\|v_0\|_{L^2(I_n)} + \|\mathbf{P}L\|\|v_0\|\Big) \\ \le& C\Big(N^{-(k+1)} + \varepsilon^{1/2}N^{-(k+1)} + N^{-(k+1)} + N^{-(k+1)}\Big)\|v_N\|_S \\ \le& CN^{-(k+1)}||v_N||_S.\end{aligned} \tag{2.84}$$

From (2.73), (2.83), and (2.84), we have

$$|T(u,v_N)| \le N^{-k}||v_N||_S,$$

which is the desired result (2.70). Thus we complete the proof. □

Theorem 2.16. *Let $\mathcal{P}u$ be the interpolation defined by (2.40) of the exact solution $u \in H^{k+1}(\Omega)$ and $u_N \in S_N^0$ be the WG-FEM solution computed by (2.22) on the Bakhvalov-type mesh for the problem (2.1), respectively. Assume that $\sigma \ge k+1$. Then we have the following estimate*

$$||\mathcal{P}u - u_N||_S \le CN^{-k},$$

where C is independent of N and ε.

Proof. With the help of Lemma 2.8, we have

$$a(\mathcal{P}u - u_N, \mathcal{P}u - u_N) \ge C||\mathcal{P}u - u_N||_S^2. \tag{2.85}$$

Choosing $v_N = \mathcal{P}u - u_N$ in the error equation (2.67) yields

$$a(\mathcal{P}u - u_N, \mathcal{P}u - u_N) = T(u, \mathcal{P}u - u_N).$$

Using Lemma 2.15, we have

$$a(\mathcal{P}u - u_N, \mathcal{P}u - u_N) \le CN^{-k}||\mathcal{P}u - u_N||_S,$$

which together with (2.85) gives the desired result. Thus we complete the proof. □

Theorem 2.17. *Let u^I be the Lagrange interpolation of the exact solution $u \in H^{k+1}(\Omega)$ and $u_N \in S_N^0$ be the WG-FEM solution computed by (2.22) on the Bakhvalov-type mesh for the problem (2.1), respectively. Assume that $\sigma \ge k+1$. Then we have the following estimate*

$$||u^I - u_N||_S \le CN^{-k},$$

where C is independent of N and ε.

Proof. Using (2.44) and the triangle inequality, we have $||u^I - u_N||_S \le ||\mathcal{P}u - u_N||_S + ||\mathbf{P}L||_S$. Lemma 2.9 and Theorem 2.16 imply the desired result. The proof is now completed. □

Theorem 2.18. *Assume that $u \in H^{k+1}(\Omega)$ is the exact solution and u^I is the Lagrange interpolation of the solution of the problem (2.1), respectively. Assume that $\sigma \ge k+1$. Then we have the following estimate*

$$||u - u^I||_S \le CN^{-k},$$

where C is independent of N and ε.

Proof. Since $\xi = u - u^I$ is continuous in Ω, we have $|\xi|_u = s_d(\xi, \xi) = 0$. Then,

$$\|\xi\|_S^2 = \varepsilon\|\xi'\|_{L^2(\Omega)}^2 + \|\gamma\xi\|_{L^2(\Omega)}^2. \tag{2.86}$$

Using (2.49) of Lemma 2.10, we have

$$\|\xi\|^2 \le CN^{-2(k+1)}. \tag{2.87}$$

From (2.47), (2.50) and (2.51), we have

$$\begin{aligned} \varepsilon\|\xi'\|_{L^2(\Omega)}^2 &= \varepsilon\Big(\|\xi_R'\|_{L^2(\Omega)}^2 + \sum_{n=1}^{N/2}\|\xi_L'\|_{L^2(I_n)}^2 + \sum_{n=\frac{N}{2}+1}^{N}\|\xi_L'\|_{L^2(I_n)}^2\Big) \\ &\le C\varepsilon\Big(N^{-2k} + \varepsilon^{-1}N^{-2k} + N^{-2k}\Big) \le CN^{-2k}. \end{aligned} \tag{2.88}$$

Combining (2.86), (2.87) and (2.88) leads to

$$||u - u^I||_S \le CN^{-k}.$$

The proof is now completed. □

The main theorem of this section is the following.

Theorem 2.19. *Assume that $u \in H^{k+1}(\Omega)$ is the exact solution and $u_N \in S_N^0$ is the WG-FEM solution computed by (2.22) on the Bakhvalov-type mesh for the problem (2.1), respectively. Assume that $\sigma \ge k+1$. Then we have the following estimate*

$$||u - u_N||_S \le CN^{-k},$$

where C is independent of N and ε.

Proof. By Theorem 2.17, Theorem 2.18 and the triangle inequality, we conclude the desired result. □

3. Numerical experiments

In this section, we give several numerical experiments to verify computationally the theoretical convergence results obtained in Theorem 2.19. All the calculations are calculated in MATLAB R2016a and all integrals in the proposed method are approximated by using the 5-point GaussLegendre quadrature rule. Let e_N be the error between the exact solution and the approximate solution on the Bakhvalov-type mesh with N elements. Then we compute the order of convergence $OC(N)$ by the formula

$$OC(N) = \log_2(\frac{e_{N/2}}{e_N}).$$

We first present the convergence rate of the WG-FEM solution $u_N = \{u_0, u_b\}$ obtained by (2.22) and the exact solution u in the $\|\cdot\|_S$-norm given by (2.32). Besides, we investigate the error $u - u_N$ in the L^2-norm and the discrete L^∞-norm defined by, respectively,

$$\|u - u_0\|_{L^2(\mathcal{T}_N)} := \left\{\sum_{n=1}^{N}\|u - u_0\|_{L^2(I_n)}^2\right\}^{1/2},$$

and

$$\|u - u_b\|_{L^\infty(\mathcal{T}_N)} := \max_{0\le n\le N}|u(x_n) - u_b(x_n)|.$$

Example 3.1. Consider the following singularly perturbed problem adapted from [35]

$$\begin{cases} -\varepsilon u''(x) - (3-x)u'(x) + u(x) = f(x) & x \in (0,1), \\ u(x) = u(1) = 0. \end{cases} \tag{3.1}$$

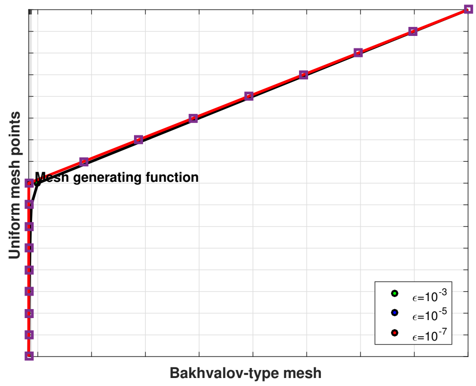


Figure 2. The Bakhvalov-type mesh (2.5) for various values of ε with $\varepsilon = 10^{-3}$, $\varepsilon = 10^{-5}$ and $\varepsilon = 10^{-7}$.

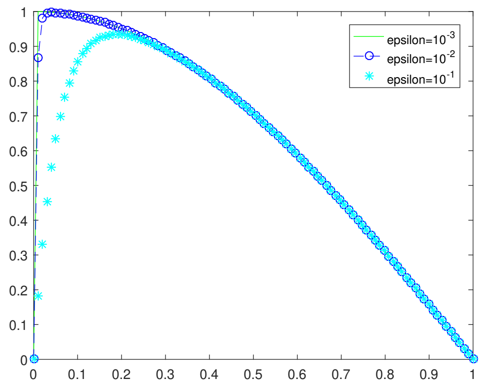


Figure 3. The exact solution of the problem in Example 3.1 with $\varepsilon = 10^{-3}$, $\varepsilon = 10^{-2}$ and $\varepsilon = 10^{-1}$.

where the function f is chosen such that the exact solution is given by

$$u(x) = \cos(\frac{\pi}{2}x)(1 - \exp(-2x/\varepsilon)). \tag{3.2}$$

For the values of $\varepsilon = 10^{-3}$, $\varepsilon = 10^{-2}$ and $\varepsilon = 10^{-1}$, we plot the exact solution of (3.1) in Figure 3. We observe that there is a boundary layer near $x = 0$ for small ε.

For the Bakhvalov-type mesh (2.5), we take $\sigma = k + 1$ and $\beta = 2$. We plot the Bakhvalov-type mesh (2.5) for various values of ε in Figure 2.

We report the history of convergence of the WG-FEM in the $\|\cdot\|_S$-norm for Example 3.1 with $\varepsilon = 10^{-3}, 10^{-5}, 10^{-7}$, $k = 1, 2, 3, 4$ and $N = 2^r, r = 3, \ldots, 8$ in Table 1. It is clear indication that the convergence rate of order k in the $\|\cdot\|_S$-norm is obtained and we numerically confirm the result of Theorem 2.19. We plot the errors in the $\|\cdot\|_S$-norm, L^2-norm and the discrete L^∞-norm for Example 3.1 with $\varepsilon = 10^{-9}$ on log-log scales in Figure 4. We observe that the order of convergence in the $\|\cdot\|_S$-norm is $\mathcal{O}(N^{-k})$ which supports the conclusion of Theorem 2.19. Furthermore, Table 2 and Table 3 indicate that the proposed WG-FEM has the optimal order of convergence of $\mathcal{O}(N^{-(k+1)})$ in the L^2-norm and the discrete L^∞-norm. Theoretical results for the optimal convergence rates in these norms can be accomplished using the techniques such as the corresponding discrete Green's function to the problem and weighted estimates. This will be investigated in a future study.

N	$\|\|u-u_N\|\|_S$	OC	$\|\|u-u_N\|\|_S$	OC	$\|\|u-u_N\|\|_S$	OC
$k=1$	$\varepsilon=10^{-3}$		$\varepsilon=10^{-5}$		$\varepsilon=10^{-7}$	
8	4.3403e-01	-	4.3445e-01	-	4.3455e-01	-
16	2.3353e-01	0.89	2.3370e-01	0.89	2.3371e-01	0.89
32	1.2117e-01	0.95	1.2126e-01	0.95	1.2126e-01	0.95
64	6.1744e-02	0.97	6.1785e-02	0.97	6.1786e-02	0.97
128	3.1170e-02	0.99	3.1191e-02	0.99	3.1191e-02	0.99
256	1.5661e-02	0.99	1.5672e-02	0.99	1.5672e-02	0.99
512	7.8496e-03	1.00	7.8549e-03	1.00	7.8550e-03	1.00
$k=2$	$\varepsilon=10^{-3}$		$\varepsilon=10^{-5}$		$\varepsilon=10^{-7}$	
8	1.4188e-01	-	1.4209e-01	-	1.4209e-01	-
16	3.9821e-02	1.83	3.9883e-02	1.83	3.9882e-02	1.83
32	1.0475e-02	1.93	1.0491e-02	1.93	1.0491e-02	1.93
64	2.6807e-03	1.97	2.6849e-03	1.97	2.6848e-03	1.97
128	6.7771e-04	1.98	6.7878e-04	1.98	6.7877e-04	1.98
256	1.7036e-04	1.99	1.7063e-04	1.99	1.7062e-04	1.99
512	4.2705e-05	2.00	4.2773e-05	2.00	4.2772e-05	2.00
$k=3$	$\varepsilon=10^{-3}$		$\varepsilon=10^{-5}$		$\varepsilon=10^{-7}$	
8	4.6352e-02	-	4.6461e-02	-	4.6462e-02	-
16	6.8032e-03	2.77	6.8199e-03	2.77	6.8200e-03	2.77
32	9.0639e-04	2.91	9.0862e-04	2.91	9.0865e-04	2.91
64	1.1640e-04	2.96	1.1669e-04	2.96	1.1669e-04	2.96
128	1.4730e-05	2.98	1.4766e-05	2.98	1.4767e-05	2.98
256	1.8520e-06	2.99	1.8566e-06	2.99	1.8566e-06	2.99
512	2.3216e-07	3.00	2.3273e-07	3.00	2.3274e-07	3.00
$k=4$	$\varepsilon=10^{-3}$		$\varepsilon=10^{-5}$		$\varepsilon=10^{-7}$	
8	1.4931e-02	-	1.4978e-02	-	1.4978e-02	-
16	1.1583e-03	3.69	1.1622e-03	3.69	1.1622e-03	3.69
32	7.8256e-05	3.89	7.8520e-05	3.89	7.8523e-05	3.89
64	5.0432e-06	3.96	5.0602e-06	3.96	5.0604e-06	3.96
128	3.1941e-07	3.98	3.2049e-07	3.98	3.2050e-07	3.98
256	2.0086e-08	3.99	2.0154e-08	3.99	2.0154e-08	3.99
512	1.2602e-09	3.99	1.2636e-09	4.00	1.2637e-09	4.00

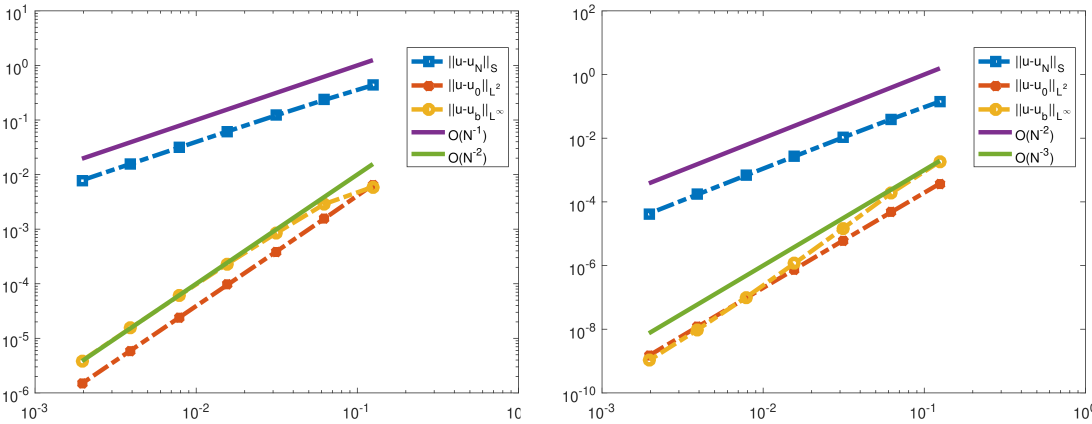
Table 1. The numerical errors in the $\|\|\cdot\|\|_S$ norm and their orders of convergence for Example 3.1

N	$\|\|u-u_0\|\|_{L^2(\mathcal{T}_h)}$	OC	$\|\|u-u_0\|\|_{L^2(\mathcal{T}_h)}$	OC	$\|\|u-u_0\|\|_{L^2(\mathcal{T}_h)}$	OC
$k=1$	$\varepsilon=10^{-3}$		$\varepsilon=10^{-5}$		$\varepsilon=10^{-7}$	
8	6.1483e-03	-	6.4609e-03	-	6.4768e-03	-
16	1.4839e-03	2.05	1.5604e-03	2.05	1.5623e-03	2.05
32	3.6509e-04	2.02	3.8449e-04	2.02	3.8488e-04	2.02
64	9.0317e-05	2.02	9.5510e-05	2.01	9.5615e-05	2.01
128	2.2227e-05	2.02	2.3802e-05	2.00	2.3833e-05	2.00
256	5.3495e-06	2.05	5.9394e-06	2.00	5.9498e-06	2.00
512	1.2283e-06	2.12	1.4825e-06	2.00	1.4864e-06	2.00
$k=2$	$\varepsilon=10^{-3}$		$\varepsilon=10^{-5}$		$\varepsilon=10^{-7}$	
8	4.2873e-04	-	3.7702e-04	-	3.7658e-04	-
16	5.4863e-05	2.97	4.8085e-05	2.97	4.8026e-05	2.97
32	6.7713e-06	3.02	6.0502e-06	2.99	6.0450e-06	2.99
64	8.0207e-07	3.08	7.5712e-07	3.00	7.5729e-07	3.00
128	9.0940e-08	3.14	9.4485e-08	3.00	9.4731e-08	3.00
256	1.0218e-08	3.15	1.1756e-08	3.01	1.1844e-08	3.00
512	1.3038e-09	2.97	1.4554e-09	3.01	1.4806e-09	3.00
$k=3$	$\varepsilon=10^{-3}$		$\varepsilon=10^{-5}$		$\varepsilon=10^{-7}$	
8	7.9959e-05	-	2.6925e-06	-	1.9916e-05	-
16	1.3509e-05	2.56	2.0709e-06	3.36	1.9149e-06	3.37
32	1.8595e-06	2.86	1.8895e-07	3.45	1.8003e-07	3.41
64	2.2406e-07	3.05	1.6457e-08	3.52	1.5171e-08	3.56
128	2.4313e-08	3.20	1.1724e-09	3.81	1.0755e-09	3.81
256	2.4613e-09	3.31	1.0916e-10	3.91	7.0681e-11	3.92
512	1.9872e-10	3.63	7.0457e-12	3.95	4.5248e-12	3.96
$k=4$	$\varepsilon=10^{-3}$		$\varepsilon=10^{-5}$		$\varepsilon=10^{-7}$	
8	1.2658e-05	-	2.6925e-06	-	2.3915e-06	-
16	1.0946e-06	3.53	1.2603e-07	4.41	1.0919e-07	4.45
32	7.5175e-08	3.86	5.6907e-09	4.46	4.9693e-09	4.45
64	4.3268e-09	4.11	2.3271e-10	4.61	2.0448e-10	4.60
128	2.2571e-10	4.26	8.2572e-12	4.81	6.9458e-12	4.87
256	8.7364e-12	4.69	2.7548e-13	4.90	2.2374e-13	4.95
512	3.1961e-13	4.77	8.8520e-15	4.95	7.1101e-15	4.97

Table 2. The numerical errors in the L^2-norm and their orders of convergence for Example 3.1

N	$\|\|u - u_b\|\|_{L^\infty(\mathcal{T}_h)}$	OC	$\|\|u - u_b\|\|_{L^\infty(\mathcal{T}_h)}$	OC	$\|\|u - u_b\|\|_{L^\infty(\mathcal{T}_h)}$	OC
$k = 1$	$\varepsilon = 10^{-3}$		$\varepsilon = 10^{-5}$		$\varepsilon = 10^{-7}$	
8	6.4371e-03	-	6.1521e-03	-	5.9496e-03	-
16	2.9132e-03	1.14	2.8997e-03	1.09	2.8870e-03	1.04
32	8.4071e-04	1.79	8.3915e-04	1.79	8.3833e-04	1.78
64	2.3173e-04	1.86	2.3133e-04	1.86	2.3126e-04	1.86
128	6.0347e-05	1.94	6.0255e-05	1.94	6.0250e-05	1.94
256	1.5401e-05	1.97	1.5379e-05	1.97	1.5378e-05	1.97
512	3.8905e-06	1.98	3.8856e-06	1.98	3.8855e-06	1.98
$k = 2$	$\varepsilon = 10^{-3}$		$\varepsilon = 10^{-5}$		$\varepsilon = 10^{-7}$	
8	1.7868e-03	-	1.8305e-03	-	1.8496e-03	-
16	1.8774e-04	3.25	1.9062e-04	3.26	1.9090e-04	3.28
32	1.4028e-05	3.74	1.4546e-05	3.71	1.4556e-05	3.71
64	1.0389e-06	3.76	1.1619e-06	3.65	1.1635e-06	3.65
128	7.1667e-08	3.86	9.9040e-08	3.55	9.9410e-08	3.55
256	3.7472e-09	4.26	9.4948e-09	3.38	9.5871e-09	3.37
512	4.8182e-10	2.96	1.0176e-09	3.22	1.0403e-09	3.20
$k = 3$	$\varepsilon = 10^{-3}$		$\varepsilon = 10^{-5}$		$\varepsilon = 10^{-7}$	
8	1.2865e-04	-	1.2983e-04	-	1.2984e-04	-
16	1.0720e-05	3.58	1.0741e-04	3.59	1.0742e-04	3.59
32	8.1671e-07	3.71	8.1704e-07	3.71	8.1705e-07	3.71
64	5.8389e-08	3.80	5.8395e-08	3.80	5.8365e-08	3.80
128	3.8467e-09	3.92	3.8469e-09	3.92	3.8469e-09	3.92
256	2.4994e-10	3.94	2.4505e-10	3.94	2.4544e-10	3.94
512	1.5740e-11	3.98	1.5790e-11	3.98	1.5781e-11	3.98
$k = 4$	$\varepsilon = 10^{-3}$		$\varepsilon = 10^{-5}$		$\varepsilon = 10^{-7}$	
8	1.1480e-05	-	1.0988e-05	-	1.0983e-05	-
16	4.7924e-07	4.58	4.4839e-07	4.61	4.4816e-07	4.61
32	1.8278e-08	4.71	1.6797e-08	4.73	1.6731e-08	4.73
64	6.3773e-10	4.86	6.0005e-10	4.80	6.0028e-10	4.80
128	2.0781e-11	4.93	1.9500e-11	4.94	1.9523e-11	4.94
256	6.7244e-13	4.94	6.1513e-13	4.98	6.1554e-13	4.98
512	2.1141e-14	4.99	1.9257e-14	4.99	1.9285e-14	4.99

Table 3. The numerical errors in the discrete L^∞-norm and their orders of convergence for Example 3.1



(a) Linear element **(b)** Quadratic element

Figure 4. Convergence rates of there norms using (a) Linear and (b) Quadratic element functions for Example 3.1 with $\varepsilon = 10^{-9}$.

4. Conclusion

In this work, we introduce and analyze a WG-FEM for the one-dimensional singularly perturbed problem of convectiondiffusion type. We introduce two stabilization terms for discretization of the diffusion term and convection term in order to derive an optimal and uniform error estimate for the convection dominated problems. The parameter-free error estimates in the corresponding energy norm of the proposed method is established on Bakhvalov-type mesh using high order elements. The present method and analyses can be extended to higher dimensional SPPs since the construction of the interpolation operator is simple and is suitable for the analysis of the WG-FEM. We will study this direction in upcoming paper.

Acknowledgment. The author would like to thank the two anonymous reviewers for their very careful reading of the manuscript and insightful comments and suggestions that improve the paper.

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